Helv. Phys. Acta 70 (1997) 578 – 589
0018-0238/97/040578-12 $ 1.50+0.20/0

Helvetica Physica Acta

Algebraic Dynamical Approach to the Linear Nonautonomous System With hw(4) Dynamical Algebra

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(22.VI.1996)

Abstract. The exact solution and the invariant Cartan operator of the linear nonautonomous system with hw(4) dynamical algebra, which describes the interaction of a quantized radiation field with a time-dependent classical current, are obtained by using the method of algebraic dynamics. Two kinds of solutions are obtained in terms of both harmonic oscillator states and coherent states. The rule (or condition) for direct quantum-classical correspondence of the solutions has been established. It has also been shown that algebraic dynamics can be generalized from the linear dynamical system with a semi-simple Lie algebra to that with a general Lie algebra.

In Ref.[1], the linear nonautonomous system with a semi-simple dynamical group has been successfully studied by means of algebraic dynamics. The exact solutions and the profound quantum-classical correspondences of the dynamical systems with SU(1,1)[2], SU(2)[3] and SP(4)[4] algebras were obtained. The advantages of algebraic dynamics are: (1) The method allows one to study the nontrivial time-dependent dynamical symmetries of linear nonautonomous quantum systems conveniently and to make full use of these symmetries; (2) By means of gauge transformation, one might readily get the analytical solution of a quantum system in various possible representations; (3) By virtue of algebraic dynamics, the number of parameters needed for solving the quantum equations of the system can be reduced to the minimum (which is equal to the rank of the algebra minus the number of its Cartan operators), and the equation of motion can be linearized; (4) Within the framework of algebraic dynamics, the quantum-classical correspondence of the solutions is exhibited clearly. The present letter serves as an attempt to generalize the algebraic dynamic from the linear dynamical system with a semi-simple Lie algebra to that with a general Lie algebra. The linear dynamical system with Heisenberg-Weyl(hw(4)) algebra is one of the simplest general Lie algebraic systems.

The study of the linear nonautonomous quantum system possessing hw(4) algebraic structure is an interesting and important subject in quantum physics[5]. On one hand, the algebraic structure of the system is simple. On the other hand, the system describes the interaction of a quantized radiation field with a time-dependent classical current[6] or multiphoton processes[7]. With the aid of algebraic dynamics[1], in this paper we present the analytical solutions of the hw(4) linear dynamical system under two different representations respectively, and find out the quantum-classical correspondence of the solutions.

Consider an hw(4) linear dynamical system described by the Hamiltonian

$$\begin{aligned} \hat{H}(t) &= \omega(t)\hat{a}^\dagger\hat{a} + \Omega^*(t)\hat{a}^\dagger + \Omega(t)\hat{a} \\ &= \omega(t)\hat{N} + \Omega^*(t)\hat{a}^\dagger + \Omega(t)\hat{a} \ , \end{aligned} \tag{1}$$

where $\omega(t)$ and $\Omega(t)$ are non-singular, real and complex functions of time respectively. The particle number operator $\hat{N} = \hat{a}^\dagger\hat{a}$, the particle creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$, as well as 1 form an hw(4) algebra, which satisfy the following commutation rules

$$[\hat{a},\ \hat{a}^\dagger] = 1 \ , \quad [\hat{N},\ \hat{a}^\dagger] = \hat{a}^\dagger \ , \quad [\hat{N},\ \hat{a}] = -\hat{a} \ . \tag{2}$$

The Lie algebra hw(4) can be decomposed into a semi-direct sum of $U(1) = \{\hat{N}\}$ and $h(3) = \{\hat{a}^\dagger, \hat{a}, 1\}$, namely

$$\mathrm{hw}(4) = \mathrm{U}(1) \oplus \mathrm{h}(3) \ . \tag{3}$$

Because h(3) is an ideal of hw(4) and solvable, it is also the radical R of hw(4). The decomposition of the Lie algebra hw(4) is thus written as

$$\mathrm{hw}(4) = \mathrm{U}(1) \oplus R \ . \tag{4}$$

Therefore, hw(4) is a general Lie algebra. A special case of (1) reads

$$\hat{H} = \frac{1}{2}\omega(t)(\hat{p}^2 + \hat{q}^2) + eE(t)\hat{q} \ , \tag{5}$$

which describes the interaction of a harmonic oscillator having time-dependent frequency with an external electric field.

