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A New Kinematical Derivation of the Lorentz Transformation and the Particle Description of Light

By J.H. Field

Département de Physique Nucléaire et Corpusculaire Université de Genève . 24, quai Ernest-Ansermet CH-1211 Genève 4.

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Abstract. The Lorentz Transformation is derived from only three simple postulates: (i) a weak kinematical form of the Special Relativity Principle that requires the equivalence of reciprocal space-time measurements by two different inertial observers; (ii) Uniqueness, that is the condition that the Lorentz Transformation should be a single valued function of its arguments; (iii) Spatial Isotropy. It is also shown that to derive the Lorentz Transformation for space-time points lying along a common axis, parallel to the relative velocity direction, of two inertial frames, postulates (i) and (ii) are sufficient. The kinematics of the Lorentz Transformation is then developed to demonstrate that, for consistency with Classical Electrodynamics, light must consist of massless (or almost massless) particles: photons.

1 Introduction

In his seminal paper of 1905 on Special Relativity [1] Einstein derived the Lorentz Transformation from two main [2] postulates:

1. The laws of physics are the same in all inertial frames (the Special Relativity Principle).
2. In any given inertial frame, the velocity of light is a constant, c, independant of the velocity of the source (Einstein's second postulate).

The first postulate was stated by Galileo [3] and was well known, before the advent of Special Relativity, to be respected by the laws of Classical Mechanics. It is clear that

Einstein regarded the second postulate as a 'law of physics' and so, in fact, a special case of the first postulate. Whether this is *necessarily* the case will be discussed below in Section 3

A study of Einstein's derivation of the Lorentz Transformation shows that the only 'law of physics' that is involved is that which is implicit in the very definition of an inertial frame: Newton's First Law. It will be demonstrated from purely kinematical arguments in Section 3. below that Einstein's 'light signals' may be identified with particles moving in straight lines with fixed momentum and energy according to Newton's First Law. That no other laws of physics are involved is crucial for the significance of Einstein's derivation of the Lorentz Transformation, and for the meaning of Special Relativity. What separates clearly Einstein's achievement from the related work of Lorentz [4], Larmor [5] and Poincaré [6] is his realisation that the Lorentz Transformation gives the relation between the space-time geometries (or, in momentum space, the kinematics) of different inertial frames. These are aspects of physics which underlie, but are quite distinct from, the actual dynamical laws. It turns out that these laws are indeed covariant under the Lorentz Transformation [7] and so respect the Special Relativity Principle as stated in Einstein's first postulate. However, as will be shown below, a much weaker statement of the Relativity Principle than Einstein's first postulate is sufficient to derive the Lorentz Transformation. More concretely it may be stated that the Lorentz Transformation describes only how space-time points or energy-momentum 4-vectors appear to different inertial observers, while the dynamical laws of physics, for example Newton's Second Law, the Lorentz Equation and Maxwell's Equations with sources, describe rather how future measurements of space-time points or, other 4-vectors, may be predicted from a knowledge of past or present ones [8]. In the works of Lorentz, Larmor and Poincaré on electrodynamics, these two different aspects, the one kinematical the other dynamical, are inextricably interwoven.

Because of this clear cut distinction in Special Relativity between kinematics and dynamics, it was recognised at an early date by Ignatowsky and Frank and Rothe [9] that Einstein's second postulate was not necessary to derive the Lorentz Transformation. The questions then arise: what are the *weakest* postulates which are sufficient to derive it and what is their *minimum* number? The first part of this paper attempts to give an answer to these questions. As in other work on the subject, a purely kinematical approach is adopted without any reference, in the derivation, to Classical Electrodynamics or any other dynamical law of physics. The kinematical consequences of the Lorentz Transformation are then compared to results of Classical Electrodynamics to establish the identity of the velocity parameter that necessarily appears in the Lorentz Transformation and the velocity of light. A consistent interpretation then requires light to consist of massless (or almost massless) particles [10].

The plan of the paper is as follows: In the following Section the three postulates on which the derivation of the Lorentz Transformation is based are introduced. Then separate derivations of the Lorentz Transformation and the Parallel Velocity Addition Relation (PVAR) are given. In Section 3 the kinematical consequences of the Lorentz Transformation are developed and it is shown that any physical object whose mass equivalent is much less than its energy will be observed to have a constant velocity V in any inertial frame. Special cases are photons ($V = c$) and massless or very light neutrinos. In the final Section the derivation of the Lorentz Transformation given here is compared with other similar work in the literature and conclusions are given.

2 Derivation of the Lorentz Transformation and the Parallel Velocity Addition Relation

The first postulate used (Postulate A) is a kinematical version of the Special Relativity Principle:

A. **Reciprocal Space-Time Measurements (STM) of similar measuring rods and clocks in two different inertial frames S and S', by observers at rest in these frames, give identical results.**

The frame S may be identified with Einstein's 'stationary system' [1], while, without loss of generality, S' may be assumed to move along the common x-axis of S and S' with velocity v. The y, z axes of S, S' are also taken to be parallel. Two examples of 'reciprocal measurements' [11], [12] are given below. As discussed in Section 4, some recent derivations of the Lorentz Transformation based upon sophisticated gedankenexperimente have, implicitly or explicitly, used Postulate A. Unlike in Einstein's first postulate there is no mention in Postulate A of the 'laws of physics'. Newton's First Law is however implicit in the term 'inertial', which means that the frames 'remain in the same state of uniform rectilinear motion' [13]. It may be objected that the 'laws of physics' are implicit in the physical processes underlying the operation of the clocks. A mechanical clock relies on the dynamical laws of Classical Mechanics, an atomic clock on those of Quantum Mechanics. By assuming spatial isotropy (see below) it can however be guaranteed that the clocks in, say, S and S', are identical even though the acceleration necessary to give S and S' their relative motion may change the operation of the clocks in some unknown way. For example, suppose that the clocks of identical construction in S and S' are originally at rest in a third inertial frame $\tilde{S}$ where they are synchronised by any convenient procedure. If the relative velocity beween S and S' is now produced by giving the frames S, S' (and their associated clocks) equal and opposite uniform accelerations for a suitably chosen time in $\tilde{S}$, it is clear, from the spatial symmetry required by the spatial isotropy postulate, that the clocks, originally identical in $\tilde{S}$, must remain so in S and S'.

