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A Promeasure on the Space of Ashtekar Connection 1-Forms

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Abstract. An SU(2) gauge-invariant promeasure on the space of Ashtekar connection 1-forms on a smooth, compact, closed, orientable, Riemannian 3-manifold will be constructed in this paper. Suggestions to implement diffeomorphism-invariance in the constructed SU(2) gauge-invariant promeasure will also be made.

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1 Introduction

The aim of this paper is two-fold: to construct a promeasure on the space $\mathcal{A}$ of Ashtekar connection 1-forms defined on a closed, smooth, compact, orientable, Riemannian 3-manifold Σ, and then to use the constructed promeasure to define a diffeomorphism-invariant promeasure on the SU(2) moduli space $\mathcal{A}[\mathrm{SU}(2)]$—the space of SU(2)-gauge orbits of $\mathcal{A}$. This is just the quotient space of $\mathcal{A}$ consisting of cosets such that any two representatives of a coset in $\mathcal{A}[\mathrm{SU}(2)]$ are related by an SU(2)-gauge transformation. The main interest in the construction of a promeasure of $\mathcal{A}[\mathrm{SU}(2)]$ lies in its application to the loop representation of quantum gravity—see, for example, [13, Eqn (3.40), p. 1644] and its accompanying footnote, or reference [14]. Briefly, given a promeasure μ on $\mathcal{A}$, the functionals $\tilde{\Psi}[A]$ on $\mathcal{A}$ and the functionals $\Psi[\gamma]$ on the space of piecewise smooth loops

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$\gamma : [0,1] \to \Sigma$ of Σ are related by

$$\Psi[\gamma] = \int_{\mathcal{A}} H[\gamma, A]\tilde{\Psi}[A]\, \mathrm{d}\mu(A),$$

where $H[\gamma, A]$ is a complexified SU(2)-holonomy around γ. A second reason for constructing a measure lies in the attempt to define a physical inner product in the physical state space of the self-dual representation theory of quantum gravity [1, 13, 14]—however, this will not be pursued here.

It should be noted that Ashtekar and Lewandowski [4] constructed an interesting class of promeasures on the space of connection 1-forms of a principal G-bundle $P(M, G)$, where G is a compact Lie group and M is an n-manifold. An appealing aspect of the promeasures they constructed pertains to their invariance under Diff(M). The construction however does rely strongly on the compactness of G. Very briefly, in reference [4], a diffeomorphism-invariant promeasure on the closure of the space of gauge-equivalent G-connection 1-forms is obtained by introducing certain equivalence classes of piecewise analytic loops called *hoops* and using the fact that a compact, Hausdorff topological group admits a (unique) normalised Haar measure.[1] Another different approach to the construction of a promeasure on $\mathcal{A}$ and its SU(2) moduli space using graphs and their vertices can be found in references [5, 6].[2]

The construction of a promeasure given in this paper will differ somewhat from the works of Ashtekar *et al.* [4, 5] and Baez [6] and so provides yet another avenue of constructing non-trivial SU(2) gauge-invariant promeasures on $\mathcal{A}$. Here, the family of spaces defining the projective limit is a family of finite-dimensional manifolds and the promeasure constructed on $\mathcal{A}$ is induced from a sequence of Gaussian-type measures defined on the family of finite-dimensional manifolds. In a way, this is a generalisation of the standard construction of a promeasure, where the projective family is a family of finite-dimensional topological vector spaces—the finite-dimensional topological vector spaces are replaced with finite-dimensional manifolds. The construction delineated in this paper does not appeal to the compact criterion of the gauge group in anyway.

The contents of this paper are organized as follows. In section 2, the construction of the Ashtekar connection 1-forms on Σ will be briefly reviewed and some topological aspects of $\mathcal{A}$ will be studied. In section 3, a promeasure on $\mathcal{A}$ will be constructed. This is followed by the construction of a promeasure on $\mathcal{A}[\mathrm{SU}(2)]$ in the fourth section. In the final section, some discussions regarding the extension of the constructed SU(2) gauge-invariant promeasures to include diffeomorphism-invariance will be sketched.

In this sequel, a *Riemannian metric* will mean a smooth, symmetric, covariant 2-tensor field that is positive-definite at each point in Σ, and Σ will denote a compact, smooth, closed, orientable, Riemannian 3-manifold. The Einstein summation convention is always implied whenever there are repeated upper and lower indices.

[1] The closure of the space of gauge-equivalent G-connection 1-forms is essentially the spectrum of the holonomy C^*-algebra generated by the Wilson loop functions on the space of gauge-equivalent G-connection 1-forms. For more details, see references [2, 4].

[2] An idea which originated from Baez [6].

2 The Space of Ashtekar Connections

Let $(T\Sigma \otimes \mathfrak{su}(2), p, \Sigma)$ be a tensor bundle over Σ and define $\mathcal{C}$ to be the space of smooth cross sections $\sigma : \Sigma \to T\Sigma \otimes \mathfrak{su}(2)$ satisfying:

(1) for each $x \in \Sigma$ and σ, $\sigma(x)$ induces a linear isomorphism $\mathfrak{su}(2) \approx T_x\Sigma$ defined by $\lambda \mapsto -\sigma(x)\cdot\lambda \overset{\text{def}}{=} \lambda^a(x)\partial_a$, where $-\sigma(x)\cdot\lambda \overset{\text{def}}{=} -\operatorname{tr}(\sigma(x)^a\lambda)\partial_a = -\sigma(x)^a{}_A{}^B\lambda_B{}^A\partial_a = \lambda^a(x)\partial_a \in T_x\Sigma$,

(2) $-\operatorname{tr}(\sigma^a\sigma^b) \overset{\text{def}}{=} q^{ab}$, where q^{ab} is the inverse matrix of q_{ab}, and q_{ab} defines (in the natural basis) a Riemannian 3-metric on Σ.