The time evolution of the system is determined by the Schrödinger equation

$$i\frac{\partial}{\partial t}|\Psi(t)\rangle = \hat{H}(t)|\Psi(t)\rangle \ , \tag{6}$$

here we have taken the natural unit, i.e., $\hbar = 1$. Eq.(6) can be solved by using the evolution operator method and Magnus expansion[8–10]. However, the drawbacks of this method are: (1) It does not give the dynamical invariant operator of the system; (2) The solution in coherent representation has not been obtained except the solution in harmonic oscillator representation; (3) The number of parameters for determining the time-evolution operator is usually more than the minimum needed for solving the quantum equations of the system; (4) The equations of motion for the parameters of the evolution operator are nonlinear, so it can not reveal the quantum-classical correspondence.

Introducing a gauge transformation for solving the Schrödinger equation (6)

$$\hat{U}_g(t) = \exp[v_1(t)\hat{a}]\exp[v_2(t)\hat{a}^\dagger]\exp[v_3(t)\hat{N}]\exp[v_4(t)] \ , \tag{7}$$

where $v_i(t)$ $(i = 1, 2, 3, 4)$ are complex functions of time. Under the gauge transformation (7), eq.(6) becomes

$$i\frac{\partial}{\partial t}|\overline{\Psi}(t)\rangle = \hat{\overline{H}}(t)|\overline{\Psi}(t)\rangle \ , \tag{8}$$

where the gauged Hamiltonian $\hat{\overline{H}}(t)$ and the gauged wave function $|\overline{\Psi}(t)\rangle$ are defined, respectively, as

$$\hat{\overline{H}}(t) = \hat{U}_g^{-1}\hat{H}(t)\hat{U}_g - i\hat{U}_g^{-1}\frac{\partial \hat{U}_g}{\partial t} \ , \tag{9}$$

$$|\overline{\Psi}(t)\rangle = \hat{U}_g^{-1}|\Psi(t)\rangle \ . \tag{10}$$

Substituting (1) into (9), after some calculation, one obtains the gauged Hamiltonian $\hat{\overline{H}}$

$$\begin{aligned}\hat{\overline{H}}(t) &= (\omega - i\dot{v}_3)\hat{N} + (\Omega^* + \omega v_2 - i\dot{v}_2)e^{-v_3}\hat{a}^\dagger \\ &+(\Omega - \omega v_1 - i\dot{v}_1)e^{v_3}\hat{a} + (\Omega v_2 - \Omega^* v_1 - v_1 v_2\omega - i\dot{v}_1 v_2 - i\dot{v}_4) \ .\end{aligned} \tag{11}$$

As has been pointed out in Ref.[1], one of the merits of algebraic dynamics is that it enables one to choose a proper gauge so as to meet his own requirement and thus the problem of a nonautonomous quantum system can be solved under different gauges. In what follows, we will solve the Schrödinger equation (6) under two different gauges which correspond to two different representations (harmonic oscillator and coherent representations) respectively. At first, let us consider the first gauge, i.e., the choice of gauge transformation is such that

$$v_3(t) = 0 \ , \tag{12a}$$

$$\dot{v}_2(t) + i\omega(t)v_2(t) + i\Omega^*(t) = 0 \ , \tag{12b}$$

$$\dot{v}_1(t) - i\omega(t)v_1(t) + i\Omega(t) = 0 \ , \tag{12c}$$

$$\dot{v}_4(t)+\dot{v}_1(t)v_2(t)+i\Omega(t)v_2(t)-i\Omega^*(t)v_1(t)-i\omega(t)v_1(t)v_2(t)=0\ . \tag{12d}$$

Within the framework of algebraic dynamics, the gauge transformation parameters $v_i(t)$ may be given any initial values. For convenience, we assume $\hat{U}_g(t=0)=1$, i.e.,

$$v_1(t=0)=0, \quad v_2(t=0)=0, \quad v_4(t=0)=0\ . \tag{13}$$

From eqs.(12a-d), one has

$$v_1(t)=-i\exp\Big[i\int_0^t\omega(\tau)\mathrm{d}\tau\Big]\int_0^t\Omega(\tau_1)\exp\Big[-i\int_0^{\tau_1}\omega(\tau_2)\mathrm{d}\tau_2\Big]\mathrm{d}\tau_1\ , \tag{14a}$$