The second postulate (Postulate B) is that of Uniqueness. This has been previously used [14] in the derivation of the PVAR. To the best of the writer's knowledge, it is here applied for the first time in the derivation of the Lorentz Transformation itself. This postulate is based on the hypothesis that if an observer in S performs a number N of STM then another, similarly equipped, observer at rest in S' can also make N STM in one-to-one correspondence with those made by the observer in S [15]. This will be so provided that the Lorentz Transformation is a single-valued function of its arguments. In the contrary case that the Lorentz Transformation is not single-valued then one STM measurement in

S may correspond to several in S' of vice-versa. Such an asymmetrical situation between two inertial frames is clearly at variance with the Principle of Relativity. The Uniqueness postulate will also be used to derive, separately, the PVAR, without assuming, as was done in Ref.[14], Einstein's second postulate. The statement of the Uniqueness postulate is:

B. **If $f(\chi, \xi, \zeta, ..) = 0$ represents either a Lorentz Transformation equation or the PVAR, then χ must be a real single-valued function of ξ, ζ,... ; ξ must be a real single-valued function of χ, ζ,... and so on for each of the arguments of f.**

For the PVAR, which has just 3 arguments, a sufficient [16] condition is that f should be trilinear in the relative velocities v_{AB}, v_{BC}, v_{CA} of three inertial frames A, B, C:

$$\begin{aligned} 0 &= P + Q_1 v_{AB} + Q_2 v_{BC} + Q_3 v_{CA} \\ &\quad + R_1 v_{AB} v_{BC} + R_2 v_{AB} v_{CA} + R_3 v_{BC} v_{CA} + S v_{AB} v_{BC} v_{CA} \end{aligned} \tag{2.1}$$

The coefficients P, Q_i, R_j, S are constants. If any two of v_{AB}, v_{BC}, v_{CD} are fixed then Eqn.(2.1) is linear in the remaining variable, and so has a unique solution [17].

The third postulate (Postulate C), spatial isotropy, requires no special comment:

C. **The Lorentz Transformation equations must be independent of the directions of the spatial axes used to specify a STM.**

These three postulates are now used to derive the Lorentz Transformation. In a first step it is assumed that the STM lie on the common x-axis of the frames S and S'. The generalisation to $y \neq 0$, $z \neq 0$ will be made subsequently. When $y = y' = z = z' = 0$ the Lorentz Transformation has the form:

$$x' = f(x, t) \tag{2.2}$$

Postulate B is satisfied provided that Eqn.(2.2) can be written as:

$$x' + a_1 x + a_2 t + b_1 x x' + b_2 x t + b_3 x' t + c x x' t = 0 \tag{2.3}$$

where a_i, b_j, c are independent of x, x', t. The velocity of S' relative to S is:

$$v \equiv \left. \frac{dx}{dt} \right|_{x' = \chi'} \tag{2.4}$$

where χ' may take any constant value. Differentiating Eqn.(2.3) with respect to t and using Eqn.(2.4) gives:

$$v = -\frac{a_2 + b_2 x + b_3 \chi' + c\chi' x}{a_1 + b_1 \chi' + b_2 t + c\chi' t} \tag{2.5}$$

Since Eqn.(2.5) must hold for all values of x, χ', t it follows that:

$$b_1 = b_2 = b_3 = c = 0 \tag{2.6}$$

so that:

$$v = -\frac{a_2}{a_1} \tag{2.7}$$

Using Eqns.(2.6),(2.7) Eqn(2.3) may be written as:

$$x' = \gamma(x - vt) \tag{2.8}$$

where

$$\gamma \equiv -a_1 \tag{2.9}$$

The Lorentz Transformation inverse to Eqn(2.2) is of the form:

$$x = f'(x', t') \tag{2.10}$$

The velocity of S relative to S', v', is defined as:

$$v' \equiv -\left.\frac{dx'}{dt'}\right|_{x=\chi} \tag{2.11}$$

where χ may take any constant value. In many derivations of the Lorentz Transformation (including Einstein's in Ref.[1]) it is assumed that:

$$v' = v \tag{2.12}$$

This hypothesis is called the Reciprocity Postulate. It has been proved by Berzi and Gorini [18] to be a consequence of the Special Relativity Principle and the usual postulates of space-time homogeneity and spatial isotropy. Since, in the present derivation, space-time homogeneity is not assumed, Eqn.(2.12) cannot be assumed to be correct. It will now be shown however, that the Reciprocity Postulate is a necessary consequence of Postulate A and Postulate B alone. That is, it is independant of the assumed properties of space-time in the case that Postulate B is true. Repeating the line of argument leading from Eqn.(2.2) to Eqn.(2.8), but starting instead with Eqn.(2.10), gives:

$$x = \gamma'(x' + v't') \tag{2.13}$$

Suppose now that a measuring rod of unit length, lying along the Ox' axis, and at rest in S' is observed, at fixed S time t, by S. It follows from Eqn.(2.8) that S will observe the length of the rod to be l where:

$$l = \frac{1}{\gamma} \tag{2.14}$$

If S' now makes a reciprocal measurement of a similar rod at rest in S the length will be observed to be l', where, from Eqn.(2.13):

$$l' = \frac{1}{\gamma'} \tag{2.15}$$

Using Postulate A:

$$l = l' \tag{2.16}$$

so that

$$\gamma = \gamma' \tag{2.17}$$

Hence, using Eqns.(2.8), (2.13), (2.17) the Lorentz Transformation may be written as:

$$x' = \gamma(x - vt) \tag{2.18}$$

$$t' = \frac{\gamma v}{v'}(t - \frac{x\delta}{v}) \tag{2.19}$$

where

$$\delta \equiv \frac{(\gamma^2 - 1)}{\gamma^2} \tag{2.20}$$

The inverse transformations are:

$$x = \gamma(x' + v't') \tag{2.21}$$

$$t = \frac{\gamma v'}{v}(t' + \frac{x'\delta}{v'}) \tag{2.22}$$

Consider now a clock at rest in S' located at $x' = 0$. As seen from S, the position of the clock, after the time $t = \tau_0$, is given by:

$$x = v\tau_o \tag{2.23}$$

From Eqns.(2.19) and (2.23) the elapsed time τ' indicated by the clock, as observed from S, during the interval τ_0 of S frame time, is:

$$\tau' = \frac{\gamma v}{v'}\tau_0(1 - \delta) \tag{2.24}$$

If an observer in S' makes now a reciprocal observation of a similar clock at rest in S, then the position of the clock, after the time $t' = \tau_0$, is given by:

$$x' = -v'\tau_o \tag{2.25}$$

Combining Eqns.(2.22) and (2.25):

$$\tau = \frac{\gamma v'}{v}\tau_0(1 - \delta) \tag{2.26}$$

where τ is the elapsed time, indicated by the clock at rest in S, as observed from S' during the period τ_0 of S' time. Because the two measurements of the elapsed time indicated by the similar clocks are reciprocal:

$$\tau = \tau' \tag{2.27}$$

It then follows from Eqns.(2.24) and (2.26) that:

$$v = v'$$

The alternative solution with $v' = -v$ may be rejected since it corresponds to the case where the clock in S' runs backwards in time ($t' \rightarrow -t'$). In the case of two inertial frames equipped with identical clocks Δt must have the same sign in both frames. Thus the Reciprocity Postulate Eqn.(2.12) is a consequence of Postulate A and Postulate B only. Using now Eqn.(2.12) the Lorentz Transformation in Eqns.(2.18) and (2.19) becomes:

$$x' = \gamma(x - vt) \tag{2.28}$$
$$t' = \gamma(t - \frac{x\delta}{v}) \tag{2.29}$$

Since the Lorentz Transformation is completely defined by the relative velocity v between the frames S and S' it follows that the unknown parameter γ, (and hence, from Eqn.(2.20) δ) must be a function of v.