The elements of $\mathcal{C}$ are called the *SU(2) soldering forms* on Σ and the inverse $\sigma(x)^{-1}$ of $\sigma(x)$ will be written as $\sigma(x)_{aA}{}^B$. Notice that as Σ is a compact orientable 3-manifold, it is parallelizable and hence $\mathcal{C}$ is a non-trivial set.

Let $C^\infty_{\text{cs}}(\Sigma, T\Sigma \otimes \mathfrak{su}(2))$ be the space of smooth cross sections of the bundle $(T\Sigma \otimes \mathfrak{su}(2), p, \Sigma)$, endowed with the compact C^∞-topology. Then, it follows from [15, §7.2, pp. 259–260] that $C^\infty_{\text{cs}}(\Sigma, T\Sigma \otimes \mathfrak{su}(2))$ is a smooth Fréchet manifold. Furthermore, let $S_2\Sigma$ be the bundle space of symmetric covariant 2-tensors on Σ and let $C^\infty_{\text{cs}}(\Sigma, S_2\Sigma)$ be the space of C^∞-cross sections on the tensor bundle equipped with the compact C^∞-topology. Then, the space of (smooth) Riemannian metrics on Σ is an open convex cone in $C^\infty_{\text{cs}}(\Sigma, S_2\Sigma)$ [8, p. 1001]. So, by (2), $\mathcal{C} \subset C^\infty_{\text{cs}}(\Sigma, T\Sigma \otimes \mathfrak{su}(2))$ is open and hence a smooth manifold.

Let $T^*\mathcal{C}$ be the cotangent bundle space over $\mathcal{C}$ and denote an element in $T^*\mathcal{C}$ by (σ, M)—it is shown in [1, p. 1592] that $M \overset{\text{def}}{=} M_{aA}{}^B$ is a densitized 1-form of weight 1 on $\mathcal{C}$ based at σ. In this section, notations consistent with [1] will be used. Let $(B, p_B, \Sigma, \mathrm{SU}(2), W)$ be a complex vector bundle of rank 2 over Σ associated with the principal SU(2)-bundle $\xi = (P_\xi, \Sigma, \mathrm{SU}(2))$, where $\dim_{\mathbb{C}} W = 2$. Fix an element $(\sigma, M) \in T^*\mathcal{C}$ and define an SU(2) connection 1-form ω_σ on P_ξ such that the covariant derivative D induced by $p^*\omega_\sigma$ on Σ is compatible with σ: $\mathrm{D}_a\sigma^a{}_A{}^B = 0$—that is, the SU(2) spin connection coefficients $\Gamma_{aA}{}^B$ of $p^*\omega_\sigma$ satisfy $\Gamma_{aA}{}^B = \frac{1}{2}\sigma_b{}^{EB}(\sigma^c{}_{AE}\Gamma^b_{ca} + \partial_a\sigma^b{}_{AE})$, where Γ^b_{ca} is the Christoffel symbol of $q = -\operatorname{tr}\sigma\cdot\sigma$. Next, following [1, p. 1593], define an $\mathfrak{su}^{\mathbb{C}}(2)$-valued connection 1-form A on Σ—the *Ashtekar connection 1-form*—of the complex vector bundle B by

$$^{\pm}A \overset{\text{def}}{=} G^{-1}s^*\omega_\sigma \pm \tfrac{\mathrm{i}}{\sqrt{2}}G^{-1}\Pi, \tag{2.1}$$

where G is the Gravitational constant, $\mathfrak{su}^{\mathbb{C}}(2)$ is the complexification[3] of $\mathfrak{su}(2)$, $s : \Sigma \to P_\xi$ a smooth cross section on Σ, and $\Pi = \Pi[\sigma, M]$ is defined in [1] by

$$\Pi_{aA}{}^B \overset{\text{def}}{=} G(\det q)^{-\frac{1}{2}}(M_{aA}{}^B + \tfrac{1}{2}\operatorname{tr}(M_b\sigma^b)\sigma_{aA}{}^B). \tag{2.2}$$

On the constraint surface in $T^*\mathcal{C}$ (which satisfies the vacuum Einstein's equations), it can be shown that $\Pi_a = K_{ab}\sigma^b$, where K_{ab} is the extrinsic curvature of Σ. The Ashtekar connection 1-form ^+A may be regarded as the anti-self-dual potential for the Weyl tensor ^3C on Σ and ^-A as the self-dual potential [1, p. 1600, eqn (19′)]:

$$^3C^{ab} \overset{\text{def}}{=} -G\ \operatorname{tr}(^{\pm}F_{cd}\sigma^a)\epsilon^{cdb} = -\sqrt{2}G(E_{ab} \mp \mathrm{i}B_{ab}),$$

[3] If V is a vector space, then the elements of $V^{\mathbb{C}}$ are precisely $u + \mathrm{i}v$, where $u, v \in V$.

where ${}^{\pm}F_{ab}$ is the curvature of ${}^{\pm}A_a$, $E_{ab} \stackrel{\text{def}}{=} C_{acbd}n^c n^d$ is the electric and $B_{ab} \stackrel{\text{def}}{=} {}^*C_{acbd}n^c n^d$ the magnetic parts (relative to Σ) of the 4-dimensional Weyl tensor C. For a detailed account, see reference [3]. In the following analysis, only the self-dual potential will be considered inasmuch as canonical quantum gravity can be formulated with either the self-dual or the anti-self-dual potential [1, 14]. Thus, in view of this restriction, denote ${}^{-}A$ for convenience by A.