$$v_2(t)=-i\exp\Big[-i\int_0^t\omega(\tau)\mathrm{d}\tau\Big]\int_0^t\Omega^*(\tau_1)\exp\Big[i\int_0^{\tau_1}\omega(\tau_2)\mathrm{d}\tau_2\Big]\mathrm{d}\tau_1\ , \tag{14b}$$

$$v_4(t)=i\int_0^t\Omega^*(\tau)v_1(\tau)\mathrm{d}\tau\ . \tag{14c}$$

It is obvious that

$$v_1(t)=-v_2^*(t)\equiv v(t)\ . \tag{15}$$

(14a-c) are the integral expressions of $v_i(t)$ under the chosen gauge. If the coefficients $\omega(t)$ and $\Omega(t)$ are given, the gauge transformation parameters $v_i(t)$ can readily be calculated from eqs.(14a-c). Let

$$\gamma(t)=\Omega(t)\exp\Big[-i\int_0^t\omega(\tau)\mathrm{d}\tau\Big]\ , \tag{16a}$$

$$\chi(t)=-i\int_0^t\gamma^*(\tau)\mathrm{d}\tau\ , \tag{16b}$$

(14a-c) may be converted into the following compact form

$$v_1(t)=v(t)=-\frac{\gamma^*(t)\chi^*(t)}{\Omega^*(t)}\ , \tag{17a}$$

$$v_2(t)=-v^*(t)=\frac{\gamma(t)\chi(t)}{\Omega(t)}\ , \tag{17b}$$

$$v_4(t)=-i\int_0^t\gamma^*(\tau)\chi^*(\tau)\mathrm{d}\tau\ . \tag{17c}$$

With the aid of (15) and the following algebraic relation

$$\exp[-v_2(t)\hat{a}^\dagger]\exp[v_1(t)\hat{a}]\exp[v_2(t)\hat{a}^\dagger]=\exp[v_1(t)\hat{a}]\exp[v_1(t)v_2(t)]\ , \tag{18}$$

the gauge transformation $\hat{U}_g(t)$ may be written as

$$\begin{aligned}\hat{U}_g(t)&=\exp[-v^*(t)\hat{a}^\dagger]\exp[v(t)\hat{a}]\exp\Big[-\frac{1}{2}|\chi(t)|^2\Big]\exp\Big[-i\ \mathrm{Re}\int_0^t\gamma(\tau)\chi(\tau)\mathrm{d}\tau\Big]\\&=\exp[-v^*(t)\hat{a}^\dagger+v(t)\hat{a}]\exp\Big[-i\ \mathrm{Re}\int_0^t\gamma(\tau)\chi(\tau)\mathrm{d}\tau\Big].\end{aligned} \tag{19}$$

It is evident from (19) that the first gauge transformation is unitary. Under the chosen gauge, the gauged Hamiltonian $\hat{\overline{H}}(t)$ becomes

$$\hat{\overline{H}}(t) = \omega(t)\hat{\overline{I}}(0) \ , \tag{20}$$

where

$$\hat{\overline{I}}(0) = \hat{N} \ . \tag{21}$$

(21) shows that under the above special gauge, the Cartan operator $\hat{\overline{I}}$ does not dependent on time explicitly, and is just the particle number operator. Therefore, this gauge corresponds to harmonic oscillator representation.

The dynamical Cartan operator reads

$$\hat{I}(t) = \hat{U}_g\hat{\overline{I}}(0)\hat{U}_g^{-1} = \hat{N} + \alpha(t)\hat{a} + \alpha^*(t)\hat{a}^\dagger + \delta(t) \ , \tag{22}$$

where the coefficients α and δ are

$$\alpha(t) = v_1(t) = v(t) \ , \quad \alpha^*(t) = -v_2(t) = v^*(t) \ , \quad \delta(t) = |\chi(t)|^2 \ . \tag{23}$$

It is easy to check that $\hat{I}(t)$ is an invariant operator of the system. The dynamical equation for $\alpha(t)$ is

$$\dot{\alpha}(t) = i\omega(t)\alpha(t) - i\Omega(t) \ , \tag{24a}$$

$$\dot{\alpha}^*(t) = -i\omega(t)\alpha^*(t) + i\Omega^*(t) \ . \tag{24b}$$

with the initial value $\alpha(t=0) = 0$. (24a, b) can also be obtained from the equation of motion for $\hat{I}(t)$, i.e., $\dfrac{\partial \hat{I}(t)}{\partial t} + i[\hat{H}(t), \hat{I}(t)] = 0$. Now we have a discussion about the quantum-classical correspondence. The classical algebraic dynamics is