Suppose now that a physical object moves with velocity u in the direction of the positive x' axis in S'. Its velocity w in the direction of the positive x axis, as observed in S, can be derived by differentiating Eqns.(2.28),(2.29) with respect to t and using the definitions:

$$w \equiv \frac{dx}{dt} \tag{2.30}$$
$$u \equiv \frac{dx'}{dt'} \tag{2.31}$$

The ratio of Eqn.(2.28) to Eqn.(2.29) after differentiation, use of Eqns.(2.30), (2.31) and some rearrangement, gives:

$$w = \frac{u+v}{1+\frac{u\delta(v)}{v}} \tag{2.32}$$

This is the Parallel Velocity Addition Relation (PVAR). By making use of the Reciprocity Postulate and Postulate C (Spatial Isotropy) it has been demonstrated [19] that the PVAR is symmetric under the exchange $u \leftrightarrow v$. As will be now shown, this symmetry is in fact a consequence of the Reciprocity Postulate alone. Introducing the notation:

$$v_{AB} = v \quad v_{BC} = u \quad v_{AC} = w$$

Eqn.(2.32) gives:

$$v_{AC} = \frac{v_{BC} + v_{AB}}{1+\frac{v_{BC}\delta(v_{AB})}{v_{AB}}} \tag{2.33}$$

Exchanging the labels A and C in Eqn.(2.33) :

$$v_{CA} = \frac{v_{BA} + v_{CB}}{1+\frac{v_{BA}\delta(v_{CB})}{v_{CB}}} \tag{2.34}$$

Using the Reciprocity Postulate, $v_{CA} = -v_{AC}$ etc, Eqn.(2.34) may be written as :

$$v_{AC} = \frac{v_{AB} + v_{BC}}{1+\frac{v_{AB}\delta(-v_{BC})}{v_{BC}}} \tag{2.35}$$

Comparing Eqns. (2.33) and (3.35) gives the relation:

$$\frac{v^2}{\delta(v)} = \frac{u^2}{\delta(-u)} = \pm V^2 \tag{2.36}$$

Since v, u are independant variables the first two members of Eqn.(2.36) must each be equal to the universal constant $\pm V^2$. By inspection, V has the dimensions of velocity. Setting $u = -u$ in Eqn.(2.36) it follows that :

$$\delta(-u) = \delta(u) \tag{2.37}$$

and hence, from Eqn.(2.36) that:

$$\frac{u}{v}\delta(v) = \frac{v}{u}\delta(u) \tag{2.38}$$

The PVAR, Eqn.(2.32), is thus a symmetric function of u and v. Two distinct possiblities now exist on combining Eqns.(2.20), (2.32) and (2.36) [21]:

a) plus sign in Eqn.(2.36):

$$\gamma(v) = \frac{1}{\sqrt{1-(\frac{v}{V})^2}} \tag{2.39}$$

$$w = \frac{u+v}{1+\frac{uv}{V^2}} \tag{2.40}$$

b) minus sign in Eqn.(2.36):

$$\gamma(v) = \frac{1}{\sqrt{1+(\frac{v}{V})^2}} \tag{2.41}$$

$$w = \frac{u+v}{1-\frac{uv}{V^2}} \tag{2.42}$$

Case b) gives no restriction on the possible values of u and v. However, in this case, the PVAR Eqn.(2.42) is not a single valued function of its arguments. It is thus excluded by Postulate B (Uniqueness). To show this, it may be noted that, for any value of v , a value of u, u_∞ may be chosen such that w is infinite:

$$u_\infty = \frac{V^2}{v} \tag{2.43}$$

defining

$$\Delta \equiv u_\infty - u \tag{2.44}$$

Eqn.(2.42) may be written as:

$$w = -\frac{(v+u_\infty+\Delta)u_\infty}{\Delta} \tag{2.45}$$

Since $w \to -\infty$ as $\Delta \to 0$ when $\Delta > 0$, whereas $w \to +\infty$ as $\Delta \to 0$ when $\Delta < 0$, Eqn.(2.45) does not give a unique solution for w when $u = u_\infty$.

In case a) the requirement that γ should be a real quantity (Postulate B) shows that, in this case, V plays the rôle of a limiting velocity:

$$u^2, v^2 \leq V^2 \tag{2.46}$$

With the definition:

$$u_{lim} \equiv -\frac{V^2}{v} \tag{2.47}$$

the restrictions on u, v given by Eqn.(2.46) imply either that no value, or a unique value, of u_{lim} exists. A consequence of Eqn.(2.47) is:

$$\frac{|u_{lim}|}{V} = \frac{V}{|v|} \tag{2.48}$$

If $|v| < V$, then $|u_{lim}| > V$, in contradiction with Eqn.(2.46). For the case $v = V = -u_{lim}$ Eqn.(2.40) gives the result: $w = -u_{lim} = V$. Hence, for all values of u, v consistent with Eqn.(2.46), the PVAR Eqn.(2.40) gives a unique value of w, and so verifies Postulate B. Thus when $v = V$, w takes also the value V, independently of the value of u. From the symmetry of Eqn.(40) this statement remains true when u and v are interchanged. A consequence is that if a physical object has velocity V in any inertial frame (say S', when $u = V$) then it has the velocity V in any other inertial frame (say S , when $w = V$). The physical interpretation of V is then the limiting velocity (independent of the choice of inertial frame) which any physical object may attain. It can already be seen that V has the same property as that ascribed to the velocity of light c in Einstein's second postulate. In Section 3. below the limiting velocity V will be related to the mass, energy and momentum of any physical object.

The Lorentz Transformation for STM lying along the common x axis of S, S' has now been completely determined by postulates A and B only. It corresponds to case a) above (plus sign in Eqn.(2.36)) and is given by the equations:

$$x' = \gamma(x - vt) \tag{2.49}$$

$$t' = \gamma(t - \frac{vx}{V^2}) \tag{2.50}$$

$$\gamma = \frac{1}{\sqrt{1-(\frac{v}{V})^2}} \tag{2.51}$$

This result is now generalised to include STM lying outside the common x-axis of S and S'. In this case Postulate C is also required. Considering first STM with $y \neq 0, z = 0$, then Postulate B implies that Eqn.(2.49) should be modified to:

$$x' = \gamma(x - vt) + yf(x', x, t) \tag{2.52}$$

where the function f is trilinear in x', x, t. Postulate C requires that Eqn.(2.52) should be invariant under the operation $y \to -y$ giving:

$$x' = \gamma(x - vt) - yf(x', x, t) \tag{2.53}$$

Subtracting Eqn.(2.53) from Eqn.(2.52):

$$yf(x', x, t) = 0 \tag{2.54}$$

Since this equation must hold for all values of x', x, y, t it follows that:

$$f(x', x, t) = 0$$

Thus Eqn.(2.49) is valid also for STM with $y \neq 0$. Using Postulate C it must also hold for STM with $z \neq 0$.