Since Σ is parallelizable, *global* connection 1-forms on Σ exist. Let $\hat{\mathcal{A}}$ be the set of *extended* Ashtekar connection 1-forms; that is, all elements $A = {}^{-}A$ of the form (2.1), where they need not satisfy the vacuum Einstein constraints $\mathrm{tr}(F_{ab}\tilde{\sigma}^a) = 0$ and $\mathrm{tr}(F_{ab}\tilde{\sigma}^a\tilde{\sigma}^b) = 0$, with $\tilde{\sigma}^a \stackrel{\text{def}}{=} (\det q)^{\frac{1}{2}}\sigma^a$ and F_{ab} the curvature of A_a. Also, let $\mathcal{S}_1 = \{\, A \in \hat{\mathcal{A}} \mid \mathrm{tr}(F_{ab}\tilde{\sigma}^a) = 0\,\}$ and $\mathcal{S}_2 = \{\, A \in \hat{\mathcal{A}} \mid \mathrm{tr}(F_{ab}\tilde{\sigma}^a\tilde{\sigma}^b) = 0\,\}$. Then, the space of Ashtekar connection 1-forms is the intersection $\mathcal{A} \stackrel{\text{def}}{=} \mathcal{S}_1 \cap \mathcal{S}_2$.

Let $E(\Sigma) = T^*\Sigma \otimes \mathfrak{su}^{\mathbb{C}}(2)$ be a tensor bundle over Σ and let $\Gamma^*(E(\Sigma))$ be the space of smooth cross sections $\Sigma \to E(\Sigma)$; that is, $\Gamma^*(E(\Sigma))$ is the space of $\mathfrak{su}^{\mathbb{C}}$-valued 1-forms on Σ. Then, $\hat{\mathcal{A}}$ is an affine subspace of the vector space $\Gamma^*(E(\Sigma))$. The space $\Gamma^*(E(\Sigma))$ will be given the compact C^∞-topology. For more details, see reference [11, p. 34–36]. In all that follows, $\hat{\mathcal{A}}$ and hence $\mathcal{A}$ will be endowed with the subspace topology of $\Gamma^*(E(\Sigma))$.

2.2. Proposition. *$\mathcal{A}$ is second countable and completely metrizable.*

Proof. It will suffice to show that $\hat{\mathcal{A}}$ is second countable and completely metrizable. However, because $\hat{\mathcal{A}} \equiv \varprojlim \hat{\mathcal{A}}_r$ is the inverse limit of $\hat{\mathcal{A}}_r$, where $\hat{\mathcal{A}}_r$ is the space of Ashtekar connections that are of class C^r (endowed with the compact C^r-topology) and each $\hat{\mathcal{A}}_r$, for $r \in \mathbb{N} \cup \{0\}$ is second countable and completely metrizable, so is $\hat{\mathcal{A}}$. □

Note that as $\hat{\mathcal{A}}$ is an affine space, from the proof of proposition 2.2, it is an infinite dimensional manifold modelled on a separable Banach space. And hence, by construction, so is $\mathcal{A}$. Indeed, using the construction given in reference [12, p. 267–268], it can be established that $\hat{\mathcal{A}}$ and $\mathcal{A}$ are smooth manifolds.

3 A Promeasure on $\mathcal{A}$

Two ways of constructing Gaussian type promeasures on $\mathcal{A}$ will be sketched in this section. The first method is to construct a system of finite dimensional manifolds whose projective limit is $\mathcal{A}$; the promeasure is then the projective limit of the Gaussian measures defined on the finite dimensional manifolds. The second way is to consider the projection of $\mathcal{A}$ onto a manifold modelled on $\Gamma^*(E(\Sigma))$ on which a Gaussian promeasure can be constructed, and then using the projection map to induce a promeasure on $\mathcal{A}$.

Let $D_\Sigma \subset \Sigma$ be a countably dense subset of Σ, that is, $\overline{D_\Sigma} = \Sigma$, and let $c[D_\Sigma]$ be a countable cover of D_Σ such that $\forall\, G \in c[D_\Sigma]$, $G \subset D_\Sigma$ and $|G| < \aleph_0$. Set $c[D_\Sigma] = \{\, G_{\alpha n} \mid \alpha, n \in \mathbb{N}\,\}$, where $|G_{\alpha n}| = n$ for each labelling index α. Each $G_{\alpha n}$ defines an equivalence relation $R_{\alpha n} \subset \mathcal{A} \times \mathcal{A}$ by

$$A \sim A' \iff A(x_i) = A'(x_i)\ \forall\, x_i \in G_{\alpha n}.$$

Let $p_{\alpha n} : \mathcal{A} \to \mathcal{A}_{\alpha n} \stackrel{\text{def}}{=} \mathcal{A}/R_{\alpha n}$ denote the natural map and endow $\mathcal{A}_{\alpha n}$ with the quotient topology. It is easy, although somewhat tedious, to show that $p_{\alpha n} : \mathcal{A} \to \mathcal{A}_{\alpha n}$ is an open and closed mapping.

Now, given any $A \in \mathcal{A}$, let $T_{[A]}\mathcal{A} \stackrel{\text{def}}{=} \bigcup\{ T_A\mathcal{A} : A \in [A] \}$, where $T_A\mathcal{A}$ is the tangent space of $A \in \mathcal{A}$, and $T_{[A]}\mathcal{A}$ is endowed with the subspace topology of the tangent bundle $T\mathcal{A}$. Then, as before, each $G_{\alpha n} \in c[D_\Sigma]$ induces an equivalence relation $\hat{R}_{\alpha n} \subset T_{[A]}\mathcal{A} \times T_{[A]}\mathcal{A}$ on $T_{[A]}\mathcal{A}$ by

$$v \sim v' \iff v(x_i) = v'(x_i) \ \forall x_i \in G_{\alpha n}.$$

Let $T_{[A]}\mathcal{A}_{\alpha n} \stackrel{\text{def}}{=} T_{[A]}\mathcal{A}/\hat{R}_{\alpha n}$ denote the quotient space and $p_{\alpha n *} : T_{[A]}\mathcal{A} \to T_{[A]}\mathcal{A}_{\alpha n}$ the natural map—$T_{[A]}\mathcal{A}_{\alpha n}$ is endowed with the quotient topology. Then, by construction, $p_{\alpha n*}(T_A\mathcal{A}) = T_{[A]}\mathcal{A}_{\alpha n} \ \forall A \in [A]$. By definition, $p_{\alpha n*}$ is an open mapping.