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \{a, \ H(a, \ a^*, \ t)\}, \quad \frac{\mathrm{d}a^*}{\mathrm{d}t} = \{a^*, \ H(a, \ a^*, \ t)\} \ , \tag{25}$$

where $\{\cdots, \cdots\}$ denotes Poisson brackets. The Poisson brackets corresponding to (2) are

$$\{a, \ a^*\} = -i, \quad \{N, \ a\} = ia, \quad \{N, \ a^*\} = -ia^* \ . \tag{26}$$

The classical Hamiltonian is

$$H(t) = \omega(t)N + \Omega^*(t)a^* + \Omega(t)a \ , \quad N = a^*a \ . \tag{27}$$

From the above equations, one gets the classical equations of motion for the system, i.e.,

$$\dot{a}(t) = \{a(t), \ H(t)\} = -i\omega(t)a(t) - i\Omega^*(t) \ , \tag{28a}$$

$$\dot{a}^*(t) = \{a^*(t), \ H(t)\} = i\omega(t)a^*(t) + i\Omega(t) \ . \tag{28b}$$

Comparing eqs.(28a,b) with eqs.(24a,b), we obtain the following remarkable relation (quantum-classical correspondence)

$$\alpha = v_1 = -a^*, \quad \alpha^* = -v_2 = -a \ , \tag{29a}$$

$$\hat{I}(t) = \hat{N} - a^*\hat{a} - a\hat{a}^\dagger + \delta \ . \tag{29b}$$

For a general Lie algebra dynamical system, a direct quantum-classical correspondence can seldom be obtained[1]. However, for the hw(4) algebra, we have arrived at such a correspondence. This is because, mathematically, the structure constant matrices appearing in the dynamical equation of the invariant Cartan operator are Hermitian, according to Ref.[1], there should be a direct quantum-classical correspondence. Physically, the algebraic dynamic freedoms related to the invariant Cartan operator $\hat{I}$ are corresponding to $\hat{a}^\dagger$ and $\hat{a}$, and their classical equations are just the equations of motion for p and q in phase space. In other word, as invariant operator $\hat{I}$ is concerned, the freedom $\hat{N}$ of the algebra hw(4) is frozen, and hw(4) is reduced to h(3). This gives rise to the above direct and excellent quantum-classical one-to-one correspondence.

In what follows, we proceed to solve the Schrödinger equation (6) of the system by using algebraic dynamics. First consider the eigen problem of the invariant operator $\hat{I}(t) = \hat{U}_g\hat{\bar{I}}(0)\hat{U}_g^{-1}$. Let $|n\rangle$ be the eigenstate of $\hat{\bar{I}}(0) = \hat{N}$, i.e.,

$$\hat{\bar{I}}(0)|n\rangle = n|n\rangle \ , \tag{30}$$

The eigen equation for the dynamical Cartan operator $\hat{I}(t)$ is

$$\hat{I}(t)\hat{U}_g|n\rangle = n\hat{U}_g|n\rangle = n|\phi_n(t)\rangle \ , \tag{31}$$

where $|\phi_n(t)\rangle = \hat{U}_g|n\rangle$ is the eigenstate of $\hat{I}(t)$ with eigenvalue n. By virtue of expression (19) of $\hat{U}_g$, one has

$$|\phi_n(t)\rangle = \sum_{m=0}^{\infty} \exp\Big[-i(m-n)\int_0^t \omega(\tau)\mathrm{d}\tau\Big]\sqrt{\frac{n!}{m!}}[\chi(t)]^{(m-n)}L_n^{m-n}(|\chi(t)|^2)$$
$$\times \exp\Big[-\frac{1}{2}|\chi(t)|^2\Big]\exp\Big[-i\ \mathrm{Re}\int_0^t \gamma(\tau)\chi(\tau)\mathrm{d}\tau\Big]|m\rangle \ , \tag{32}$$

where $L_n^{m-n}(|\chi(t)|^2)$ are the generalized Laguerre polynomials.