Consider now the transformation equations for STM along the y-axis ($z = z' = 0$) at $x' = 0$:

$$f(y', y, x, t) = 0$$

Postulate B is verified provided that f is quadrilinear in y', y, x, t i.e.

$$\begin{aligned} y' &+ A_1y + A_2x + A_3t + B_1y'y + B_2y'x + B_3y't \\ &+ B_4yx + B_5yt + B_6xt + C_1y'yx + C_2y'yt \\ &+ C_3y'xt + C_4yxt + Dy'yxt = 0 \end{aligned} \tag{2.55}$$

Invariance under $x \to -x$ (Postulate C) implies that the coefficients of all terms containing x must vanish. Similarly, invariance under the combined transformation: $y \to -y$, $y' \to -y'$, gives the further conditions:

$$A_3 = B_1 = C_2 = 0$$

so that Eqn.(2.55) reduces to

$$y' + A_1y + B_3y't + B_5yt = 0 \tag{2.56}$$

Fixing y' to be equal to ξ' and differentiating Eqn.(2.56) with respect to t gives:

$$A_1 \left.\frac{dy}{dt}\right|_{y'=\xi'} + B_3\xi' + B_5(\left.\frac{dy}{dt}\right|_{y'=\xi'} + y) = 0 \tag{2.57}$$

But, because the velocity of S' is perpendicular to y:

$$\left.\frac{dy}{dt}\right|_{y'=\xi'} = 0$$

so that Eqn.(2.57) becomes:

$$B_3\xi' + B_5y = 0$$

As this equation must be true for arbitary ξ', y then:

$$B_3 = B_5 = 0$$

giving, with Eqn.(2.56):

$$y' = -A_1y \tag{2.58}$$

The proof that $A_1 = -1$ was given in Ref.[1]. Denote by $A_1(v)$ the coefficient in Eqn.(2.58) corresponding to the Lorentz Transformation of Eqn.(2.49). The corresponding coefficient for the Lorentz Transformation inverse to Eqn.(2.49) is then $A_1(-v)$. Applying, in succession, the Lorentz Transformation of Eqn.(2.49) and its inverse then Eqn.(2.58) gives:

$$A_1(-v)A_1(v) = 1 \tag{2.59}$$

As the direction of the relative velocity of S and S' is perpendicular to the y, y' axes $A_1(v)$ cannot depend on the spatial orientation of the relative velocity (Postulate C). Hence:

$$A_1(v) = A_1(-v) \tag{2.60}$$

Eqns.(2.59) and (2.60) give

$$A_1(v) = \pm 1 \tag{2.61}$$

Since, evidently, Eqn(2.57) reduces to $y = y'$ in the limit $v = 0$, the minus sign must be taken in Eqn.(2.61) so that, for arbitary v, Eqn.(2.58) becomes:

$$y' = y$$

Application of Postulate C then implies that:

$$z' = z$$

The final result for the Lorentz Transformation of STM at an arbitary spatial position in S is then:

$$x' = \gamma(x - vt) \tag{2.62}$$

$$y' = y \tag{2.63}$$

$$z' = z \tag{2.64}$$

$$t' = \gamma(t - \frac{vx}{V^2}) \tag{2.65}$$

where γ is defined in Eqn.(2.51). The derivation has used Postulates A, B and C. However, as mentioned above, for the case $y = y' = z = z' = 0$ Eqns.(2.62),(2.65) may be derived from Postulates A and B only.

Finally, in this Section, an alternative derivation of the PVAR is given using only Postulate B, and the Reciprocity Postulate. As already noted, Postulate B is verified if the PVAR has the trilinear form of Eqn.(2.1).The argument given above to show that the PVAR is symmetric under the exchange $u \leftrightarrow v$ is easily extended to prove that it is symmetric under the exchange of any two of u, v, w (see also Ref.[19]). This has the consequence that, in Eqn.(2.1):

$$Q_1 = Q_2 = Q_3 \equiv Q \tag{2.66}$$

$$R_1 = R_2 = R_2 \equiv R \tag{2.67}$$

Imposing now the condition that $v_{CA} = 0$ when $v_{AB} = v_{CB}$ (the Reciprocity Postulate) gives:

$$P - v_{AB}^2 R = 0 \tag{2.68}$$

Since Eqn.(2.68) must hold for all values of v_{AB} then

$$P = R = 0 \tag{2.69}$$

Using Eqns.(2.66), (2.67), (2.69) the PVAR may be written:

$$v_{AB} + v_{BC} + v_{CA} + \frac{S}{Q} v_{AB} v_{BC} v_{CA} = 0 \tag{2.70}$$

From dimensional analysis of Eqn.(2.70) it follows that:

$$\frac{Q}{S} = \pm V^2 \tag{2.71}$$

where V is a universal constant with the dimensions of velocity. From the definitions of u, v, w given above, and using again the Reciprocity Postulate, Eqn.(2.70) becomes identical to Eqn.(2.40) or (2.42) according as the plus or minus sign respectively is chosen in Eqn.(2.71). The argument given above, using the Uniqueness postulate (Postulate B) then eliminates the solution with the minus sign. Finally it may be remarked that if the Reciprocity Postulate is regarded, as is often the case in the literature, as 'obvious' it would follow that the PVAR has been derived here purely from Postulate B, i.e. without recourse to the Special Relativity Principle itself. In fact, as shown here and in Ref.[18], the Reciprocity Postulate is actually a necessary consequence of the Relativity Principle and other postulates (space-time homogeneity and spatial isotropy in Ref.[18], the Relativity Principle and Uniqueness in the present paper).

3 Kinematical Consequences of the Lorentz Transformation

Introducing the notation $s \equiv Vt$, the Lorentz Transformation in Eqns.(2.62) to (2.65) may be written in the form:

$$x' = \gamma(x - \beta s) \tag{3.1}$$
$$y' = y \tag{3.2}$$
$$z' = z \tag{3.3}$$
$$s' = \gamma(s - \beta x) \tag{3.4}$$

where

$$\beta \equiv \frac{v}{V} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

The four component quantity $X \equiv (s; x, y, z)$ is a 4-vector [22] whose 'length' r_X is defined by the relation:

$$r_X^2 \equiv V^2\tau^2 \equiv s^2 - x^2 - y^2 - z^2 \tag{3.5}$$

As may be shown directly, using Eqns.(3.1)-(3.4), Eqn.(3.5) is invariant under the Lorentz Transformation S $\rightarrow$ S' corresponding to the replacement [23]:

$$(s; x, y, z) \rightarrow (s'; x', y', z')$$

The quantities r_X, τ, with dimensions of length and time respectively, defined in Eqn.(3.5) thus have the same value in all inertial frames, i.e. they are Lorentz invariant.

