New developments in numerical relativity

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New Developments in Numerical Relativity

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Abstract. Numerical Relativity is gaining momentum and acceptance as a tool for investigating the both the Einstein equations themselves, and their application to astrophysics. While this research field is not new, there are many reasons to believe that it is achieving a certain maturity that will make it useful to the larger relativity and astrophysics communities. I touch on a number of new developments in numerical relativity, including new collaborations, explosive growth in computer power, new formulations of the Einstein equations, tools for analysis in numerical studies, and applications to 2D and 3D black hole spacetimes.

1 Introduction

For all their mathematical beauty, Einstein's equations of general relativity form a set of fourteen highly nonlinear, coupled, hyperbolic and elliptic partial differential equations that are not easily amenable to analytic study, except in a handful of idealized cases involving specialized symmetries or approximations. They are among the most complicated equations in mathematical physics. For this reason, and despite many years of investigation, the solution space of the general set of Einstein equations remains largely unknown. As a result, many researchers have turned to computers to aid in the study of these equations. In particular, numerical relativity has emerged and matured over the past few decades. This approach draws upon synergistic developments in high performance computing, computer science, theoretical physics, and mathematics to investigate the Einstein equations and their application to astrophysics and cosmology.

However, due to the enormous complexity of the equations, and the subtle numerical techniques needed to obtain reliable solutions, the promise of numerical relativity has been slow to be realized. For many relativists, untrained in computers and numerical methods, it remained something of a black art. On the other hand, for those attempting to find numerical solutions, inadequate computer power, coupled with uncertainties in which mathematical formulation of the equations, which gauge conditions, and which analysis tools are most appropriate for obtaining and interpreting numerical solutions, slowed progress. In this article, which is based on more extensive treatments found in [1, 2, 3], I will describe exciting developments in many areas that are impacting on numerical relativity.

2 New Collaborations and Computational Powers

2.1 Collaborations

For many years numerical relativity has been carried out by a small number of relatively independent research groups. Typically even a 1D or 2D numerical code can take years of careful development and testing by one group. Furthermore, the codes are often specialized and robust only for specific problems, so they are not widely useable by other research groups. Hence each group developed its own codes for its own problems, leading to considerable duplication of effort. These codes also required expert users, and hence were not useful to other groups in the community as research tools. Finally, these codes were developed primarily by relativists, and not by computer scientists or applied mathematicians who have years of expertise in developing efficient algorithms for solving complex partial differential equations on computers.

Fortunately this outlook is changing rapidly! Modern code development tools, coupled with symbolic manipulators that manipulate the Einstein equations and generate efficient and accurate computer code, speed the development of codes for numerical relativity while increasing their robustness. Furthermore, a number of large collaborations have developed between established computational relativity and astrophysics groups and computer scientists to develop powerful codes to solve the Einstein equations. For example, the Binary Black Hole Grand Challenge Alliance, which includes researchers at Texas, Syracuse, Illinois, North Carolina, Pittsburgh, Cornell, Penn State, and Northwestern, is an NSF funded collaboration of leading numerical relativity and computer science groups developing 3D codes to study the problem of two black holes coalescing, and the gravitational waves they will emit. A similar grand challenge program funded by NASA includes groups at Illinois, Washington University, Argonne, Livermore, Stony Brook, and Potsdam, aim at developing high performance codes for general relativistic hydrodynamics to study the problem of coalescing neutron stars. Also, a large numerical relativity group has recently developed in Potsdam with close ties to the NCSA and Washington University groups.

These collaborations pull together expertise from many related disciplines in relativity, computational science, astrophysics, and nuclear physics. In addition to Einstein equation solvers and symbolic manipulator packages to generate them, complex codes are under development for hydrodynamics, adaptive mesh refinement, parallel I/O, visualization, etc. Further, the codes they develop will be made available to their communities. As codes are developed and tested, they are made available on the world wide web. At present, many useful codes (some even come with documentation!) can be found on the servers of the NCSA-Potsdam-WashU numerical relativity group at http://jean-luc.ncsa.uiuc.edu or its European mirror site at http://www.aei-potsdam.mpg.de, the Black Hole grand challenge server at http://www.npac.syr.edu/projects/bh/, and the Neutron Star grand challenge server, which can also be found at http://jean-luc.ncsa.uiuc.edu. These collaborations and others are creating powerful tools for exploring the Einstein equations and their application to many problems in astrophysics and cosmology. These tools will also help smaller groups get involved through code distribution and collaborations.

2.2 The Supercomputing Revolution

The computational requirements for solving the full set of Einstein equations, without symmetries, are immense. Several years ago, given the numerical algorithms in use at that time, we estimated that a single calculation of a 3D black hole coalescence would take about 100,000 hours on a multimillion dollar Cray Y-MP, at that time the world's fastest computer! However, relatively inexpensive microprocessors are becoming ever more powerful, now rivaling the speed of the specialized processors found on conventional supercomputers. Computer manufacturers have begun to connect them together in parallel, multiplying the performance on many problems of physical interest by the number of processors.