Under the special gauge (12a-d), the gauged Schrödinger equation is

$$i\frac{\partial}{\partial t}|\overline{\Psi}(t)\rangle = \omega(t)\hat{\bar{I}}(0)|\overline{\Psi}(t)\rangle \ , \tag{33}$$

which has the following solution

$$|\overline{\Psi}_n(t)\rangle = \exp[i\Theta_n(t)]|n\rangle \ , \tag{34}$$

where

$$\Theta_n(t) = -n\int_0^t \omega(\tau)\mathrm{d}\tau \ . \tag{35}$$

From (10), the orthonormal diabatic basis, which is an exact solution of the original equation (6), can be obtained

$$|\Psi_n(t)\rangle = \exp[i\Theta_n(t)]|\phi_n(t)\rangle = \sum_{m=0}^{\infty}\exp\Big[-im\int_0^t \omega(\tau)\mathrm{d}\tau\Big]\exp\Big[-\frac{1}{2}|\chi(t)|^2\Big]$$

$$\times \exp\Big[-i\,\mathrm{Re}\int_0^t \gamma(\tau)\chi(\tau)\mathrm{d}\tau\Big]\sqrt{\frac{n!}{m!}}[\chi(t)]^{(m-n)}L_n^{m-n}(|\chi(t)|^2)|m\rangle\ . \tag{36}$$

The diabatic energy levels are

$$\begin{aligned} E_n(t) &= \langle\Psi_n|\hat{H}(t)|\Psi_n\rangle \\ &= \langle\overline{\Psi}_n|\hat{U}_g^{-1}\hat{H}\hat{U}_g|\overline{\Psi}_n\rangle \\ &= n\omega(t) + |\chi(t)|^2\omega(t) + 2\mathrm{Re}[\chi(t)\gamma(t)]\ . \end{aligned} \tag{37}$$

An arbitrary solution of the Schrödinger equation (6) can be written as

$$\Psi(t) = \sum_n C_n\Psi_n(t) = \sum_n C_n \exp[i\Theta_n(t)]\phi_n(t)\ , \tag{38}$$

where C_n is independent of time and only determined by the initial state of the system, while the whole dynamical information is contained in the phase $\Theta_n(t)$ and the eigen states $\phi_n(t)$ of the invariant operator $\hat{I}(t)$. It is easy to check that the expectation value of $\hat{I}(t)$ is a constant of motion.

$$\langle\Psi(t)|\hat{I}(t)|\Psi(t)\rangle = \sum_n n|C_n|^2\ . \tag{39}$$

Up to now, we have obtained the analytical expression of an arbitrary solution of the system studied. In the following, we discuss some special cases:

(1) The initial state is an eigenstate of the particle number operator $\hat{N}$, i.e., $|\Psi(t=0)\rangle = |n\rangle$. For this initial condition, the solution of the Schrödinger equation is

$$|\Psi(t)\rangle = \sum_{m=0}^{\infty} \exp\Big[-im\int_0^t \omega(\tau)\mathrm{d}\tau\Big]\exp\Big[-\frac{1}{2}|\chi(t)|^2\Big]$$

$$\times \exp\Big[-i\,\mathrm{Re}\int_0^t \gamma(\tau)\chi(\tau)\mathrm{d}\tau\Big]\sqrt{\frac{n!}{m!}}[\chi(t)]^{(m-n)}L_n^{m-n}(|\chi(t)|^2)|m\rangle\ . \tag{40}$$

which recovers the result in Ref.[8].

(2) $|\Psi(t=0)\rangle = |n\rangle$, and $\omega = \mathrm{const.}$, $\Omega^* = \Omega = \mathrm{const.}$. For this case, the wave function of the system is

$$|\Psi(t)\rangle = \sum_{m=0}^{\infty} \exp[-in\omega t]\exp\Big[-\frac{1}{2}|\beta(t)|^2\Big]\exp\Big[i\frac{\omega}{2}\int_0^t |\beta(\tau)|^2\mathrm{d}\tau\Big]$$

$$\times\sqrt{\frac{n!}{m!}}[\beta(t)]^{(m-n)}L_n^{m-n}(|\beta(t)|^2)|m\rangle\ , \tag{41}$$

where

$$\beta(t) = -i\Big[\Omega\frac{\sin(\omega t/2)}{\omega/2}\Big]\exp[-i\omega t/2]\ . \tag{42}$$