Consider now a physical object of Newtonian inertial mass (referred to subsequently simply as 'mass') situated at the space-time point X in S and moving with an arbitary uniform velocity $\vec{u}$ in that frame. Suppose that the object is at rest at the origin of the inertial frame S". Using Eqn.(3.5) in the frame S", it follows that if t'' is the time at the object as observed in S" then:

$$\tau = t'' \tag{3.6}$$

That is the Lorentz invariant time τ is the *proper time* (the time in its own rest frame) of the physical object. A new 4-vector p may now be defined as:

$$p \equiv m\frac{dX}{d\tau} \tag{3.7}$$

Because τ is Lorentz invariant p transforms in the same way under the Lorentz Transformation as X [24]. Indeed this is the property which defines, in general, a 4-vector. Thus p' observed in S' is related to p observed in S by :

$$p'_x = \gamma(p_x - \beta p_s) \tag{3.8}$$
$$p'_y = p_y \tag{3.9}$$
$$p'_z = p_z \tag{3.10}$$
$$p'_s = \gamma(p_s - \beta p_x) \tag{3.11}$$

By considering the Lorentz Transformation parallel to $\vec{u}$, the infinitesimal time increments δt and $\delta\tau$ are related by the expression:

$$\delta t = \gamma_u \delta\tau \tag{3.12}$$

where

$$\beta_u \equiv \frac{u}{V} \qquad \gamma_u = \frac{1}{\sqrt{1-\beta_u^2}}$$

Using Eqns.(3.7), (3.12) and taking the limits as δt, $\delta\tau$, $\delta\vec{x}$, $\to 0$:

$$p_x = m\frac{dx}{d\tau} = m\gamma_u\frac{dx}{dt} = m\gamma_u u_x \tag{3.13}$$
$$p_y = m\gamma_u u_y \tag{3.14}$$
$$p_z = m\gamma_u u_z \tag{3.15}$$
$$p_s = mV\frac{dt}{d\tau} = m\gamma_u V \tag{3.16}$$

Analagously to r_X, the length of p, r_p, is defined as:

$$r_p^2 \equiv p_s^2 - p_x^2 - p_y^2 - p_z^2 = m^2V^2\gamma_u^2(1-\beta_u^2) = m^2V^2 \tag{3.17}$$

The mass m is then proportional to the length r_p of the energy momentum 4-vector p and $U \equiv (\gamma_u V; \gamma_u u_x, \gamma_u u_y, \gamma_u u_z)$ is also a 4-vector, the relativistic generalisation of the velocity vector of classical mechanics. Introducing the definitions:

$$p^2 \equiv p_x^2 + p_y^2 + p_z^2 \tag{3.18}$$
$$E \equiv Vp_s \tag{3.19}$$

then Eqns.(3.13)-(3.17) lead to the relations

$$\beta_u = \frac{Vp}{E} \tag{3.20}$$
$$\gamma_u = \frac{E}{mV^2} \tag{3.21}$$
$$E^2 = p^2V^2 + m^2V^4 \tag{3.22}$$

To see the connection between p, E and the physical quantities of Classical Mechanics, consider the limit where $u \ll V$ i.e. $\beta_u \ll 1$:

$$p = m\gamma_u u = mu(1 + \frac{1}{2}\beta_u^2 + ...) \tag{3.23}$$
$$\simeq mu = p^{(N)} \tag{3.24}$$
$$E = (p^2V^2 + m^2V^4)^{\frac{1}{2}} = mV^2(1 + \frac{p^2}{2m^2V^2} + ...) \tag{3.25}$$
$$\simeq mV^2 + \frac{p^2}{2m} = mV^2 + T^{(N)} \tag{3.26}$$

Here $p^{(N)}$, $T^{(N)}$ denote the Newtonian momentum and kinetic energy respectively. The success of Newtonian mechanics in the everyday world then indicates that V must be very large as compared to the typical velocities encountered on the surface of the earth. The physical meaning of the relativistic energy E is given by setting p = 0 in Eqn.(3.22):

$$E_0 = E(p = 0) = mV^2 \tag{3.27}$$

This equation states the equivalence of mass and energy. *The mass m of a physical object is equivalent to its relativistic energy in its own rest frame E_0 .*

As a consequence of the conservation of relativistic momentum $\vec{p} = (p_x, p_y, p_z)$ the relativistic energy E, defined in Eqn.(3.19) is also a conserved quantity. Suppose that an ensemble of N physical objects with energy-momentum 4-vectors $p^{i,IN}$, $(i = 1, 2, ..N)$ interact in an inelastic way so as to produce a different ensemble of M physical objects with energy-momentum 4-vectors $p^{j,OUT}$, $(j = 1, 2, ..M)$. Define 4-vectors P, $\tilde{P}$, ΔP as:

$$P \equiv \sum_{i=1}^{N} p^{i,IN} \tag{3.28}$$
$$\tilde{P} \equiv \sum_{j=1}^{M} p^{j,OUT} \tag{3.29}$$
$$\Delta P \equiv \tilde{P} - P \tag{3.30}$$

Momentum conservation gives:

$$\Delta P_x = \Delta P_y = \Delta P_z = 0 \tag{3.31}$$

Applying the Lorentz Transformation Eqn.(3.8) to ΔP_x :

$$\Delta P_{x'} = \gamma(\Delta P_x - \beta \Delta P_s) \tag{3.32}$$

In order that $\Delta P_{x'}$ is zero for arbitary γ, β (i.e. that momentum is always conserved in the frame S') it follows that $\Delta P_s = 0$. Use of Eqns.(3.28-3.30) and (3.19) gives:

$$\sum_{i=1}^{N} E^{i,IN} = \sum_{j=1}^{M} E^{j,OUT} \tag{3.33}$$

so that conservation of relativistic energy, E, is a consequence of the conservation of relativistic momentum.

Another, related, conserved quantity is the Lorentz invariant common 'effective mass' of the N incoming or M outgoing physical objects, defined by the relation:

$$M_{eff}^2 V^2 \equiv P_s^2 - P_x^2 - P_y^2 - P_z^2 = \tilde{P}_s^2 - \tilde{P}_x^2 - \tilde{P}_y^2 - \tilde{P}_z^2 \tag{3.34}$$

In a way analagous to Eqn(3.27) for a single physical object, M_{eff} is equivalent to the total energy in the overall centre-of-mass system, where:

$$\sum_{i=1}^{N} P^{i,IN} = \sum_{j=1}^{M} P^{j,OUT} = 0$$

so that:

$$\sum_{i=1}^{N} E_0^{i,IN} = \sum_{j=1}^{M} E_0^{j,OUT} = M_{eff} V^2 \tag{3.35}$$

By using the Lorentz Transformation for energy-momentum 4-vectors Eqns.(3.8)-(3.11), and the conservation laws of relativistic energy and momentum, it is not difficult to devise simple particle physics experiments to determine the velocity parameter V. Two such examples follow:

(i) A proton with measured velocity v_{IN} collides with a proton at rest. The recoil and scattered particles are required to have momentum vectors making equal angles θ with the direction of the incoming proton. V is then given by the relation:

$$V = \frac{v_{IN}[\cos^2\theta + \cos 2\theta]}{2\cos\theta\sqrt{\cos 2\theta}} \tag{3.36}$$

For example, with $v_{IN} = 0.5V$, (proton momentum of 541 MeV/c, assuming that $V = c$) then $\theta = 43.93°$.