3.1. Lemma. *$T_{[A]}\mathcal{A}_{\alpha n}$ is a finite-dimensional linear space.*

Proof. First, observe that $T_{[A]}\mathcal{A}_{\alpha n}$ can be endowed with a vector space structure: define $\tilde{+}$ and $\cdot$ by

$$[v]\tilde{+}[u] \stackrel{\text{def}}{=} [v+u] \quad \text{and} \quad c\cdot[v] \stackrel{\text{def}}{=} [cv] \ \forall c \in \mathbb{R}.$$

These operations are well-defined since $p_{\alpha n*}(u+v) = [u+v] = \{ u'+v' : u|G_{\alpha n} = u'|G_{\alpha n} \text{ and } v|G_{\alpha n} = v'|G_{\alpha n} \} = p_{\alpha n*}(u)\tilde{+}p_{\alpha n*}(v)$ and $p_{\alpha n*}(cu) = \{ cu' : u'|G_{\alpha n} = u|G_{\alpha n} \} = c \cdot p_{\alpha n*}(u) \ \forall c \in \mathbb{R}$.

To check that this vector space structure is compatible with the quotient topology, it must be verified that $\tilde{+} : T_{[A]}\mathcal{A}_{\alpha n} \times T_{[A]}\mathcal{A}_{\alpha n} \to T_{[A]}\mathcal{A}_{\alpha n}$ and $\cdot : \mathbb{R} \times T_{[A]}\mathcal{A}_{\alpha n} \to T_{[A]}\mathcal{A}_{\alpha n}$ are both continuous. Because $(p_{\alpha n*}|T_A\mathcal{A})(T_A\mathcal{A}) = T_{[A]}\mathcal{A}_{\alpha n}$, there is no loss of generality in fixing some $A \in [A]$ and considering $T_A\mathcal{A}$ instead of $T_{[A]}\mathcal{A}$ in the discussion that follows. So, consider the following diagram:

$$\begin{array}{ccc}
T_A\mathcal{A} \times T_A\mathcal{A} & \xrightarrow{\ +\ } & T_A\mathcal{A}_{\alpha n} \\
{\scriptstyle p_{\alpha n*}\times p_{\alpha n*}}\big\downarrow & & \big\downarrow{\scriptstyle p_{\alpha n*}} \\
T_{[A]}\mathcal{A}_{\alpha n} \times T_{[A]}\mathcal{A}_{\alpha n} & \xrightarrow{\ \tilde{+}\ } & T_{[A]}\mathcal{A}_{\alpha n}.
\end{array}$$

It is clear from the definition that the diagram commutes: $\tilde{+} \circ (p_{\alpha n*} \times p_{\alpha n*}) = p_{\alpha n*} \circ +$. Hence, the continuity of $\tilde{+}$ follows from the surjectivity of $p_{\alpha n*} \times p_{\alpha n*}$ and the continuity of $p_{\alpha n*}$, $+$ and $p_{\alpha n*} \times p_{\alpha n*}$. In a similar way, it can be verified easily that $\cdot$ is continuous. Whence, $T_{[A]}\mathcal{A}_{\alpha n}$ is a linear space.

Finally, set $V_{\alpha n} = \prod_{i=1}^{n}(T^*_{x_i}\Sigma \otimes \mathfrak{su}^{\mathbb{C}}(2))$, which is linearly isomorphic to $\mathbb{R}^{18n}$ (and hence finite-dimensional), and define $L_{\alpha n} : T_{[A]}\mathcal{A}_{\alpha n} \to V_{\alpha n}$ by

$$[v] \mapsto (v(x_1), \dots, v(x_n)),$$

where v is any representative of $[v]$ and $(T^*_{x_i}\Sigma \otimes \mathfrak{su}^{\mathbb{C}}(2))$ is identified with its tangent space for each i. It is trivial to check that $L_{\alpha n}$ defines a linear isomorphism. $\square$

Note that as metrizability is an invariant under closed surjective mappings [10, p. 285, theorem 4.4.18], $\mathcal{A}_{\alpha n}$ is metrizable and hence paracompact and Hausdorff. Also, because

$p_{\alpha n}$ is open and $\mathcal{A}$ is second countable by proposition 2.2, so is $\mathcal{A}_{\alpha n}$. In fact, it can be verified without too much difficulty that $\mathcal{A}_{\alpha n}$ is a paracompact, Hausdorff, second countable manifold modelled on $V_{\alpha n}$ and hence finite dimensional by lemma 3.1. This means in particular that Gaussian measures can be constructed on $\mathcal{A}_{\alpha n}$.

Now, for any $G_{\alpha n}, G_{\alpha' n'} \in c[D_\Sigma]$ with $G_{\alpha' n'} \subset G_{\alpha n}$ and $n' < n$, let $p^{\alpha n}_{\alpha' n'} : \mathcal{A}_{\alpha n} \to \mathcal{A}_{\alpha' n'}$ be the natural map $[A]_{\alpha n} \mapsto [A]_{\alpha' n'}$, where $[A]_{\beta m} \stackrel{\text{def}}{=} \{ A' \mid A'(x_i) = A(x_i) \ \forall x_i \in G_{\beta m} \}$. This map is clearly a continuous surjection. It is not difficult to see that it is also an open map as $p^{\alpha n}_{\alpha' n'} \circ p_{\alpha n} = p_{\alpha' n'}$ and $p_{\alpha n}$ is onto. Order $c[D_\Sigma]$ by inclusion according to: $G' \preccurlyeq G \Leftrightarrow G' \subseteq G$. Then, $\{(\mathcal{A}_{\alpha n}, p^{\alpha n}_{\alpha' n'}) : G_{\alpha' n'} \preccurlyeq G_{\alpha n},\ n' \leqq n,\ n', n \in \mathbb{N}\}$ defines an inverse sequence.

