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Some exact results on the CGHS black-hole radiation*

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Abstract. Theorems on the emission of massless scalar particles by the CGHS black hole are presented. The convergence of the mean number of particles created spontaneously in an arbitrary state is studied and shown to be strongly dependent on the infrared behavior of this state. A bound for this quantity is given and its asymptotic forms close to the horizon and far from the black hole are investigated. The physics of a wave packet is analysed in some detail in the black-hole background. It is also shown that for some states the mean number of created particles is *not* thermal close to the horizon. These states have a long queue extending far from the black hole, or are unlocalised in configuration space.

1 Introduction

The quantum physics of black holes has been a field of extensive research since Hawking discovered that, due to quantum mechanical effects, black holes emit spontaneously particles with a thermal spectrum [1]. In order to understand better the physical outcomes of this

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discovery, as for example the evaporation and entropy of black holes, two-dimensional black-hole models have been studied for simplicity in the literature. One of these is the CGHS black hole [2] which is again considered here.

The present paper is mainly concerned by the study, in the CGHS black-hole background, of the mean number $\bar{N}[f]$ of massless scalar particles created spontaneously in an *arbitrary* state f. The investigation of this issue has first been done by Wanders [3] for the Dirac massless field. In that case, this quantity is reinterpreted as the probability $W[f]$ of detecting a fermion in the state f. Wanders showed that this probability tends to the thermal probability $W_\beta^{Th}[f]$ when the state f is translated towards the event-horizon, where β is the inverse temperature of the black hole. He obtained furthermore a bound for the difference $\mid W[f] - W_\beta^{Th}[f] \mid$, which exhibits a strong dependence on the queue of f extending far away from the horizon.

The massless scalar field will be studied here along similar lines. In this case, however, infrared issues are of primordial importance, so I shall also concentrated on them. Although the Wightman function is in general not positive definite in two-dimensional spacetimes because of its bad IR behaviour [4], the massless scalar field may still be considered if the set of states is reduced in an appropriate way [5]. In this framework, one of the relevant problem is then the influence of the infrared behavior of the state f on the mean number $\bar{N}[f]$ and on the difference $\mid \bar{N}[f] - \bar{N}_\beta^{Th}[f] \mid$, where $\bar{N}_\beta^{Th}[f]$ is the average number of particles in the state f for an outgoing thermal flux of radiation of temperature β^{-1}. It is shown in this paper that these quantities may diverge or not depending on the IR properties of the considered state. This imply in particular that *there exist states for which the mean number of created particles is not thermal close to the horizon.* The IR properties of f are related to the properties of its queue in configuration space. As in the fermionic case, a bound is obtained for the difference $\mid \bar{N}[f] - \bar{N}_\beta^{Th}[f] \mid$, which depends strongly on the queue of f extending far away from the black hole.

The queues of states in configuration space play thus a relatively important role for the black-hole physics. When one restricts oneself to positive momentum modes only, states cannot be well localised. There is a theorem of Paley and Wiener (see appendix A.1) which asserts that if the Fourier transform of a function of one variable vanishes for all negative values of its argument, then it does not decrease at infinity faster that an exponential function. One is thus led to study states whose wave function decreases at infinity in an algebraic way or which are unlocalised. One may expect that the global properties of these states come into play when the physics of the black hole is analysed close to its event-horizon. This issue is considered in the present paper.

The physics of a wave packet in the CGHS black-hole background is also studied. This wave packet depends on a parameter δ, and its Fourier transform becomes narrower in momentum space when δ vanishes. In this limit, the variance of the momentum operator vanishes in this state and the wave packet is completely delocalised. In consequence, it is not justified to make the approximation of the horizon to calculate the mean number $\bar{N}[f]$, as it is usually done, and an exact calculation must be performed. This is done here for the first time and leads to unexpected results. In particular, the mean number of particles created by the black hole in this delocalised state is equal to the *half* of the thermal average $\bar{N}_\beta^{Th}[f]$, and is invariant under a translation of the state f.

Section 2 is devoted to a review of the CGHS black hole and of the quantum field formalism [6]. In section 3, some conditions under which the mean number $\bar{N}[f]$ of created particles diverges are investigated and a bound for this quantity is given. The asymptotic behaviors of the mean number $\bar{N}[f]$ close to the horizon and far from the black hole are given in section 4. These last results are applied to the physics of a wave packet in the black-hole background in section 5.