(ii) An annihilation photon from a para-positronium atom at rest Compton scatters on a free electron. The recoil electron and the scattered photon have momentum vectors making equal angles with the incoming photon direction. If $\beta_{OUT} = v_{OUT}/V$ where v_{OUT} is the velocity of the recoil electron, then:

$$\frac{1 + \beta_{OUT}}{\sqrt{1 - \beta_{OUT}^2}} = 2 \tag{3.37}$$

Solving Eqn.(3.37), it is found that:

$$V = 1.412 v_{OUT} \tag{3.38}$$

For any physical object whose relativistic momentum is much larger than the equivalent of its mass:

$$p \gg mV \tag{3.39}$$

it follows from Eqns.(3.20),(3.22) that:

$$u \simeq V \tag{3.40}$$

all highly relativistic objects, in the sense of Eqn.(3.39) then have a velocity slightly smaller than, but very close to, V. This has been demonstrated here for such objects in observed from the frame S. It was shown in the previous Section however, by use of the PVAR, that the same is true in all inertial frames. A special case of Eqn.(3.40), corresponding to exact equality, is a massless particle, such as a photon or a neutrino, when:

$$u = V = c \tag{3.41}$$

If photons are massless particles then Einstein's second postulate has indeed been shown to be a necessary consequence of Postulates A, B and C. However, as previously pointed out [25], the actual value of the photon mass is an experimentally determined parameter [26], like the mass of any other particle. If it is admitted that the photon may have a non-zero mass, smaller than the experimental upper limit, then, for sufficiently low energy photons, Einstein's second postulate would no longer hold, and so his derivation of the Lorentz transformation would not, in this case, be valid. No such restrictions apply to purely kinematical derivations, such as those originally reported in Ref.[9], or that presented in the present paper.

It may be objected that the 4-vector definition in Eqn.(3.7) makes no sense if $m = 0$, as in this case the rest frame S" of the particle cannot be defined. Actually, however the quantity $\gamma_u m$ which occurs as a factor in all the components of p (see Eqns.(3.13) -(3.16)) has a finite limit as $m \to 0$, $\gamma_u \to \infty$. Using Eqns.(3.20),(3.22) one obtains:

$$\begin{aligned} \gamma_u m &= m[1-\beta_u^2]^{-\frac{1}{2}} \\ &= m\left[1-\frac{V^2p^2}{V^2p^2+m^2V^4}\right]^{-\frac{1}{2}} \\ &= [\frac{p^2}{V^2}+m^2]^{\frac{1}{2}} \end{aligned} \tag{3.42}$$

The right hand side of Eqn.(3.42) is finite as $m \to 0$. It is crucial in discussing the massless limit of Eqn.(3.42) that m is identified with the Lorentz invariant Newtonian inertial mass. As emphasised by Okun [27] many texts and popular books on Special Relativity still introduce a velocity dependent mass [28] that is, in fact defined as the quantity on the left hand side of Eqn.(3.42). One then arrives at the somewhat paradoxical conclusion that the relativistic mass of an object whose Newtonian mass is zero, is not zero. In fact, for highly relativistic objects, the 'relativistic mass' is simply equal to p/V, an already well defined kinematical quantity, so that the use of the term 'relativistic mass' becomes redundant. In all cases, (and, suprisingly, in contradiction with what is claimed in Ref.[28]), the physical meaning of the equations of relativistic kinematics is much more transparent when only the Newtonian inertial mass is used in them [27].

A simple conclusion on the nature of light, and its relation to all other physical objects existing in nature can now drawn. If photons are indeed massless (or effectively massless in the sense of Eqn.(3.39)) objects moving with constant momentum according to Newton's First Law, then:

(1) The 'aether' becomes completely redundant. Since light is not (any more than any other physical object in motion) a wave phenomenon in the classical sense, then no wave-carrying medium is required. The Wave Theory of Light in the 19th Century, and the confusion over 'wave particle duality' in the early part of the 20th, arose because the mathematical description of large ensembles of photons, each of which is *individually* described by the laws of Quantum Mechanics, has a structure very similar to that of transverse waves in a classical medium. The mathematical descriptions are isomorphic even though the physical systems are quite distinct. This point is further elaborated elsewhere [29].

(2) The counter-intuitive nature of the PVAR, which, as Einstein realised, (see for example the popular book referred to in Ref.[12]) is a major stumbling block for the understanding of Special Relativity, is not due to some specific and mysterious property of light, as seems to be the case when it embodied in Einstein's second postulate, but a property of the geometry of space-time that relates observations of *all physical objects* with $p \gg mV$. This fact may appear just as mysterious. It is, however, an inevitable logical consequence of simple, apparently self-evident, postulates such as those used above or in other purely kinematical derivations of the Lorentz Transformation.

4 Comparison with Previous Work and Conclusions

The literature on the derivation of the Lorentz Transformation is vast (see Ref.[2] of Ref[18] for a partial list of work published before 1968). Here only a brief survey of some more recent work is presented to put in perspective the derivation given in the present paper. Firstly, a list of the different postulates that have been used in the literature is given. The postulates used in several different derivations are then presented in Table I. For reference, Einstein's derivation of 1905 is included as the first entry. The other derivations in Table I do not make use of Einstein's second postulate.

The postulates, and the abbreviation by which they are referred to in Table I are:

- The Special Relativity Principle (SRP)
- The Reciprocity Postulate (RP)
- Space-Time Homogeneity (STH)

- Spatial Isotropy (SI)
- Relativistic Transverse Momentum Conservation (TMC)
- The Group Property (GP)
- Relativistic Mass Increase (RMI)
- The Constancy of the Velocity of Light (CVL)
- The Sign of the Limiting Velocity Squared (SLV2)
- The Causality Postulate (CP)
- The Uniqueness Postulate (UP)

It can be seen from Table I that all authors cited require the Relativity Principle. Similarly, all authors, prior to the present paper, require Space-Time Homogeneity. It has been shown [25, 30, 34] that this requires the Lorentz Transformation to be linear. Since Einstein assumed linearity, without proof, Space-Time Homogeneity is included among Einstein's necessary postulates in Table I. In the present paper it is shown that the linearity of the Lorentz Transformation (Eqns.(2.8),(2.13)) follows from the Uniqueness postulate and the definitions (Eqns.(2.4),(2.11) of relative velocity alone, i.e. no explicit postulates on space-time geometry are required. All the derivations in Table I require, however, Spatial Isotropy to derive the Lorentz Transformation for space time points lying outside the x, x' axis. Since the Reciprocity Postulate has been shown [18] to be a consequence of the Special Relativity Principle, Space-Time Homogeneity and Spatial Isotropy, it is not included in Table I as these three postulates are included in all the derivations cited, except that of the present paper.