(3) $|\Psi(t=0)\rangle = |0\rangle$, and $\omega = \text{const.}$, $\Omega^* = \Omega = \text{const.}$. In this case, the time evolution of the system is determined by

$$|\Psi(t)\rangle = \sum_{m=0}^{\infty} \frac{[\beta(t)]^m}{\sqrt{m!}} \exp\Big[i\frac{\omega}{2}\int_0^t |\beta(\tau)|^2 \mathrm{d}\tau\Big] \exp\Big[-\frac{1}{2}|\beta(t)|^2\Big]|m\rangle \ . \tag{43}$$

As has been pointed out that by virtue of algebraic dynamic, one may readily choose a proper gauge at his disposal to study a linear quantum system. To illustrate this statement, now we consider the second gauge. The second choice of the gauge transformation is such that

$$v_1(t) = 0 \ , \tag{44a}$$

$$\dot{v}_2(t) + i\omega(t)v_2(t) + i\Omega^*(t) = 0 \ , \tag{44b}$$

$$\dot{v}_3(t) + i\omega(t) = 0 \ , \tag{44c}$$

$$\dot{v}_4(t) + i\Omega(t)v_2(t) = 0 \ . \tag{44d}$$

Assume $\hat{U}_g(t=0) = 1$, one has

$$v_2(t=0) = 0 \ , \quad v_3(t=0) = 0 \ , \quad v_4(t=0) = 0 \ . \tag{45}$$

From (44a-d), we get

$$v_2(t) = \frac{\gamma(t)\chi(t)}{\Omega(t)} \ , \tag{46a}$$

$$v_3(t) = -i\int_0^t \omega(\tau)\mathrm{d}\tau \ , \tag{46b}$$

$$v_4(t) = -i\int_0^t \gamma(\tau)\chi(\tau)\mathrm{d}\tau \ , \tag{46c}$$

where γ and χ are defined by (16a, b). (46a-c) are the integral expressions of $v_i(t)$ under the second chosen gauge. Once the coefficients $\omega(t)$ and $\Omega(t)$ are specified, the transformation parameters $v_i(t)$ can be readily calculated from (46a-c). Under the second gauge, the gauge transformation $\hat{U}_g(t)$ becomes

$$\hat{U}_g(t) = \exp\Big[\frac{\gamma(t)\chi(t)}{\Omega(t)}\hat{a}^\dagger\Big] \exp\Big[-i\int_0^t \omega(\tau)\mathrm{d}\tau \hat{N}\Big] \exp\Big[-i\int_0^t \gamma(\tau)\chi(\tau)\mathrm{d}\tau\Big] \ . \tag{47}$$

Under this gauge, the gauged Hamiltonian $\hat{\overline{H}}(t)$ is

$$\hat{\overline{H}}(t) = f(t)\hat{\overline{I}}(0) \ , \tag{48}$$

where

$$f(t) = \Omega(t)\exp\Big[-i\int_0^t \omega(\tau)\mathrm{d}\tau\Big] = \gamma(t) \ , \tag{49a}$$

$$\hat{\overline{I}}(0) = \hat{a} \; . \tag{49b}$$

It is evident that under the second gauge choice, the Cartan operator $\hat{\overline{I}}$ does not dependent on time explicitly, and is just the particle annihilation operator $\hat{a}$. Therefore, this gauge corresponds to coherent state representation.

It should be noted that the gauge transformation $\hat{U}_g(t)$ corresponding to the second gauge, (47), is not unitarity. As a result, the gauged Hamiltonian $\hat{\overline{H}}(t)$, (48), is not Hermitian. However, the gauge transformation only serves as a method to study the system conveniently. The physical solution of the Schrödinger equation (6) is naturally irrelevant to the choice of gauge transformation.

The invariant Cartan operator is

$$\hat{I}(t) = \hat{U}_g \hat{\overline{I}}(0) \hat{U}_g^{-1} = -\frac{1}{\chi^*(t)}[\alpha(t)\hat{a} + \delta(t)] \; , \tag{50}$$

where the coefficients α and δ are defined by

$$\alpha(t) = -\frac{\gamma^*(t)\chi^*(t)}{\Omega^*(t)} \; , \qquad \delta(t) = |\chi(t)|^2 \; . \tag{51}$$