2 Quantum field theory in the black-hole background

2.1 The CGHS black hole

The CGHS black hole [2] is a vacuum solution of the dilatonic gravity theory defined by the action

$$S = \frac{1}{2\pi}\int d^2x\,\sqrt{-g}\,\left\{e^{-2\phi}\left[R+4(\nabla\phi)^2+4\,\lambda^2\right]-\frac{1}{2}(\nabla f)^2\right\}, \tag{2.1}$$

where g is the metric, ϕ the dilatonic field, f a classical massless matter field and λ^2 the cosmological constant. This black hole may be created from a shock wave of f-matter, whose only non-vanishing energy-momentum tensor $T^f_{\mu\nu}(x)$ component is

$$T^f_{++}(x) = \tfrac{1}{2}(\partial_+ f)^2 = M\,\delta(x^+), \tag{2.2}$$

where $M > 0$. For simplicity, one assumes that $\lambda = M = 1$ without loss of generality. If the line element is Minkowskian for $x^+ \leq 0$, then from the equations of motion one gets for $x^+ \geq 0$

$$ds^2 = \frac{dx^+dx^-}{1+e^{x^-}\left(e^{-x^+}-1\right)}. \tag{2.3}$$

The x coordinates are the incoming coordinates, the outgoing coordinates $(y^+,y^-)\in \mathbb{R}^2$ are defined by the transformation

$$\begin{cases} x^+(y^+) = y^+, \\ x^-(y^-) = -\log\left(1+e^{-y^-}\right). \end{cases} \tag{2.4}$$

These coordinates parametrise only the lower half-plane $x^- < 0$ of spacetime where the line element (2.3) is given by

$$ds^2 = \frac{dy^+dy^-}{1+e^{y^- - y^+}}, \tag{2.5}$$

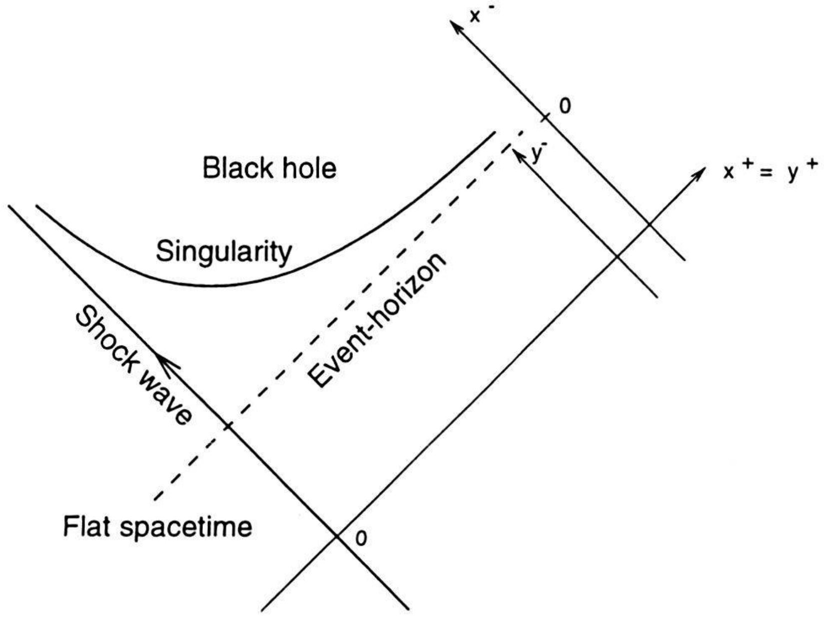

Figure 1: Spacetime diagram of the CGHS black hole in x coordinates. The event-horizon is located at $x^- = 0$ or $y^- = +\infty$.

if $x^+ \geq 0$. This tends to the Minkowski metric in the limit $y^+ \to +\infty$. The spacetime diagram in shown in fig. 1.