The postulates of Constancy of the Velocity of Light, and Conservation of Relativistic Momentum used in Refs.[1,31] respectively are 'strong' consequences of the Lorentz Transformation that must be guessed or derived directly from experiment. If the Group Property [35] of the transformation is used, as in Refs.[9,25,30,32] then such strong postulates are not necessary. The present paper gives however a counter example to show that the Lorentz Transformation may be derived, in the absence of strong postulates, without assuming the Group Property. The remaining postulates cited in Table I (Relativistic Mass Increase, Sign of the Limiting Velocity Squared, and the Causality Postulate) are introduced to circumvent a problem which arises in all derivations that do not use Einstein's second postulate. For dimensional reasons, a constant with the dimensions of velocity squared must occur in the equations (Eqn.(2.36) in the present paper), but its sign is not specified. One choice leads to the Lorentz Transformation, when the constant is identified with c^2, the other to another transformation (Eqns.(2.41),(2.42)) which, apparently, has unphysical properties. For example, there is no limiting velocity and the sum of two finite velocities may yield an infinite result. There are then two possible approaches to select the solution corresponding to the Lorentz Transformation:

Author	Ref.	SRP	STH	SI	TMC	GP	RMI	CVL	SLV2	CP	UP
Einstein	[1]	Y	Y	Y	N	N	N	Y	N	N	N
Terletskii	[30]	Y	Y	Y	N	Y	Y	N	N	N	N
Weinstock	[31]	Y	Y	Y	Y	N	Y	N	N	N	N
Lee and Kalotas	[32]	Y	Y	Y	N	Y	N	N	Y	N	N
Lévy-Leblond	[25]	Y	Y	Y	N	Y	N	N	N	Y	N
Sen	[33]	Y	Y	Y	N	N	N	N	Y	N	N
This paper	-	Y	N	Y	N	N	N	N	N	N	Y

Table 1: Postulates required for different derivations of the Lorentz Transformation. Y $\equiv$ 'YES' (postulate used) , N $\equiv$ 'NO' (postulate not used). See the text for the definitions of the postulates. All derivations except that of this paper and Lévy-Leblond also assumed the Reciprocity Postulate. Since however the latter has been shown [18] to be a consequence of SRP, STH and SI which are assumed in every case in these derivations it is omitted from the list of *required* postulates in the Table.

(i) Reject the undesirable solution on the grounds that it is 'unphysical'. This is denoted by SLV2 in Table I.

(ii) Introduce another postulate that has the effect of rejecting the 'unphysical' solution.

This is the case for 'Relativistic Mass Increase' in Refs.[30,31] and the 'Causality Postulate' in Ref.[25]. In fact the 'unphysical' solution is not a single valued function of its arguments, so it is rejected, in the present paper, by the Uniqueness Postulate.

The recent derivation of Sen [33], based on an ingenious gedankenexperiment involving three parallel-moving inertial frames is remarkable in that it uses neither strong postulates nor the Group Property. However both the Reciprocity and SLV2 Postulates were used. In fact if the derivation of the PVAR by Mermin [19] (also based on a sophisticated gedankenexperiment) and the subsequent derivation of the Lorentz Transformation from the PVAR by Singh [20] are combined, the same set of postulates as used by Sen (the Special Relativity Principle, the Reciprocity Postulate, Space-Time Homogeneity, Spatial Isotropy, and the Sign of the Limiting Velocity Squared) are invoked.

It is clear that the Uniqueness Postulate replaces the STH postulate of previous derivations, that requires the transformation equations to be linear. Indeed in the direct derivation of the Lorentz Transformation above the UP in the form of the trilinear equation (2.3) yields immediately, on applying the condition of constant relative velocity, Eqn.(2.4), linear transformation equations. Uniqueness does not however imply linearity of the PVAR. The final equation (2.70) here retains the trilinear form of the initial anstaz Eqn.(2.1). Writing the relative velocities as derivatives of the form $\delta x/\delta t$ it can be seen that the PVAR is in fact the ratio of two linear space-time transformation equations for δx and δt.

The argument used in this paper to reject the solution with $-V^2$ in Eqn.(2.35), the non single-valued nature of Eqn.(2.42) for particular choices of u and v, replaces the SLV2 assumption of say Sen [33], which rejects the solution rather because a singular result is obtained for the same choices of u and v. The economy of postulates obtained in the derivation presented in this paper is a consequence of the fact that the UP replaces the space-time postulate STH and also requires rejection the $-V^2$ solution.

The Uniqueness postulate is actually implemented by assuming a multilinear form of the equations (trilinear in Eqns.(2.1),(2.3); quadrilinear in Eqn.(2.55)). This is however not even a sufficient condition that the PVAR is single valued [17]. Indeed Eqn.(2.42) is trilinear but does not, in all cases, give a unique solution for w. All that has been shown is that a single valued multilinear solution can be found that is completely determined by consistency with the remaining postulates (effectively RP only, equivalent, as shown in Section 2 above, to SRP and UP). The interesting but unaswered question that remains is whether a single-valued solution can be found that does *not* have a multilinear form, i.e. are the Lorentz Transformations and the PVAR of Special Relativity the *only* solutions consistent with the inital postulates.

Inspection of Table I shows that the minimum number of postulates necessary to derive the Lorentz Transformation before the present paper is four, actually the same number that Einstein required for his original derivation. The new derivation presented here requires, in general, only three postulates and for the limited class of space-time points lying along the common x, x' axis of the frames S, S' only *two* postulates the Special Relativity Postulate and the Uniqueness Postulate are sufficient.

By developing the kinematical consequences of the Lorentz Transformation it has been shown that, for consistency with Classical Electrodynamics, light must be described by particles whose mass-equivalent is much smaller than their energy. That this conclusion follows from the Lorentz Transformation *alone* was pointed out, though not explicitly demonstrated, in Ref.[25]. Finally a remark on Pauli's discussion of kinematical derivations of the Lorentz Transformation [36]. Pauli pointed out that the Lorentz Transformation may be derived from the Group Property, the Reciprocity Postulate and the Special Relativity Postulate in the form of Postulate A used here, but applied only to length measurements [37]. He also states that 'From the group theoretical assumption it is only possible to derive the transformation formula but not its physical content'. In fact it has been demonstrated above that the physical content of the Lorentz Transformation actually does becomes transparent when its kinematical consequences are developed in detail, as in Section 3 above. In particular, the interpretation of V as the universal limiting velocity of any physical object is quite general, while the identification $c = V$ for light leads naturally to the particle description of the latter.

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References

[1] A.Einstein,'Zur Elektrodynamik bewegter Körper' ' Annalen der Physik **17** 891 (1905)

[2] A number of other postulates were also necessary. The most important of these were: linearity of the equations, Spatial Isotropy and the Reciprocity Postulate. These are discussed in relation to the present and other derivations in Section 4 .

[3] G.Galileo, 'Dialogues Concerning Two New Sciences' (1538)

[4] H.A.Lorentz, 'Electromagnetic Phenomena in a System Moving with any Velocity less than that of Light' Proc. Acad. Sci. Amst. **6** 809 (1904). Re-printed in English translation in: C.W.Kilmister, 'Special Theory of Relativity', Pergamon Press (1970)

[5] J.Larmor, 'Aether and Matter', Cambridge University Press (1900), Chapter XI.