(50) shows that the varying of $\hat{I}(t)$ is constrained in the subalgebra h(3) and the subalgebra $\hat{N}$ of hw(4) is frozen. This again leads to a direct quantum-classical correspondence. It is proved that $\hat{I}(t)$ is an invariant of the system. $\alpha(t)$ satisfies the following dynamical equation

$$\dot{\alpha}(t) = i\omega(t)\alpha(t) - i\Omega(t) \; , \tag{52}$$

(52) can also be obtained from the equation of motion for the invariant operator $\hat{I}(t)$, i.e., $\frac{\partial \hat{I}(t)}{\partial t} + i[\hat{H}(t), \hat{I}(t)] = 0$. Comparing (52) with (28b), one gets the quantum-classical correspondence which is similar to (29a)

$$\alpha = -a^*, \quad \alpha^* = -a \; . \tag{53}$$

Now, we proceed to give the exact solution of the system studied in coherent state representation. Let $|\lambda\rangle$ be the eigen state of $\hat{\overline{I}}(0) = \hat{a}$, i.e.

$$\hat{\overline{I}}(0)|\lambda\rangle = \lambda|\lambda\rangle \; , \tag{54a}$$

where λ is an eigenvlaue. The eigen state $|\lambda\rangle$ of $\hat{\overline{I}}(0)$ is a coherent state which can be expressed as

$$|\lambda\rangle = \exp\left[\lambda\hat{a}^\dagger - \lambda^*\hat{a}\right]|0\rangle = N_\lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}}|n\rangle = N_\lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}(\hat{a}^\dagger)^n|0\rangle \; , \tag{54b}$$

where $N_\lambda = \exp[-|\lambda|^2/2]$. The eigen problem of the dynamical Cartan operator $\hat{I}(t)$ is

$$\hat{I}(t)|\phi_\lambda(t)\rangle = \lambda|\phi_\lambda(t)\rangle \; , \tag{55}$$

where $|\phi_\lambda(t)\rangle$ is the eigen state of $\hat{I}(t)$.

$$|\phi_\lambda(t)\rangle = \hat{U}_g|\lambda\rangle = N_\lambda \exp\Big[-i\int_0^t \chi(\tau)\gamma(\tau)\mathrm{d}\tau\Big]\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\exp\Big[-in\int_0^t \omega(\tau)\mathrm{d}\tau\Big]$$

$$\times\Big[\frac{\gamma(t)\chi(t)}{\Omega(t)}\Big]^m \frac{1}{m!}\frac{\lambda^n}{\sqrt{n!}}\sqrt{\frac{(n+m)!}{n!}}|n+m\rangle\ , \tag{56}$$

λ is the corresponding eigen value. The eigen state $|\phi_\lambda(t)\rangle$ can also be written as a more compact form

$$|\phi_\lambda(t)\rangle = \exp\Big\{\frac{1}{2}\Big[|\Lambda(t)|^2 - |\lambda|^2\Big]\Big\}\exp\Big\{-i\int_0^t \gamma(\tau)\chi(\tau)\mathrm{d}\tau\Big\}|\Lambda(t)\rangle\ . \tag{57}$$

In (57), $|\Lambda(t)\rangle$ is the eigen state of $\hat{a}$ with eigen value $\Lambda(t)$, i.e.,

$$|\Lambda(t)\rangle = \exp\Big[\Lambda(t)\hat{a}^\dagger - \Lambda^*(t)\hat{a}\Big]|0\rangle = \exp\Big[-\frac{1}{2}|\Lambda(t)|^2\Big]\sum_{n=0}^{\infty}\frac{[\Lambda(t)]^n}{\sqrt{n!}}|n\rangle\ , \tag{58}$$

where

$$\Lambda(t) = v_2(t) + \lambda\exp[v_3(t)] = \frac{\gamma(t)}{\Omega(t)}[\lambda + \chi(t)]\ . \tag{59}$$

Under the second gauge choice (44a-d), the gauged Schrödinger equation is

$$i\frac{\partial}{\partial t}|\overline{\Psi}(t)\rangle = f(t)\hat{\overline{I}}(0)|\overline{\Psi}(t)\rangle\ , \tag{60}$$

which has the following solution

$$|\overline{\Psi}_\lambda(t)\rangle = \exp[i\Theta_\lambda(t)]|\lambda\rangle\ , \tag{61}$$

where

$$i\Theta_\lambda(t) = -i\lambda\int_0^t \gamma(\tau)\mathrm{d}\tau = -\lambda\chi^*(t)\ . \tag{62}$$

Under the second gauge, the diabatic basis is

$$|\Psi_\lambda(t)\rangle = \hat{U}_g|\overline{\Psi}_\lambda(t)\rangle = \exp[i\Theta_\lambda(t)]|\phi_\lambda(t)\rangle\ . \tag{63}$$