Equation (2.3) implies that the scalar curvature is singular on the curve $x^- = x_S^-(x^+)$ where [7]

$$x_S^-(x^+) \quad = \quad -\log\left(1 - e^{-x^+}\right). \tag{2.6}$$

The signature of the line element (2.3) is reversed there, i.e. the conformal factor is only positive if $x^- < x_S^-(x^+)$. This line element may be rewritten in terms of the function x_S^-:

$$ds^2 \quad = \quad \frac{dx^+\,dx^-}{1 - e^{x^- - x_S^-(x^+)}}. \tag{2.7}$$

In the Minkowskian region close to the singularity, i.e. in the limits $x^- \approx x_S^-$ and $x^+ \gg 1$, eq. (2.7) tends to [6, 8]

$$ds^2 \quad = \quad -\frac{dx^+\,dx^-}{x^-}, \tag{2.8}$$

since $x_S^-(x^+) \approx 0$ if $x^+ \gg 1$. For this new line element, the transformation $x = x(y)$ is redefined by

$$\left\{\begin{array}{lcl} x^+(y^+) & = & y^+, \\ x^-(y^-) & = & -e^{-y^-}, \end{array}\right. \tag{2.9}$$

and this line element is Minkowskian in these new y coordinates. In the limit $x^- \approx 0$, the two transformations (2.4) and (2.9) coincide.

2.2 Quantum field theory

2.2.1 Fields and test functions

In a two-dimensional spacetime, the line element can always be written in a conformal form in an appropriate set of coordinates, at least locally. In these coordinates, the left and right moving modes of the massless scalar field decouple. If the transformation relating the incoming and outgoing coordinates, denoted by x and y respectively, takes the form $x^{\pm} = x^{\pm}(y^{\pm})$, these modes do not mix up when the change of coordinates is made, i.e. left (right) moving modes in incoming coordinates are still left (right) moving modes in outgoing coordinates. The physics of the left moving modes is then trivial in the CGHS black-hole background (see eqs (2.4)), and so we will concentrate from now on only on the right moving modes, and the subscripts $\pm$ will be dropped.

The incoming and outgoing field distributions, denoted by ϕ and $\hat{\phi}$, are related through the equation $\phi[\hat{f}] = \hat{\phi}[f]$ [6]. The incoming test function $\hat{f}(x)$ is given in terms of the outgoing test function $f(y)$ by

$$f(y) \;=\; x'(y)\,\hat{f}(x(y)), \qquad \forall\, y \in \mathbb{R}, \tag{2.10}$$

or by $\tilde{\hat{f}} = U\tilde{f}$, where the kernel of the operator U is defined by

$$U(k,p) \;=\; \frac{1}{2\pi}\int_{-\infty}^{+\infty} dy\; e^{-ikx(y)}\, e^{ipy}. \tag{2.11}$$

The incoming and outgoing momenta are denoted by k and p respectively. The outgoing wave function space is $L^2(\frac{dp}{2p}, \mathbb{R}_+)$, and is the completion of the set[‡]

$$\mathcal{S}(\mathbb{R}_+) \;=\; \{\, \tilde{f} \in \mathcal{S}(\mathbb{R}) \mid \tilde{f}(p) = \theta(p)\tilde{f}(p),\; \forall\, p \in \mathbb{R}\,\}, \tag{2.12}$$

where $\mathcal{S}(\mathbb{R})$ is the Schwartz space.

In the CGHS black-hole background, one has from eqs (2.4) and (2.11),

$$U(k,p) \;=\; \frac{1}{2\pi}\, B(\mathrm{i}p - \mathrm{i}k + 0^+, -\mathrm{i}p + 0^+). \tag{2.13}$$

Here B is the beta function defined by [9]

$$B(q,r) \;=\; B(r,q) \;=\; \frac{\Gamma(q)\,\Gamma(r)}{\Gamma(q+r)} \;=\; \int_{-\infty}^{+\infty} dy\, \frac{e^{qy}}{(1+e^{y})^{q+r}}, \tag{2.14}$$

where Γ is the gamma function satisfying

$$|\,\Gamma(\mathrm{i}p)\,|^2 \;=\; \frac{\pi}{p\,\sinh(\pi p)}. \tag{2.15}$$

[‡] θ denotes the step function defined by $\theta(p) = 0$ if $p < 0$ and $\theta(p) = 1$ otherwise.