Suppose $m < n$ and $G_{\alpha(m)m} \subset G_{\alpha(n)n}$. Define $\hat{p}^{\alpha(n)n}_{\alpha(m)m} : V_{\alpha(n)n} \to V_{\alpha(m)m}$ by

$$\hat{p}^{\alpha(n)n}_{\alpha(m)m} \stackrel{\text{def}}{=} L_{\alpha(m)m} \circ (p^{\alpha(n)n}_{\alpha(m)m})_* \circ L^{-1}_{\alpha(n)n},$$

where $(p^{\alpha(n)n}_{\alpha(m)m})_* : T_{[A]}\mathcal{A}_{\alpha(n)n} \to T_{[A]}\mathcal{A}_{\alpha(m)m}$ is a map induced by $p^{\alpha(n)n}_{\alpha(m)m} : \mathcal{A}_{\alpha(n)n} \to \mathcal{A}_{\alpha(m)m}$. In all that follows, $(p^{\alpha(n)n}_{\alpha(m)m})_*$ will be identified with $\hat{p}^{\alpha(n)n}_{\alpha(m)m}$ and $T_{[A]}\mathcal{A}_{\alpha(n)n}$ with $V_{\alpha(n)n}$ for each $[A] \in \mathcal{A}_{\alpha(n)n}$.

Observe from the construction of $\mathcal{A}_{\alpha n}$ that $\mathcal{A}$ is the projective limit of the sequence $\{(\mathcal{A}_{\alpha(n)n}, p^{\alpha(n)n}_{\alpha(m)m})\}$, and furthermore, notice that it is an easy, albeit somewhat tedious, exercise to generalise the theory of promeasures on locally convex vector spaces to differentiable manifolds modelled on separable Banach spaces. Hence, the sequence of Gaussian measures $\lambda_{\alpha(n)n}$ on $\mathcal{A}_{\alpha(n)n}$ induces a promeasure $\lambda = \{\lambda_{\alpha(n)n}, \mathcal{A}_{\alpha(n)n}\}$ on $\mathcal{A}$, as required. Another way of constructing promeasures on $\mathcal{A}$ will be given below.

3.2. Theorem. *A Gaussian promeasure exists on $\mathcal{A}$.*

Proof. Let $V(\Sigma)$ be the vector space of smooth cross sections $\Sigma \to T^*\Sigma \otimes \mathfrak{su}(2)$ (endowed with the compact C^∞-topology) and $V_0(\Sigma) = \{ \sqrt{2}\,\mathrm{im}\, A \mid A \in \mathcal{A} \}$ be a subspace of $V(\Sigma)$, where $\mathrm{im}\, A$, the imaginary part of A, is defined by equation (2.2). Then, $V_0(\Sigma)$ is an open subset in $V(\Sigma)$. To see this, it will suffice to note from §2 that the fibre $T^*_\sigma \mathcal{C} = \{M\}$ over σ is a linear space. Hence, given $\Pi_0 \in V_0(\Sigma)$, by taking $r > 0$ to be large enough, $\varepsilon > 0$ to be small enough and choosing any finite compact covering $\{K_i\}^n_{i=1}$ of Σ, the neighbourhood $N_{\varepsilon,r}(\Pi_0, \{K_i\}^n_{i=1}) = \{ f \in V(\Sigma) : \|D^k f(x) - D^k \Pi_0(x)\| < \varepsilon,\ k = 0, 1, \ldots, r,\ x \in \Sigma \}$ of Π_0 is contained in $V(\Sigma)$ (note that $\bigcup_i K_i = \Sigma$ by definition). Since $V(\Sigma)$ is locally convex with respect to the C^∞-topology—see [9, p. 774]—by [7, p. 582, theorem], there exists a Gaussian promeasure on $V(\Sigma)$ of variance Q defined by a positive definite quadratic form $Q : V(\Sigma)' \to \mathbb{R}$ on the topological dual $V(\Sigma)'$ of $V(\Sigma)$. Let μ_0 denote the promeasure on $V(\Sigma)$ that is restricted to $V_0(\Sigma)$.

It should be noted as a side comment that the choice of the projection map is rather apt here as Π encodes the extrinsic curvature of Σ and hence the information contained in the Ashtekar connection 1-forms is not completely lost. By construction, $p_0 : \mathcal{A} \to V_0(\Sigma)$ defined by $A \mapsto \sqrt{2}\,\mathrm{im}(A) = \Pi$, is a continuous open surjection (from equation (2.2) and the fact that both spaces are endowed with the C^∞-topology). Hence, p_0 induces a promeasure λ on $\mathcal{A}$ by $\lambda = \mu_0 \circ p_0$, as desired. $\square$

3.3. Remark. From the definition of $V_0(\Sigma)$, it is obvious that the projection $p_0 : \mathcal{A} \to V_0(\Sigma)$ can be extended to the whole of $\hat{\mathcal{A}}$ by $A \mapsto \sqrt{2}\,\mathrm{im}\, A$. Hence, a promeasure trivially exists on $\hat{\mathcal{A}}$. It is also of interest to note in the above construction that the Einstein constraints were not needed in the construction of the promeasure. What is important to note here is that $\mathcal{A}$ is a smooth submanifold by definition as both $\mathcal{S}_1$ and $\mathcal{S}_2$ are, and this enabled a promeasure to be constructed on $\mathcal{A}$.

4 A Promeasure on the Ashtekar Moduli Space

Let $\mathrm{Diff}(P_\xi)$ denote the group of smooth diffeomorphisms (endowed with the compact C^∞-topology) on the bundle space P_ξ and let $\mathcal{G} \subset \mathrm{Diff}(P_\xi)$ be the set of elements $f \in \mathrm{Diff}(P_\xi)$ satisfying

(1) $f(ug) = f(u)g \ \forall\, u \in P_\xi$ and $g \in \mathrm{SU}(2)$,
(2) $\pi_\xi \circ f = \pi_\xi$, where $\pi_\xi : P_\xi \to \Sigma$ is the bundle projection.