Let the system is initially at its coherent state $|\lambda\rangle$. At time t, the state of the system becomes

$$|\Psi(t)\rangle = |\Psi_\lambda(t)\rangle = \exp[i\Theta_\lambda(t)]|\phi_\lambda(t)\rangle = \exp[i\theta_\lambda(t)]|\Lambda(t)\rangle\ , \tag{64}$$

where $|\Lambda(t)\rangle$ and $\Lambda(t)$ are defined respectively by (58) and (59). $\theta_\lambda(t)$ is a real function of time, which is defined as

$$\theta_\lambda(t) = -\Big\{\mathrm{Im}\Big[\lambda\chi^*(t)\Big] + \mathrm{Re}\int_0^t \gamma(\tau)\chi(\tau)\mathrm{d}\tau\Big\}\ . \tag{65}$$

It is easy to check that $|\Psi(t)\rangle$ is the solution of the Schrödinger equation with the initial condition $|\Psi(t=0)\rangle = |\lambda\rangle$. (64) shows that for the system studied, if the initial state is a coherent state, the time evolution of the system conserves its coherence within a phase factor. Before concluding the paper, we give some expectation value of quantum quantities.

(1) The wave function $|\Psi(t)\rangle$ is normalized, i.e.,

$$\langle\Psi(t)|\Psi(t)\rangle = 1 \ . \tag{66}$$

(2) The expectation value of $\hat{a}$

$$\langle\hat{a}\rangle = \Lambda(t) \ , \quad \langle\hat{a}^2\rangle = \Lambda^2(t) \ . \tag{67}$$

(3) The expectation value of $\hat{a}^\dagger$

$$\langle\hat{a}^\dagger\rangle = \Lambda^*(t) \ , \quad \langle\hat{a}^{\dagger 2}\rangle = \Lambda^{*2}(t) \ . \tag{68}$$

(3) The energy of the system is

$$\begin{aligned} E(t) = \langle\hat{H}(t)\rangle &= \omega(t)|\Lambda(t)|^2 + \Omega^*(t)\Lambda^*(t) + \Omega(t)\Lambda(t) \\ &= \omega(t)|\lambda + \chi(t)|^2 + 2\mathrm{Re}\{\gamma(t)[\lambda + \chi(t)]\} \ . \end{aligned} \tag{69}$$

(4) Define coordinate and momentum operators as follows

$$\hat{q} = \frac{1}{\sqrt{2}}(\hat{a}^\dagger + \hat{a}), \quad \hat{p} = \frac{i}{\sqrt{2}}(\hat{a}^\dagger - \hat{a}), \tag{70}$$

then one has

$$\langle\hat{q}\rangle = \sqrt{2}\mathrm{Re}[\Lambda(t)], \quad \langle\hat{p}\rangle = \sqrt{2}\mathrm{Im}[\lambda(t)], \tag{71a}$$

$$\langle\hat{q}^2\rangle = 2\Big\{\mathrm{Re}[\Lambda(t)]\Big\}^2 + \frac{1}{2} \ , \quad \langle\hat{p}^2\rangle = 2\Big\{\mathrm{Im}[\Lambda(t)]\Big\}^2 + \frac{1}{2} \ . \tag{71b}$$

The uncertainty relation is

$$\Delta p(t)\Delta q(t) = \Delta p(0)\Delta q(0) = \frac{1}{2} \tag{72}$$

Because $|\Psi(t)\rangle$ is a coherent state, it conserves the minimum uncertainty of the system.

Up to now, we have obtained the exact solution of the linear nonautonomous system with hw(4) dynamical algebra under two different gauges (or two different representations). The new results in the present paper are: 1) It has generalized the method of algebra dynamics from the linear dynamical system with a semi-simple Lie algebra to that with a general Lie algebra. 2) It has established the rule or condition for a direct quantum-classical correspondence in general Lie algebraic dynamical systems: the equation of quantum motion is only related to the structure constant matrices which are Hermitian; 3) Using the gauge transformation approach, the exact solution in coherent state representation is also obtained besides that in harmonic oscillator representation; 4) The invariant Cantan operator and the quantum-classical correspondence in both representations are obtained.

Acknowledgement

This work was supported by the National Science Foundation and the Doctoral Education Fund of the State Education Commission of China.

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