[6] H.Poincaré, 'The Dynamics of the Electron' Rend. del Circ. Mat. di Palermo **21** 129-146, 166-175 (1906)

[7] This was shown for the first time in Einstein's paper (Ref.[1]) by his derivation of the Lorentz Equation of motion of the electron from the covariance of the free-space Maxwell Equations and Newton's Second Law in the electron rest frame. The covariance of the free-space Maxwell equations themselves, also demonstrated in Ref[1], is more in the nature of a consistency check, as the Wave Equation for light and hence the propagation of light at the fixed velocity c (Einstein's second postulate) is a consequence of these equations. Indeed the Lorentz Transformation (up to a constant factor) had already been derived in 1887 by W.Voigt, by requiring covariance of the Wave Equation for light. (W.Voigt, Goett. Nachr. P47 (1887)). The fact that the Maxwell Equations in the presence of sources are also covariant under the Lorentz Transformation, as first demonstrated by Einstein in Ref.[1] and independently by Poincaré in Ref.[6], is however another non-trivial example of the application of the Principle of Special Relativity to a dynamical law.

[8] This clear separation of kinematics and dynamics is not present in General Relativity, so it is understood that the 'laws of physics' discussed in this paper exclude those governing gravitation.

[9] W.v Ignatowsky Arch. Math. Phys. Lpz. **17** 1 (1910) and **18** 17 (1911) Phs. Z. **11** 972 (1910) and **12** 779 (1911). P.Frank and H.Rothe, Annalen der Physik **34** 825 (1911) and Phys. Z. **13** 750 (1912). The subequent literature on 'lightless' derivations of the Lorentz Transformation is very extensive. A long list (prior to 1968) may be found in Ref.[18] below. More recent papers of particular interest are specifically referred to in the discussion of Section 4.

[10] In the paper that Einstein published earlier in 1905 that did predict the existence of light quanta he used thermodynamical arguments based on the equivalence of the Planck radiation formula in the Wein regime ($h\nu \gg kT$) to the properties of an ideal gas. Thus he showed that cavity radiation behaved as though it consisted of particles

with energy $h\nu$. It is amusing to note, that, by comparing the prediction of Parallel Velocity Addition Relation, derived in Ref.[1], with the properties of light described in his second postulate, Einstein might have been lead also to infer the existence of photons uniquely from results contained in the Special Relativity paper ! See also Section 3.

[11] That is to say that S (S') perform similar measurements of similar measuring rods or clocks at rest in S' (S)

[12] The Special Relativity Principle as stated in Postulate A was used by Einstein in a derivation of the Lorentz Transformation presented in a popular exposition of Relativity Theory published in 1916. An English translation may be found in 'Relativity the Special and General Theory' published by Routledge (London) 1994.

[13] As pointed out by Mach, such motion has physical meaning only in relation to other, similar, inertial frames. See for example 'Relativity Theory, its Origins and Impact on Modern Thought' Ed. L.Pearce Williams, John Wiley and Sons (1968) P16.

[14] E.T.Whittaker, Tarner Lectures at Cambridge University (1947). Published as: 'From Euclid to Eddington, a Study of Conceptions of the External World' Dover (New York) 1958, P49. Whittaker assumed the constancy of the velocity of light in his derivation of the PVAR.

[15] If the STM are associated with single quantum events (say the detection of a single photon reflected near the end of a measuring rod) this will clearly not be the case, since observation by S excludes observation by S' and vice-versa. It is understood that the STM are made within suitably short space and time intervals on an ensemble of 'equivalent' quantum events. Depending on the method of measurement, one or more such 'equivalent quantum events' constitute a STM.

[16] Perhaps other single valued functions can be defined. Polynomials of degree n are, of course, n-valued.

[17] By a special choice of the coefficients certain 'pathological' multilinear functions may be defined that are not single-valued. For example:

$$\alpha + \beta + \gamma - k\alpha\beta\gamma = 0$$

has the same form as Eqn(2.1). Choosing $\alpha = 1/(k\beta)$ it is seen that the equation is verified for all values of γ, so it is not single-valued in this variable. Except for such special cases, the requirement of multilinearity is sufficient to ensure Uniqueness in the sense of Postulate B.

[18] V.Berzi and V.Gorini, 'Reciprocity Principle and Lorentz Transformations', Journ. Math. Phys. **10** 1518-1524 (1969)

[19] N.D.Mermin, 'Relativity without Light', Am. J. Phys. **52** 119-124 (1984)

[20] S.Singh, 'Lorentz Transformations in Mermin's Relativity without Light', Am. J. Phys. **54** 183-184 (1986)

[21] Since, clearly, $\gamma(0) = 1$, the plus sign is taken on solving Eqns.(2.20),(2.36) for $\gamma(v)$.

[22] H.Minkowski, Phys. Zeitschr. **10** 104 (1909)

[23] Since $x^2 + y^2 + z^2$ is rotationally invariant, this is true for a Lorentz Transformation in any direction, not only along the x,x' axis as for the case S $\rightarrow$ S'.

[24] M.Planck, Verh. Deutsch. Phys. Ges. **4** 136 (1906)

[25] J.M. Lévy-Leblond, 'One more Derivation of the Lorentz Transformation', Am. J. Phys. **44** 271-277 (1976)

[26] The current experimental upper limit on the mass of the photon is $3 \times 10^{-33} MeV$. (Review of Particle Properties. Phys. Rev. D**50** Part I (1994)). This implies that for a photon of mass equal to the limit, whose wavelength is equal to the distance from the Earth to the Sun, (and so has an energy of $1.31 \times 10^{-24} MeV$) will have a velocity that is smaller than V by only 2.62×10^{-16} % !

[27] L.B.Okun, 'The Concept of Mass', Physics Today, June 1989, 31-36.

[28] 'The Feynman Lectures in Physics', Mechanics I, Chapter 16. Addison Wesley, Reading Massachusetts (1963).

[29] J.H.Field, 'Quantum Mechanics from Special Relativity and Classical Electromagnetism. A new Pedagogical Approach'.(Paper in preparation).

[30] Y.P.Terletskii, 'Paradoxes in the Theory of Relativity', Plenum Press, New York, 1968, P17.

[31] R.Weinstock, 'New Approach to Special Relativity', Am. J. Phys. **33** 640-645 (1965)

[32] A.R.Lee and T.M.Kalotas, 'Lorentz Transformation from the First Postulate', Am. J. Phys. **43** 434-437 (1975)

[33] A.Sen, 'How Galileo could have derived the Special Theory of Relativity', Am. J. Phys. **62** 157-162 (1994).

[34] L.J.Eisenberg, 'Necessity of the linearity of relativistic transformations between inertial systems', Am. J. Phys. **35** 649 (1967).

[35] The Group Property has 3 essential aspects:

1) If the operations A and B are members of a Group, then so must the operation C consisting of the combined operations of A, then B. In symbols $BA = C$.

2) The existence of the identity operation I such that $IA = A$.

3) For each member of the group A, the existence of an inverse A^{-1} such that $A^{-1}A = I$.

[36] W.Pauli, 'Theory of Relativity' Pergamon Press 1958 P11.

[37] Although not explicitly stated by Pauli, the 'Sign of the Limiting Velocity Squared' postulate is also needed to reject the 'unphysical' solution with the opposite sign of V^2.