2.2.2 Local observables

The two-point function for the incoming vacuum is given in y coordinates by [6]

$$\widehat{W}(y,y') = -\frac{1}{4\pi}\log\left[x(y') - x(y) + \mathrm{i}\,0^+\right]. \tag{2.16}$$

In the CGHS black-hole background, it is periodic in the imaginary direction for all $y, y' \in \mathbb{R}$,

$$\widehat{W}(y,y') = \widehat{W}(y, y' + \mathrm{i}\,2\pi n), \qquad \forall n \in \mathbb{Z}. \tag{2.17}$$

Since the thermal two-point function is given by [6]

$$W_\beta^{Th}(y,y') = -\frac{1}{4\pi}\log\left\{\frac{\beta}{\pi}\sinh\left[\frac{\pi}{\beta}\left(y' - y + \mathrm{i}\,0^+\right)\right]\right\}, \tag{2.18}$$

one obtains in this case

$$\widehat{W}(y,y') \approx W_{2\pi}^{Th}(y,y'), \qquad \textit{when } y, y' \gg 1, \tag{2.19}$$

$$\widehat{W}(y,y') \approx W_{\infty}^{Th}(y,y'), \qquad \textit{when } -y, -y' \gg 1, \tag{2.20}$$

if two-point functions are considered as kernel of distributions on $\mathcal{S}(\mathbb{R}_+) \times \mathcal{S}(\mathbb{R}_+)$. This means that, in outgoing coordinates, the incoming vacuum is a thermal state of temperature $(2\pi)^{-1}$ close to the horizon, and of temperature zero far from the horizon. Since the energy-momentum tensor $\widehat{T}(y)$ in y coordinates may be obtained from the two-point function $\widehat{W}(y,y')$, these results imply that $\widehat{T}(y)$ is also thermal close the horizon and far from the CGHS black hole,

$$\lim_{y\to+\infty}\widehat{T}(y) = T_{2\pi}^{Th}, \qquad \lim_{y\to-\infty}\widehat{T}(y) = T_{\infty}^{Th}, \tag{2.21}$$

where the thermal energy-momentum tensor of temperature β^{-1} is given by

$$T_\beta^{Th} = \frac{\pi}{12\beta^2}. \tag{2.22}$$

2.2.3 Mean number of created particles and implementability

The mean number of particles created spontaneously in a normalised state $\tilde{f} \in L^2(\frac{dp}{2p}, \mathbb{R}_+)$ is given by [6]

$$\bar{N}[f] = \int_0^\infty \frac{dk}{2k}\,|\hat{\tilde{f}}(-k)|^2\,. \tag{2.23}$$

In the CGHS black-hole background one has from eq. (2.15),

$$|U(-k,p)|^2 = \frac{1}{4\pi}\frac{k}{p\,(p+k)}\frac{\sinh\pi k}{\sinh\pi p\,\sinh\pi(p+k)} \tag{2.24}$$

$$\approx \frac{1}{2\pi p}\left[\frac{\theta(k)}{e^{2\pi p}-1} + \frac{\theta(-k)}{1-e^{-2\pi p}}\right], \qquad \textit{if } |k| \gg |p| + 1. \tag{2.25}$$

The mean number of particles created in a given mode f_p, defined by $\tilde{f}_p(p') = 2p'\delta(p-p')$ where $p > 0$, is UV divergent in k,

$$\bar{N}[f_p] = \infty. \tag{2.26}$$

The total mean number of created particles is clearly also infinite, with an additional IR divergence in p. The incoming and outgoing vacuums can thus not be related by an unitary transformation, i.e. the problem is not implementable.

2.2.4 The thermal case

In ref. [6], it is shown that the theory of a massless scalar field interacting with an outgoing thermal flux of radiation is equivalent to the theory of this field in the spacetime background defined by eq. (2.8), in the sense that expectation values of observables in the incoming vacuum are equal to their thermal averages in the outgoing Hilbert space. In particular, the incoming-vacuum two-point function and energy-momentum tensor are given everywhere in this spacetime by eqs (2.18) and (2.22) with $\beta = 2\pi$, i.e. they coincide with the thermal averages. Similarly, the mean number of particles spontaneously created in this spacetime and in the normalised state $\tilde{f} \in L^2(\frac{dp}{2p}, \mathbb{R}_+)$ is given by $\bar{N}[f] = \bar{N}^{Th}_\beta[f]$ with $\beta = 2\pi$, where $\bar{N}^{Th}_\beta[f]$ is the thermal average given by

$$\bar{N}^{Th}_\beta[f] = \int_0^\infty \frac{dp}{2p} \frac{|\tilde{f}(p)|^2}{e^{\beta p} - 1}. \tag{2.27}$$