Note that the bundle is actually trivial and hence $P_\xi \cong \Sigma \times \mathrm{SU}(2)$ (which is compact).

Identify P_ξ with $\Sigma \times \mathrm{SU}(2)$ and let $p_2 : \Sigma \times \mathrm{SU}(2) \to \mathrm{SU}(2)$ be the projection of the second factor, $(x, g) \mapsto g$. Set $\phi_f \stackrel{\text{def}}{=} p_2 \circ f$ for any $f \in \mathcal{G}$. Define an equivalence relation $\mathcal{R} \subset \mathcal{A} \times \mathcal{A}$ on $\mathcal{A}$ by

$$A \sim A' \iff A'(x) = \phi_f(x)^{-1} A(x) \phi_f(x) + \phi_f(x)^{-1}\, \mathrm{d}\phi_f(x),$$

and denote A' by A^{ϕ_f}. Then, the equivalence class $\hat{A}$ of A is the set $\{\, A^{\phi_f} \mid f \in \mathcal{G} \,\}$. Let $\pi_{\mathrm{SU}(2)} : \mathcal{A} \to \mathcal{A}[\mathrm{SU}(2)] \stackrel{\text{def}}{=} \mathcal{A}/\mathcal{R}$ be the natural map and $\mathcal{A}[\mathrm{SU}(2)]$ be given the quotient topology. It can be verified that this topology is Hausdorff. Observe that ϕ_f induces a homeomorphism $\Phi_f : \mathcal{A} \to \mathcal{A}$ by $A \mapsto A^{\phi_f}$ and hence, if $N^r_\delta(A)$ is any neighbourhood of A in $\mathcal{A}$, then $\Phi_f(N^r_\delta(A)) \stackrel{\text{def}}{=} \{\, A^{\phi_f} \mid A \in N^r_\delta(A) \,\}$ is open. Consequently, $\pi_{\mathrm{SU}(2)}$ is an open mapping as

$$\pi^{-1}_{\mathrm{SU}(2)} \circ \pi_{\mathrm{SU}(2)}(N^r_\delta(A)) = \bigcup_{f \in \mathcal{G}} \Phi_f(N^r_\delta(A)).$$

To construct a promeasure on $\mathcal{A}[\mathrm{SU}(2)]$, one must define the spaces $\mathcal{A}_{\alpha n}[\mathrm{SU}(2)]$, the analogues of $\mathcal{A}_{\alpha n}$. To this end, observe firstly that if $A(x_i) = B(x_i)$ for $i = 1, \ldots, n$, then $A^{\phi_f}(x_i) = B^{\phi_f}(x_i)$ for each $i = 1, \ldots, n$ and $f \in \mathcal{G}$. Hence, each $G_{\alpha n} \in c[D_\Sigma]$ generates an equivalence relation $R_{\alpha n} \subset \mathcal{A}[\mathrm{SU}(2)] \times \mathcal{A}[\mathrm{SU}(2)]$ by

$$\hat{A} \sim \hat{B} \iff A(x_i) = B(x_i) \text{ on } G_{\alpha n},$$

where $A \in \hat{A}$ and $B \in \hat{B}$ are any fixed representatives such that $A^{\phi_f}(x_i) = B^{\phi_f}(x_i)$ on $G_{\alpha n}$ for each $f \in \mathcal{G}$. Denote the quotient space by $\mathcal{A}_{\alpha n}[\mathrm{SU}(2)]$ and $\tilde{p}_{\alpha n} : \mathcal{A}[\mathrm{SU}(2)] \to \mathcal{A}_{\alpha n}[\mathrm{SU}(2)]$ its natural map. Moreover, define a map $\pi_{\alpha n} : \mathcal{A}_{\alpha n} \to \mathcal{A}_{\alpha n}[\mathrm{SU}(2)]$ so that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{p_{\alpha n}} & \mathcal{A}_{\alpha n} \\ {\scriptstyle \pi_{\mathrm{SU}(2)}}\downarrow & & \downarrow{\scriptstyle \pi_{\alpha n}} \\ \mathcal{A}[\mathrm{SU}(2)] & \xrightarrow{\tilde{p}_{\alpha n}} & \mathcal{A}_{\alpha n}[\mathrm{SU}(2)], \end{array}$$

that is, $\pi_{\alpha n} \circ p_{\alpha n} = \tilde{p}_{\alpha n} \circ \pi_{\mathrm{SU}(2)}$. It is easy to verify that $\pi_{\alpha n}$ is a continuous surjection.

A promeasure on $\mathcal{A}[\mathrm{SU}(2)]$ can now be constructed. First, $p^{\alpha n}_{\beta m} : \mathcal{A}_{\alpha n} \to \mathcal{A}_{\beta m}$ induces a map $\tilde{p}^{\alpha n}_{\beta m} : \mathcal{A}_{\alpha n}[\mathrm{SU}(2)] \to \mathcal{A}_{\beta m}[\mathrm{SU}(2)]$ given by $[\hat{A}]_{\alpha n} \mapsto [\hat{A}]_{\beta m}$ so that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{A}_{\alpha n} & \xrightarrow{p^{\alpha n}_{\beta m}} & \mathcal{A}_{\beta m} \\ {\scriptstyle \pi_{\alpha n}}\downarrow & & \downarrow{\scriptstyle \pi_{\beta m}} \\ \mathcal{A}_{\alpha n}[\mathrm{SU}(2)] & \xrightarrow{\tilde{p}^{\alpha n}_{\beta m}} & \mathcal{A}_{\beta m}[\mathrm{SU}(2)]. \end{array}$$