Based on scalable, massively parallel technology, machines are now available that can achieve tens of GFlops on actual application codes. This is about two orders of magnitude faster than could be achieved just a few years ago on, say, a Cray Y-MP, due to both hardware and algorithm improvements. Perhaps more importantly for numerical relativity, where 3D codes require dozens or hundreds of variables, the memory of the machines is increasing along with the speed; we now have access to dozens of GBytes of memory, allowing much larger problems to be solved.

The present parallel machines are dramatically more powerful than the machines of just a few years ago, but we can expect much more capability very soon. By the turn of the century we should have access to machines capable of Teraflop performance and with Terabyte memories, making highly resolved, full 3D calculations of the Einstein equations routine. These developments are changing the face of numerical relativity and computational science dramatically in a very short time.

Although the explosion in computer power is very important for 3D numerical relativity, a huge supercomputer is not needed for one to make contributions in 1D and 2D studies. Workstations, with roughly the power of the multimillion dollar Y-MP of a few years ago, are able to run codes for studying many important problems in 1D and 2D spacetimes. These machines are within the budgets of many research groups. I think this is a very exciting change in the field that will have wide impact.

3 Mathematical Formulation

I will now turn to a few specific examples of what I feel are important new developments for numerical relativity. This is just a small sample of the work that is going on worldwide, but representative of exciting things to come.

First I discuss new developments in the underlying mathematical formulation of the Einstein equations for numerical relativity. The basic separation of the initial data problem and its subsequent evolution has been known for years, and one may refer to the classic article by York[4] for a review. Although the basic algorithm for solving the constraints for initial data has been around for a long time, only in the last few years has it been used to produce full 3D numerical solutions for 2 black holes with arbitrary masses, spin, and angular momentum[5, 6]. This work generalized the early, classic work of Misner[7] and others on time symmetric, axisymmetric two-black hole data sets; in other words, two black holes at rest, ready to fall together along the axis of symmetry at $t = 0$. In yet another development, a series of interesting axisymmetric black hole data sets have been developed, corresponding to an "adjustable" gravitational wave surrounding a single black hole. These are useful for studying highly distorted black holes and their horizons, and their interactions with highly nonlinear gravitational waves. These black holes are sometimes referred to as the "Brill wave plus black hole" data sets[8]. Finally, these distorted black hole data sets were recently generalized to include angular momentum[9, 10]. These data sets will keep numerical relativists busy for some time in developing numerical codes to *evolve* these black holes.

The basic evolution equations for such initial data sets have also been known for many years[4], but they have recently been completely overhauled by a number of authors. The original "ADM" formulation for numerical relativity has been used by nearly all practitioners of numerical relativity for the last 30 years. However, although this formulation does provide a practical scheme for evolving initial data, it is extremely complicated; the physical and mathematical interpretation of the many thousands of terms in the equations is simply not possible. In contrast, for hydrodynamics the equations are in a well studied flux conservative, hyperbolic system of balance laws having the form

$$\partial_t \mathbf{u} + \partial_k F^k_- \mathbf{u} = S_- \mathbf{u} \tag{3.1}$$

where the vector $\mathbf{u}$ displays the set of variables and both "fluxes" F^k and "sources" S are vector valued functions. In such a system, the evolution matrix can be diagonalized, so that the fields which actually propagate can be identified (the eigenfields), and their speeds are known (the eigenvalues). All spatial derivatives are contained in the flux terms, which have a well understood physical interpretation. Finally the source terms in the equations contain no derivatives of the eigenfields. All of these features can be exploited in numerical finite difference schemes that treat each term in an appropriate way to preserve important physical characteristics of the solution.

Amazingly, the complete set of Einstein equations can also now be put in this "simple" form. Building on earlier work by Choquet-Bruhat and Rugerri[11], Bona and Massó began to study this problem in the late 1980's, and by 1992 they had developed a fully hyperbolic system for the Einstein equations with harmonic lapse and zero shift[12] (by fully hyperbolic I mean that the system has a complete set of eigenfields and can be diagonalized). This work was generalized recently to apply to a large family of slicing conditions and arbitrary shift[13]. Independently, another system was developed by Abrahams, Anderson, Choquet-Bruhat, and York[14]. This system contains an extra time derivative, so that a second order hyperbolic system for the extrinsic curvature K_{ab} is found. 1D and 3D codes based on these two new formulations are under development at present. These works have sparked considerable interest in the relativity community, and now I am aware of several more systems of hyperbolic equations for the Einstein system (see, e.g., [15, 16]and references therein).

The numerical properties of these systems are being investigated by many groups, and although it is not clear at present which of these will prove best suited for numerical study, there is considerable excitement about the possibilities. There are a number of potential advantages to these treatments. In particular, two points stand out:

- (*i*) *Causal structure.* First, the ability of finding the eigenfields explicitly allows one to write the system as a set of "uncoupled" wave equations, each with its own characteristic speed. (Of course, there is coupling in the source terms, and the speeds depend on the fields as well.) Knowing which fields are propagating, and knowing their speeds, can be of great value in developing boundary conditions. This information may be particularly helpful at both the outer boundary, where for example one generally wants to impose that radiation leaves the system, and at the horizon of a black hole, where one wants to impose that all information must be ingoing to the black hole interior.
- (*ii*) *Treating large gradients.* Second, in numerical relativity one often encounters large gradients, particularly near singularities or black hole horizons. Some of the gradients may represent physical effects, and some may be due to coordinate pathologies that develop. In any case, such peaks and gradients are very troublesome to handle numerically with finite resolution, and are presently the main difficulty in evolving black holes for long times[17, 18]. Finally, actual shocks may develop in the fluid variables, which may lead to large gradients in the metric functions as well. Very advanced numerical methods have been developed over the last 30 years in hydrodynamics for treating just such large gradients, and this numerical "technology" can be borrowed directly for numerical relativity for the first time. In a particular example of the power of this approach, using such hydro techniques a single black hole in spherical treatment has been evolved beyond $t = 1000M$, where M is the mass of the black hole, with errors of less than a few percent[13]. By contrast, standard techniques using the

non-hyperbolic ADM approach develop serious errors after only about $t = 150M$, and the computations cannot be continued.