In the spacetime defined by eq. (2.8), the incoming test function will thus be denoted by $\hat{f}^{Th}(x)$, which is defined by eq. (2.10), where the transformation $x(y)$ is given by eq. (2.9). This spacetime is the dynamical counterpart of the $\eta - \xi$ spacetime of Gui [10], which corresponds to the thermal *equilibrium* case. Since the CGHS line element (2.7) coincides with the line element (2.8) in the region defined by $x^- \approx 0$ and $x^+ \gg 1$, the outgoing radiation emitted by the CGHS black hole is thermal in that region.

3 The mean number of created particles

3.1 Convergence of the mean number $\bar{N}[f]$

A quick look at eq. (2.23) shows that the mean number $\bar{N}[f]$ of particles created by the black hole may diverge or not depending on the infrared and ultraviolet behaviors of the incoming test function $\hat{\tilde{f}}(k)$. These behaviors are related to the properties of the outgoing test function $f(y)$ through eq. (2.10). It is therefore interesting to study the influence of $f(y)$ on the convergence of the mean number $\bar{N}[f]$ in the CGHS black-hole background, and in the spacetime defined by eq. (2.8) as well, in order to understand the physics of the black hole close to the horizon. Of particular interest are the infrared behavior of $\tilde{f}(p)$ and the decreasing properties of $f(y)$ at infinity. Their influence on the function $\tilde{\hat{f}}(k)$ is explored in the two following lemmas.

Lemma 1 *Let $\tilde{\chi} \in C^\infty(\mathrm{I\!R})$ be a function satisfying $\tilde{\chi}(0) \neq 0$, vanishing at infinity and such that all its derivatives vanish at the origin and at infinity. If one defines the function f_α by $\tilde{f}_\alpha(p) \equiv \theta(p)\, p^\alpha\, \tilde{\chi}(p)$, $\forall\, p \in \mathrm{I\!R}$, where $\alpha > 0$, then one has in the CGHS black-hole background*

$$|\,\tilde{\hat{f}}_\alpha(-k)\,| \;\leq\; C_\alpha\, k^{q(\alpha)} + \mathcal{O}(k), \qquad \textit{if}\;\; k \approx 0^+, \tag{3.1}$$

$$\tilde{\hat{f}}_\alpha(-k) \;=\; \frac{C_{\alpha,-}}{\sqrt{2\pi}} \frac{B(k)}{\alpha\,(\log k)^\alpha} + \mathcal{O}\left[\frac{1}{(\log k)^{1+\alpha}}\right], \qquad \textit{if}\;\; k \gg 1, \tag{3.2}$$

and in the spacetime defined by eq. (2.8)

$$\tilde{\hat{f}}{}^{Th}_\alpha(-k) \;=\; \begin{cases} \dfrac{C_{\alpha,+}}{\sqrt{2\pi}} \dfrac{A(k)}{\alpha\,(-\log k)^\alpha} + \mathcal{O}\left[\dfrac{1}{(-\log k)^{1+\alpha}}\right], & \textit{if}\;\; k \approx 0^+, \quad (3.3)\\[2ex] \dfrac{C_{\alpha,-}}{\sqrt{2\pi}} \dfrac{B(k)}{\alpha\,(\log k)^\alpha} + \mathcal{O}\left[\dfrac{1}{(\log k)^{1+\alpha}}\right], & \textit{if}\;\; k \gg 1, \quad (3.4)\end{cases}$$

where the functions A and B are both bounded by above and below, C_α and $C_{\alpha,\pm}$ are three constants depending on α, and where the function q is defined by

$$q(\alpha) \;=\; \begin{cases} \alpha, & \textit{if}\;\; 0 < \alpha \leq 1/2, \\ 1/2, & \textit{if}\;\; 1/2 < \alpha. \end{cases} \tag{3.5}$$

Proof See appendix A.2.