So, define $\mu_{\alpha n}$ by $\mu_{\alpha n} \stackrel{\text{def}}{=} \lambda_{\alpha n} \circ \pi^{-1}_{\alpha n}$. Then, $\mu_{\alpha n}$ defines a measure on $\mathcal{A}_{\alpha n}[\mathrm{SU}(2)]$ as $\lambda_{\alpha n}$ is a bounded measure. Thus $\mu \stackrel{\text{def}}{=} \{(\mathcal{A}_{\alpha(n)n}[\mathrm{SU}(2)], \mu_{\alpha(n)n})\}$ defines a promeasure on $\mathcal{A}[\mathrm{SU}(2)]$. To verify this, recall that $\lambda_{\alpha(m)m} = \lambda_{\beta(n)n} \circ p^{\beta(m)m}_{\alpha(n)n}$, and for each n, $\lambda_{\alpha(n)n}$ is bounded. Hence, the relation $\pi_{\beta(m)m} \circ p^{\alpha(n)n}_{\beta(m)m} = \tilde{p}^{\alpha(n)n}_{\beta(m)m} \circ \pi_{\alpha(n)n}$ together with the surjectivity of the maps imply that $\mu_{\beta(m)m} = \lambda_{\beta(m)m} \circ \pi^{-1}_{\beta(m)m} = \lambda_{\alpha(n)n} \circ p^{\beta(m)m}_{\alpha(n)n} \circ \pi^{-1}_{\beta(m)m} = \mu_{\alpha(n)n} \circ \tilde{p}^{\beta(m)m}_{\alpha(n)n}$, as required.

5 Diffeomorphism-invariant Promeasure

This paper will end with a tentative sketch of the construction of a promeasure on $\mathcal{A}$ that is simultaneously SU(2) gauge-invariant and $\mathrm{Diff}^+(\Sigma)$-invariant. The construction in fact turns out to be amazingly simple. First, some notations will be introduced. Let $\hat{\pi}_\Sigma : \mathcal{A} \to \mathcal{A}^\Sigma$ denote the quotient map, where $\mathcal{A}^\Sigma \stackrel{\text{def}}{=} \mathcal{A}/\mathrm{Diff}^+(\Sigma)$ is the space of $\mathrm{Diff}^+(\Sigma)$-equivalent connection 1-forms:

$$A \sim A' \quad \Longleftrightarrow \quad A' = f^* A \text{ for some } f \in \mathrm{Diff}^+(\Sigma).$$

Now, fix any $f \in \mathrm{Diff}^+(\Sigma)$ and consider $A, B \in \mathcal{A}$ such that $A(x_i) = B(x_i)\ \forall i = 1, \ldots, n$. Since under a coordinate transformation induced by f, $A_a(x_i) \to A'_a(y_i) = \frac{\partial x^b(x_i)}{\partial f^a(x_i)} A_b(x_i)$ and $B_a(x_i) \to B'_a(y_i) = \frac{\partial x^b(x_i)}{\partial f^a(x_i)} B_b(x_i)$ for each $i = 1, \ldots, n$, where $x_i = f(y_i)$, it follows at once that $f^*A(y_i) = f^*B(y_i)$ for each i. Hence, the following equivalence relation $\sim$ on $\mathcal{A}_{\alpha n}$ given by

$$[A]_{\alpha n} \sim [A']_{\alpha n} \quad \Longleftrightarrow \quad \exists\, f \in \mathrm{Diff}^+(\Sigma) \text{ such that } f^*A(x_i) = A'(x_i)\ \forall i = 1, \ldots, n,$$

where A (resp. A') is any representative of $[A]_{\alpha n}$ (resp. $[A']_{\alpha n}$) is well-defined.

Denote the coset of $[A]_{\alpha n}$ under $\sim$ by $[A]^{\Sigma}_{\alpha n}$ and the quotient space $\mathcal{A}_{\alpha n}/\operatorname{Diff}^+(\Sigma)$ by $\mathcal{A}^{\Sigma}_{\alpha n}$. Furthermore, let $\hat{\pi}_{\alpha n} : \mathcal{A}_{\alpha n} \to \mathcal{A}^{\Sigma}_{\alpha n}$ denote the quotient map. Then, the following diagram commutes:

$$\begin{array}{ccccc}
\mathcal{A} & \xrightarrow{p_{\alpha n}} & \mathcal{A}_{\alpha n} & \xrightarrow{p^{\alpha n}_{\alpha m}} & \mathcal{A}_{\alpha m} \\
\downarrow{\scriptstyle \hat{\pi}_{\Sigma}} & & \downarrow{\scriptstyle \hat{\pi}_{\alpha n}} & & \downarrow{\scriptstyle \hat{\pi}_{\alpha m}} \\
\mathcal{A}^{\Sigma} & \xrightarrow{p'_{\alpha n}} & \mathcal{A}^{\Sigma}_{\alpha n} & \xrightarrow{(p')^{\alpha n}_{\alpha m}} & \mathcal{A}^{\Sigma}_{\alpha m},
\end{array}$$

where $p'_{\alpha n}$ (resp. $(p')^{\alpha n}_{\alpha m}$) is a projection induced by $p_{\alpha n}$ (resp. $p^{\alpha n}_{\alpha m}$). Finally, given $f \in \operatorname{Diff}^+(\Sigma)$ and an SU(2) gauge transformation ϕ_g, where $g \in \mathcal{G}$, set $f^*D \stackrel{\text{def}}{=} \{\, f^*A \mid A \in D \,\}$ and $D^{\phi_g} = \{\, A^{\phi_g} \mid A \in D \,\}$, where $D \subset \mathcal{A}$ and $A^{\phi_g} = \phi_g^{-1} A \phi_g + \phi_g^{-1}\, d\phi_g$.