4 New Tools for Understanding Numerical Evolutions

Up to now I have discussed how new collaborations and supercomputers are injecting much needed expertise and computational capabilities into numerical studies of the Einstein equations, and furthermore, that the initial value problem is basically solved, and new and promising formulations of the evolution equations should aid greatly in their numerical evolution. However, once one has a numerical solution, one still faces the difficult task of understanding the physical content of the results. For example, the numerical solution consists typically of the metric and extrinsic curvature functions on a spacetime grid. These quantities of course depend fundamentally on the coordinate system and gauge conditions used. But one would like to ask physical questions like "What is the radiation waveform emitted?", or "Did a black hole horizon form?" Furthermore, one often needs to appeal to simpler analytic models to understand the physical processes uncovered by the numerical results. In this section, of many new results in this area I will sketch two important advances: (*i*) finding horizons in numerical spacetimes, and (*ii*) perturbative approaches to numerical black hole studies.

4.1 Tools for Analyzing Horizon Dynamics

There are two types of horizons that are usually discussed in black hole physics. Both are useful in numerical relativity. The first is the apparent horizon (AH), defined as the outermost trapped surface on a given spacelike slice, if it exists, in a spacetime. Since such a surface can be defined completely on a single slice of spacetime, it is very convenient for studies of numerically generated black hole spacetimes. Methods for finding apparent horizons, and their use in dynamic spacetimes, are described in various references[19, 20].

In contrast to the AH, the event horizon (EH) is a global object in time; it is traced out by the path of outgoing light rays that *never* propagate to future null infinity, and *never* hit the singularity. (It is the boundary of the causal past of future null infinity $\dot{\mathcal{J}}^-(\mathcal{I}^+)$.) For this reason, in principle one needs to know the entire time evolution of a spacetime in order to know the precise location of the EH. However, in spite of the global properties of the EH, hope is not lost for finding it very accurately, even in a numerical simulation of finite duration.

In principle, as the EH is a null surface, it can be found by tracing the path of null rays through time. Outward going light rays emitted just outside the EH will diverge away from it, escaping to infinity, and those emitted just inside the EH will fall away from it, towards the singularity. In a numerical integration it is difficult to follow accurately the evolution of a horizon generator forward in time, as small numerical errors cause the ray to drift and diverge rapidly from the true EH. It is a physically unstable process. But we can actually use this property to our advantage by considering the time-reversed problem. In a global sense in time, any outward going photon that begins near the EH will be *attracted* to the horizon if integrated *backward* in time [21, 20]. In integrating backwards in time, it turns out that it suffices to start the photons within a fairly broad region where the EH is expected to reside. Such a horizon-containing region, as we call it, is often easy to determine after the spacetime has settled down to approximate stationarity. The crucial point is that when integrated backward in time along null geodesics, this horizon-containing region shrinks rapidly in "thickness", leading to a very accurate determination of the location of the EH at earlier times. Although one can integrate individual null geodesics backward in time, we find that there are various advantages to integrate the entire null surface, which we denote by $f(t, x^i) = 0$, backward in time. For details see [21].

Once the AH or EH has been located, there are many ways to analyze it to study black hole physics. The function $f(t, x^i)$, together with the metric induced on the 2-surface, gives both the topology and the intrinsic geometry of the surface, from which important physical properties can be determined. The topology can, for example, determine whether there is one black hole, or more. Through the metric induced on the surface, one can also compute the area (or the corresponding "mass" of the horizon), the Gaussian curvature, and also various circumferences and their ratios. For example, in axisymmetric spacetimes one can measure the horizon's intrinsic polar and equatorial circumferences, labeled C_p and C_e respectively, which is often a useful indicator of the geometry of the surface. These measurements can be made by computing straightforward integrals of the metric induced on the 2D horizon surface.

I will show a few examples of how these simple tools can be used. We have performed numerical evolutions of various axisymmetric black hole spacetimes discussed in section 3 above. The "Brill wave plus black hole" solutions, with and without rotation, provide examples of dynamic, distorted single black holes, while the Misner data sets provide examples of two black holes merging to form a single black hole. What can these evolutions tell us about the dynamics of black hole horizons? As a first example, we evolve a perturbed "Brill wave plus black hole" spacetime and also the head-on collision of two black holes (Misner data), and find the event horizons as explained above. The Misner data is for the case $\mu = 2.2$ in the language of Ref.[22]. In Fig. 1, for both simulations I show the ratio C_r

$$C_r = \frac{C_p}{C_e} \tag{4.1}$$

and compare the time development of C_r to the quasi-normal frequencies of the spacetime. Clearly the black horizons themselves are oscillating with the quasi-normal frequencies of the spacetime. One can also construct directly from $f(t, x^i)$ its embedding diagram on a given slice, and embedding "histories" showing its evolution. This technique has been described fully in Ref. [19].