5.1. Remark. It is easy to see that given any subset D of $\mathcal{A}_{\alpha n}$ such that $D \neq \mathcal{A}_{\alpha n}$, $\hat{\Pi}_{\alpha n} \circ \Pi_{\alpha n}(D) \neq \mathcal{A}_{\alpha n}$, where $\hat{\Pi}_{\alpha n} \stackrel{\text{def}}{=} \hat{\pi}^{-1}_{\alpha n} \circ \hat{\pi}_{\alpha n}$ and $\Pi_{\alpha n} \stackrel{\text{def}}{=} \pi^{-1}_{\alpha n} \circ \pi_{\alpha n}$.

5.2. Lemma. *For each $D \subseteq \mathcal{A}_{\alpha n}$, $\hat{\Pi}_{\alpha n} \circ \Pi_{\alpha n}(D) = \Pi_{\alpha n} \circ \hat{\Pi}_{\alpha n}(D)$.*

Proof. Now, given $f^*(A^{\phi}) \in \hat{\Pi}_{\alpha n} \circ \Pi_{\alpha n}(D)$, where $A \in D$ and $D \subseteq \mathcal{A}_{\alpha n}$, $f^*(A^{\phi}) = (\phi \circ f)^{-1} f^*A(\phi \circ f) + (\phi \circ f)^{-1} f^*\, d\phi$. So, set $B = f^*A$ and $\varphi = \phi \circ f$. Then, $d\varphi = f^*\, d\phi$ and

$$\begin{aligned}
f^*(A^{\phi}) &= (\phi \circ f)^{-1} f^*A(\phi \circ f) + (\phi \circ f)^{-1} f^*\, d\phi \\
&= \varphi^{-1} B \varphi + \varphi^{-1}\, d\varphi \\
&= B^{\varphi} \in \Pi_{\alpha n} \circ \hat{\Pi}_{\alpha n}(D).
\end{aligned}$$

Hence, $\hat{\Pi}_{\alpha} \circ \Pi_{\alpha n}(D) \subseteq \Pi_{\alpha n} \circ \hat{\Pi}_{\alpha n}(D)$.

Conversely, consider any element $(f^*A)^{\phi} \in \Pi_{\alpha n} \circ \hat{\Pi}_{\alpha n}(D)$, where $A \in D$, set $\varphi = \phi \circ f^{-1}$. Then, $d\varphi = (f^{-1})^*\, d\phi$ and

$$\begin{aligned}
(f^*A)^{\phi} &= \phi^{-1} f^*A\phi + \phi^{-1}\, d\phi \\
&= \phi^{-1} f^*A\phi + \phi^{-1} f^* \circ (f^{-1})^*\, d\phi \\
&= f^*(\varphi^{-1} A \varphi + \varphi^{-1}\, d\varphi) \\
&= f^*(A^{\varphi}) \in \hat{\Pi}_{\alpha n} \circ \Pi_{\alpha n}(D).
\end{aligned}$$

Hence, $\hat{\Pi}_{\alpha n} \circ \Pi_{\alpha n}(D) \subseteq \Pi_{\alpha n} \circ \hat{\Pi}_{\alpha n}(D)$ and the assertion thus follows. □

5.3. Theorem. *$\mathcal{A}_{\alpha n}$ admits a (bounded) measure $\nu_{\alpha n}$ that is simultaneously* $\operatorname{Diff}^+(\Sigma)$*-invariant as well as SU(2) gauge-invariant.*

Proof. Set $\nu_{\alpha n} \stackrel{\text{def}}{=} \lambda_{\alpha n} \circ \hat{\Pi}_{\alpha n} \circ \Pi_{\alpha n}$. By remark 5.1, $\nu_{\alpha n}$ is not a "trivial" measure in the sense that the equality $\nu_{\alpha n}(D) = \nu_{\alpha n}(\mathcal{A}_{\alpha n})$, for each measurable D with non-empty interior, will not hold in general. And by lemma 5.2, $\nu_{\alpha n} = \lambda_{\alpha n} \circ \Pi_{\alpha n} \circ \hat{\Pi}_{\alpha n}$ is well-defined. Hence, for each $\lambda_{\alpha n}$-measurable subset $D \subset \mathcal{A}_{\alpha n}$, $\nu_{\alpha n}(f^*(D^{\phi})) = \nu_{\alpha n}(D) = \nu_{\alpha n}((f^*D)^{\phi})$ $\forall f \in \operatorname{Diff}^+(\Sigma)$ and any SU(2) gauge transformation ϕ, and it is thus the desired $\operatorname{Diff}^+(\Sigma)$- and SU(2) gauge-invariant promeasure on $\mathcal{A}_{\alpha n}$. □

The following result, which is now immediate, will conclude this paper.

5.4. Corollary. *A* $\mathrm{Diff}^+(\Sigma)$*-invariant and* $SU(2)$ *gauge-invariant promeasure* ν *exists on* $\mathcal{A}$.

Proof. Set $\nu = \{(\nu_{\alpha(n)n}, \mathcal{A}_{\alpha(n)n})\}$. Then, the proof that ν is a promeasure follows from the commutativity of the following two diagrams:

$$\begin{array}{ccc} \mathcal{A}_{\alpha(n)n} & \longrightarrow & \mathcal{A}_{\alpha(m)m} \\ \downarrow & & \downarrow \\ \mathcal{A}^{\mathrm{SU}(2)}_{\alpha(n)n} & \longrightarrow & \mathcal{A}^{\mathrm{SU}(2)}_{\alpha(m)m} \end{array} \qquad \begin{array}{ccc} \mathcal{A}_{\alpha(n)n} & \longrightarrow & \mathcal{A}_{\alpha(m)m} \\ \downarrow & & \downarrow \\ \mathcal{A}^{\Sigma}_{\alpha(n)n} & \longrightarrow & \mathcal{A}^{\Sigma}_{\alpha(m)m}, \end{array}$$

for each $m < n$, where $\mathcal{A}^{\mathrm{SU}(2)}_{\alpha k} \stackrel{\mathrm{def}}{=} \mathcal{A}_{\alpha k}[\mathrm{SU}(2)]^{\bullet}$ for typesetting convenience. □

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