Finally, there are new tools that can be applied to the study of event horizons. The EH is generated by null geodesics, and with the above techniques accurate trajectories of the "horizon generators", which contain all the information of the dynamics of the EH, can be obtained. The shear, expansion, and other properties of these generators can be measured

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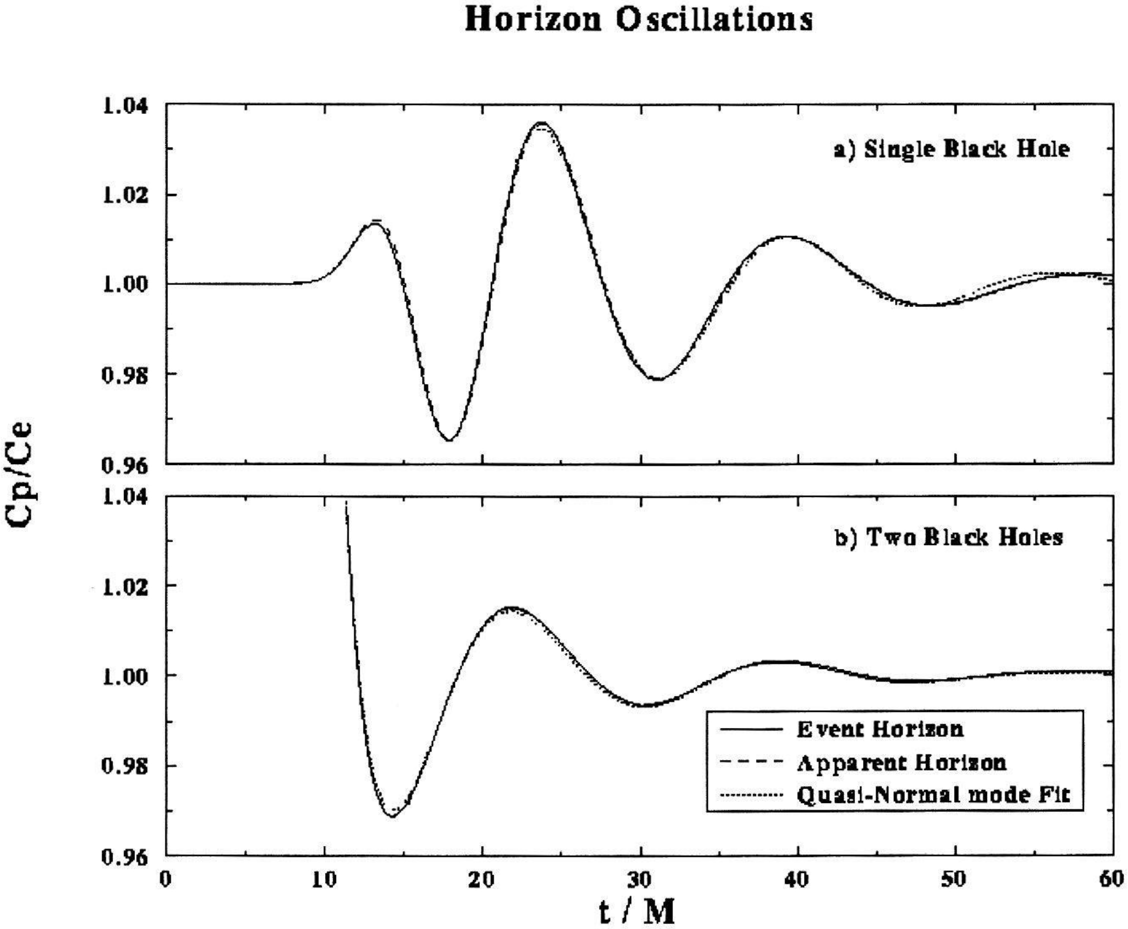

Figure 1: Quasi-Normal Mode Oscillations of the Event Horizon. The top figure shows the evolution of a distorted single black hole, while the bottom figure shows the evolution of the colliding black hole case discussed above. The quasi-normal modes, as determined analytically, of black holes with the same ADM masses are plotted as dashed lines. We see that the horizons are oscillating with precisely the analytically determined quasi-normal mode frequencies.

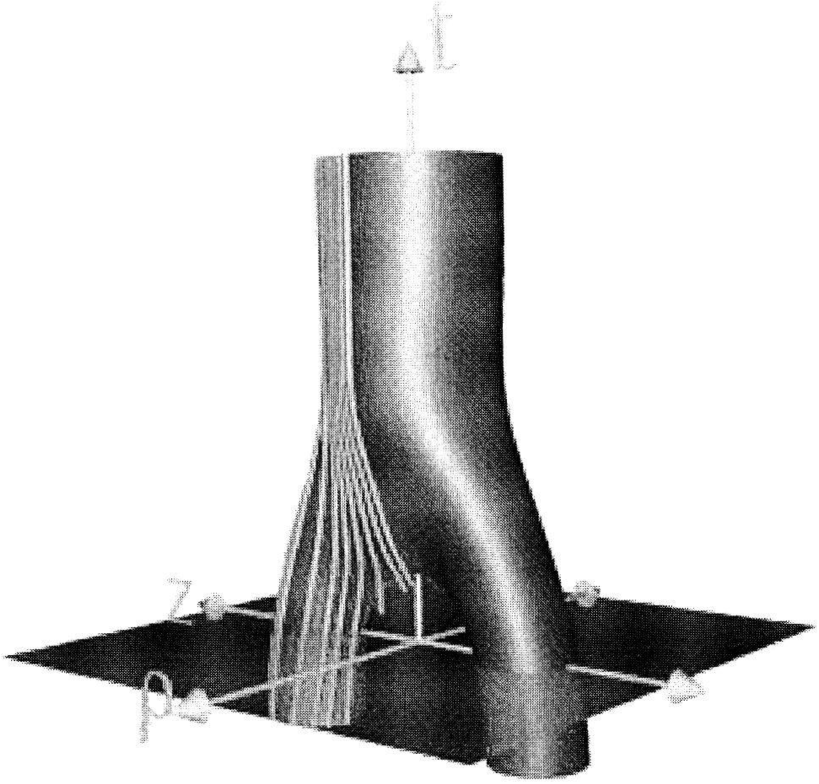

Figure 2: The Pair of Pants. The figure shows the embedding of the horizon computed for Misner two black hole data, with horizon generators superimposed.

numerically and translated into physical properties of the horizon surface, as described in [24]. We can combine these embedding and generator finding techniques to illustrate the dynamics of the EH surface. In Fig. 2 I show the embedding of the "pair of pants" (two colliding black holes) spacetimes. Such a diagram has been so familiar in the minds of relativists for the last 25 years. It is interesting to note that in the "pair of pants" diagram, there are the generators leaving the horizon (going backwards in time) at the inner seam of the pants. There is a line of caustic points extending backward from the "crotch" point where the two horizons merge [21, 23, 24, 25]. It is at these points along the caustic line in the history diagram that photons originally travelling in the causal past of null infinity ($J^-(\mathcal{I}^+)$) join the horizon. However, as only the surface of the horizon has been embedded; the photons that have left the embedding diagram have also left the embedding space, and their paths are not shown in the figure. The same techniques have been applied to distorted, and rotating black holes, as discussed in [24]. These tools should be useful for many studies of numerically generated black hole spacetimes.

4.2 Perturbation Theory

With so many numerical capabilities for evolving black hole spacetimes, it is very important to be able to develop methods that provide physical understanding of the numerical results, as well as checks on their accuracy. Recently, a particularly promising technique called the "close-limit" has been developed, based on black hole perturbation theory[28, 29]. In principle, this method applies when the holes are initially very close together. In this case, the horizon is initially only slightly nonspherical and the spacetime that evolves outside the horizon can be treated as a perturbed single black hole. The highly nonspherical nature of the spacetime inside the horizon is causally disconnected from the exterior, and from the generation of outgoing gravitational waves. The exterior spacetime can be evolved forward in time from the initial data hypersurface with the linearized equations of perturbation theory.

This method turns out to be remarkably successful[28, 30, 31, 32]. The details of this success has provided insights into the nature of collisions of holes, and should also apply to many systems of dynamical black holes. For holes that are momentarily stationary, the close-limit predictions of radiated energy and waveforms are quite good (i.e., in agreement with the results of numerical relativity) even when the initial horizon is highly distorted, violating the assumptions underlying the method. The close limit has also been used [33, 32] for the head-on collision of holes with initial momenta towards each other. The waveforms and energies agree remarkably well with numerical simulations. The success of these estimates suggests, among other things, that to a large extent the role of the early weak-field phase of the evolution is only to determine what the momentum of the holes will be when they start to interact nonlinearly.

So far, the method has only been applied to axisymmetric spacetimes, but should also be very useful in studies of 3D black hole spacetimes. This technique is also being applied to rotating black hole spacetimes, including the collision of rotating black holes. This is one important example of an emerging synergism between numerical relativity and approximation theory that will play an increasingly important role in the future of this field.

5 Colliding Black Holes in 2D and 3D

There is much exciting work that is beginning in numerical relativity, including pure gravitational wave studies, critical phenomena, and general relativistic hydrodynamics. Fully relativistic hydrodynamics is just beginning, but is one of the main directions I see numerical relativity moving. But in this paper I will have to focus on just one area that has received much attention during the last years. In this section I discuss results obtained by evolving various black hole initial data sets described in section 3, using the methods and analysis tools discussed above. The study of distorted black hole initial data sets with and without rotation has taught us much about the behavior of highly distorted black holes, the radiation they emit, and the structure of their horizons. Here I focus on evolutions of the classic Misner two black hole data set, consisting of two axisymmetric, equal mass black holes at rest.

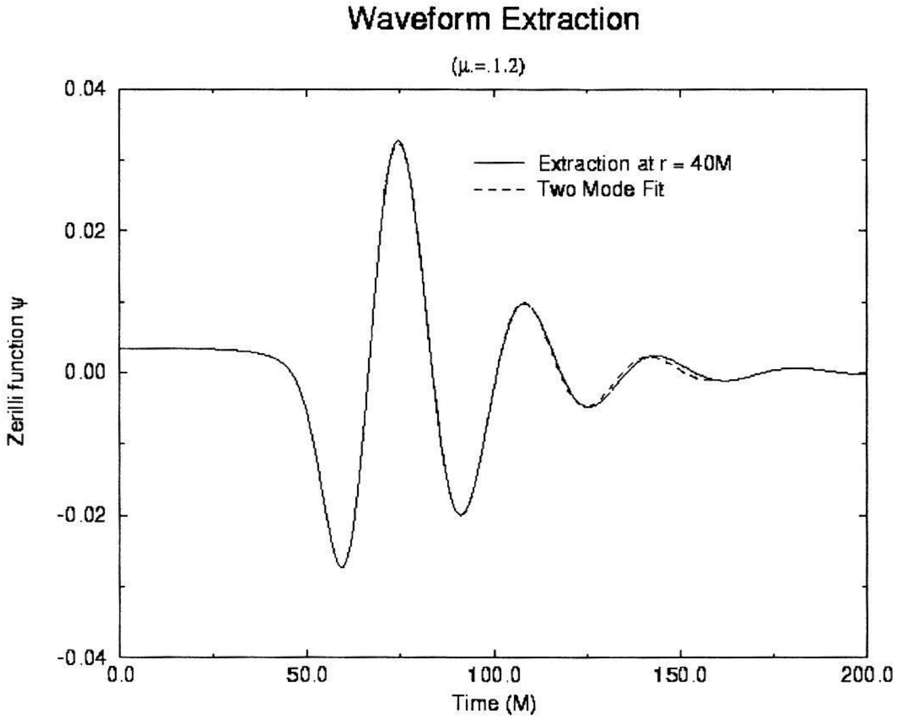

Figure 3: The $\ell = 2$ waveforms for the case $\mu = 1.2$. The solid line is the numerically generated waveform extracted at $r = 40M$. The dashed line is a fit of the two lowest $\ell = 2$ quasi-normal modes, over the domain $70 < t/M < 160$, to the extracted waveform.

Evolutions of this data set have been attempted by many groups over the years, beginning with Hahn and Lindquist[34], and later Smarr [35], and most recently by the NCSA/WashU (and now Potsdam) group[36, 37, 22]. Although this is one of the simplest possible two black hole data sets, this important stepping stone to more general cases provides important clues to the kinds of problems to be encountered, and the physics to be expected in the general 3D case.

These simulations show that the quasi-normal modes[38] of the final black hole are strongly excited as a result of the collision. The total energy emitted by the waves was shown to be on the order of 0.1% of the total mass of the system. These results are discussed in detail in [36, 22]. For a particular case where the black holes are close, and the black holes are already surrounded by a common horizon (corresponding to the Misner parameter $\mu = 1.2$), I show the $\ell = 2$ waveform in Fig. 3. This waveform was obtained by using a gauge-invariant waveform extraction technique developed by Abrahams[39].

The quasi-normal ringing mode of the black black hole is clearly excited, and fits the numerically obtained waveform with great accuracy. For cases where the black holes are farther apart, we see a similar phenomenon. The radiation emitted by the infalling of the black holes is minimal compared to the excitation of the quasi-normal ringing of the final

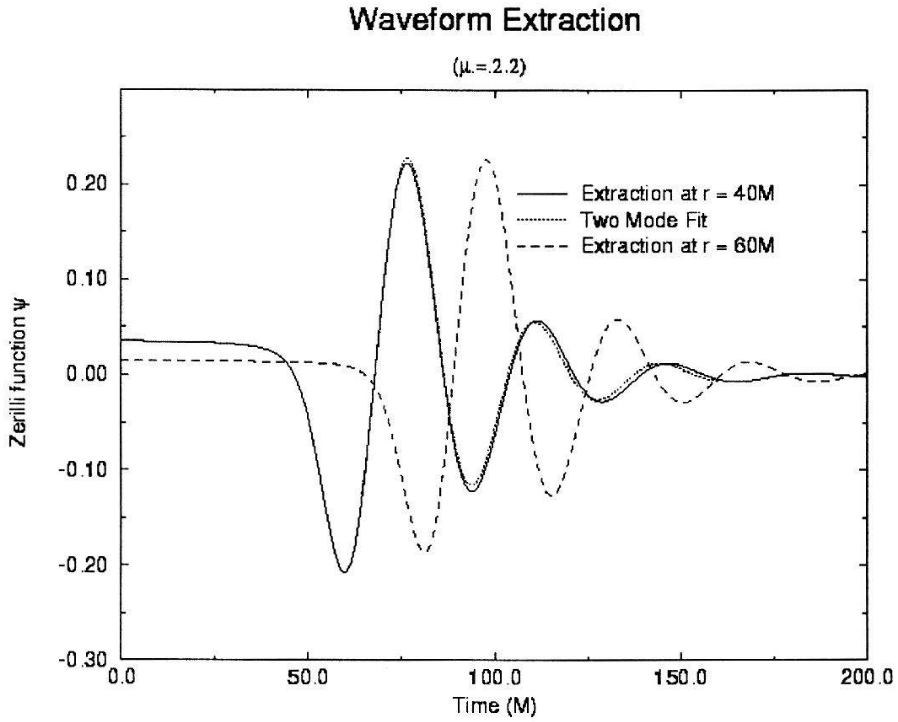

Figure 4: The $\ell = 2$ waveforms for the case $\mu = 2.2$. The solid line is the waveform extracted at $r = 40M$ and the long dashed line is the waveform at $r = 60M$. The dotted line is the quasi-normal mode fit.

black hole formed during the merger process. In Fig. 4 we show the $\ell = 2$ waveform obtained from numerical simulations of the case with Misner parameter $\mu = 2.2$. In this case we know from studies of the event horizons that the two black holes are separate initially.

In Fig. 5 we show the collision of two black holes for the case shown above in Fig. 4 (Misner parameter $\mu = 2.2$). The image shows the system at a late time after the holes have collided. The grayscale map shows the variation in the radiation field Ψ_4, and the height of the graph is related to the value of the lapse. The apparent horizon is shown as a black ring, while the two individual black hole throats can be seen inside the horizon. At this point, the holes have collided and most of the radiation has been emitted in the form of quasi-normal modes of the final black hole.

Based on the experience developed with this system and the rotating black hole system discussed in section 3 above, a logical next step would be the collision of spinning, rotating black holes, where spin-spin effects can be studied for the first time. In the case of counter rotating black holes, one might expect more radiation to be emitted because all rotational energy in each hole must be converted to radiation, as the final system contains no angular momentum. The question is how much of this energy goes into the holes and how much is radiated away? This question can be settled using the numerical techniques and codes

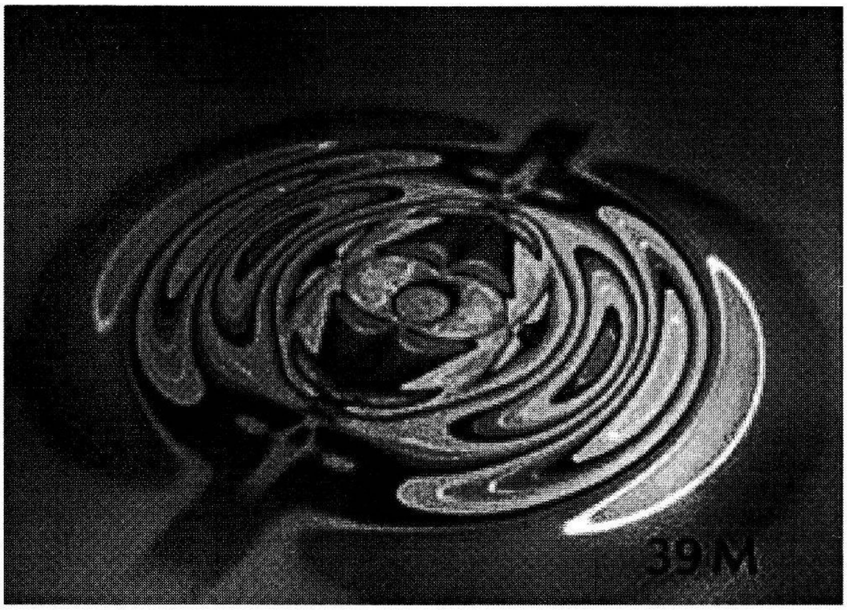

Figure 5: Collision of 2 black holes. The spatial distribution of the strength of the radiation field Ψ_4 is shown in grayscale, at a time $39M$ into the evolution, where M is the ADM mass of the system. The height of the graph is proportional to the difference of the lapse from unity. The apparent horizon is shown as a black ring, while the two individual black hole throats can be seen inside the horizon.

developed as discussed above.

5.1 3D Black Hole Studies

Beginning first with spherical black holes, and then considering axisymmetric, colliding, rotating and distorted black holes, in the last several years we are finally moving on to the true 3D case.

At this time a number of 3D codes have been constructed for solving the complete set of Einstein equations in the absence of symmetries [40, 41, 42, 43]. I will briefly describe results obtained by the NCSA/WashU/Potsdam collaboration on colliding black holes in 3D. In our 3D studies, we began with spherical and axisymmetric systems, but now treated in a general 3D Cartesian coordinate grid. Where possible we take advantage of reflection symmetries of the spacetimes so that we can perform computations only on the needed portion of the 3D spacetime, say, for equatorial plane symmetric, axisymmetric spacetimes. This allows one to evolve only a single octant of the full spacetime, providing eight times the resolution one could achieve in treating the complete spacetime. However, no symmetry assumption is used to simplify the evolution equations, and in all cases the systems are treated in a true 3D fashion. In extensive 3D studies of spherical black holes [41] we described what can be achieved with present resolution (200^3 zones in 3D) using various slicing conditions. With geodesic slicing, one can accurately reproduce results from 1D codes, and with maximal slicing one can evolve the black hole until about $t = 35M$ with standard techniques. We have also explored a number of algebraic slicing conditions, as described above, and found that using these we can evolve until about $t = 50M$, but the "grid stretching" inherent in all singularity avoiding time slicings prevents us from going further. Adaptive mesh refinement, which automatically adds grid zones as they are needed during a calculation, would clearly help this situation, but even so the evolution will eventually generate unresolvable gradients that can lead to numerical instabilities. To address this problem, we have implemented a so-called apparent horizon boundary condition (AHBC) in our 3D code, that excises the region inside the horizon from the calculation. At present the use of this technique brings the evolution to over $t = 100M$, with hamiltonian violation on the order of 10^{-3} throughout the evolution. Work is continuing to refine this technique at present[41].

We have also computed the head-on collision of two equal mass black holes in the 3D code, and compared with the extensive work performed in 2D as described above. Preliminary results for the case $\mu = 2.2$ agree very well with 2D, although we cannot yet evolve the 3D system as far into the future. In Fig. 6 we show the evolution of the radiation field Ψ_4 as a grayscale map, and the coordinate position of the event horizon, traced out using the techniques described above. Notice the "banana" shaped quadrupole lobes of radiation propagating out from the colliding holes, now in 3D, just as in the 2D calculations. Quantitative studies of the wavelength and damping time of the radiation and the coalescence of the horizons show excellent agreement with the 2D studies.

6 Conclusion

Numerical relativity has been under development for the last 30 year. The coming of age of numerical relativity is driven partly by the stunning increases in computer power, and partly by the great promise of gravitational wave astronomy made possible by the Laser Interferometer Gravitational Wave Observatories (LIGO and VIRGO, to begin operation in a few years). This increase in computer power is important not only for 3D numerical relativity, but also for the many groups who now have the capability to purchase workstations that are powerful enough for 1D and 2D calculations.

The most intense research effort in numerical relativity is on the study of the gravitational radiation emitted by violent astrophysical events, with one of the most interesting case being the coalescences of black holes in binary systems. The dynamical behavior of such strongly gravitating astrophysical systems, governed by the non-linear partial differential equations of classical general relativity, is beyond the reach of analytic treatments. Through numerical relativity, it may be possible to calculate accurately the waveforms of the gravitational radiation coming from these events.

One particular point to be made about numerical relativity is that it is still a very young and developing field. For this reason, simple ideas can have great impact. A good example is the backward integration methods that enable studies of the dynamics of event horizons in numerical relativity. I expect that within a few years, many other important ideas will emerge, which when combined with the immense computing power to emerge, and

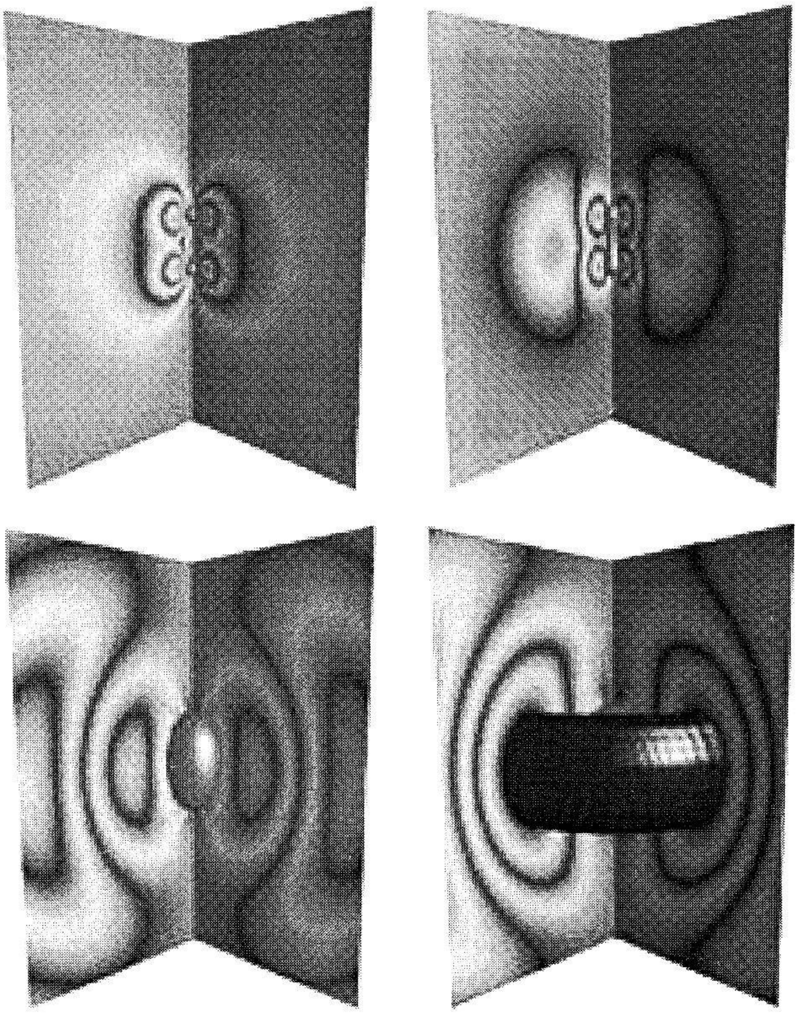

Figure 6: 3D Evolution of the Head-on Collision of Two Black Hole. The radiation field Ψ_4 is shown as a grayscale map. The event horizon is shown as a solid object in the center. Features are compared to the results obtained in 2D evolutions, showing excellent agreement.

the observational data, will make this field very exciting indeed.

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