This first lemma shows in a generic example that both the IR and UV behaviors of $\tilde{\hat{f}}(-k)$ are determined by the IR properties of $\tilde{f}(p)$. The smoothness assumption on $\tilde{\chi}$ is necessary in order that the decreasing property of $f_\alpha(y)$ at infinity is well determined for in that case one has [11]

$$f_\alpha(y) \;=\; \begin{cases} \dfrac{C_{\alpha,-}}{y^{1+\alpha}} + \mathcal{O}\left(\dfrac{1}{y^{2+\alpha}}\right), & \textit{if}\;\; -y \gg 1, \quad (3.6)\\[2ex] \dfrac{C_{\alpha,+}}{y^{1+\alpha}} + \mathcal{O}\left(\dfrac{1}{y^{2+\alpha}}\right), & \textit{if}\;\; y \gg 1, \quad (3.7)\end{cases}$$

where $C_{\alpha,\pm}$ are the same constants as in lemma 1 (if $\tilde{\chi} \notin C^\infty(\mathrm{I\!R})$, the discontinuity of one of the derivatives of $\tilde{f}_\alpha(p)$ may imply that $f(y)$ decreases more slowly). Equations (3.6) and (3.7) determine then the behavior of $\hat{f}_\alpha(x)$ for $-x \gg 1$ and $x \approx 0^-$ respectively through eq. (2.10), on which the properties of $\tilde{\hat{f}}_\alpha(-k)$ depend. It can actually be shown, in a more general context, that the IR behavior of $\tilde{\hat{f}}(-k)$ depends solely on the behavior of $f(y)$ far from the black hole, and that the UV behavior of $\tilde{\hat{f}}(-k)$ depends solely on the behavior of $f(y)$ close to the horizon. This is done in the following lemma which gives bounds instead of asymptotic behaviors for weaker assumptions, in particular it is assumed that the *modulus* of f decreases at infinity faster than the inverse of an algebraic function. From theorem A.1, if f is squared integrable, it is always possible to choose the phase of f in such a way that its Fourier transform $\tilde{f}$ vanishes for all negative values of its argument.

Lemma 2 *If f is an integrable function satisfying $\tilde{f}(0) = 0$, whose derivative f' exists and is integrable, and such that its modulus $|f|$ satisfies*

$$|f(y)| \le \begin{cases} \dfrac{C_-}{(-y)^{1+\varepsilon}}, & \text{if } y \le -L, \qquad (3.8) \\[2ex] \dfrac{C_+}{y^{1+\alpha}}, & \text{if } y \ge L, \qquad (3.9) \end{cases}$$

where $C_\pm > 0$, $L > 1$, $\alpha > 0$ and $\varepsilon > 0$ are five constants, then

$$|\tilde{\hat{f}}(-k)| \le \begin{cases} C_\varepsilon \, k^{q(\varepsilon)} + \mathcal{O}(k), & \text{if } k \approx 0^+, \qquad (3.10) \\[2ex] \dfrac{2^{1+\alpha} C_+}{\alpha \, (\log k)^\alpha} + \mathcal{O}\left(\dfrac{1}{\sqrt{k}}\right), & \text{if } k \gg 1, \qquad (3.11) \end{cases}$$

and

$$|\tilde{\hat{f}}^{Th}(-k)| \le \begin{cases} \dfrac{3\, C_-}{\varepsilon \, (-\log k)^\varepsilon} + \mathcal{O}(k), & \text{if } k \approx 0^+, \qquad (3.12) \\[2ex] \dfrac{2^\alpha C_+}{\alpha \, (\log k)^\alpha} + \mathcal{O}\left(\dfrac{1}{\sqrt{k}}\right), & \text{if } k \gg 1, \qquad (3.13) \end{cases}$$

where the function q is defined by eq. (3.5), and where $C_\varepsilon > 0$ is a constant which diverges when $\varepsilon \to 0^+$.

Proof See appendix A.3.

Under the assumptions of this lemma, the IR and UV behaviors of $\tilde{\hat{f}}^{Th}(-k)$ and the UV behavior of $\tilde{\hat{f}}(-k)$ are thus at least inversely proportional to a power of the logarithm of k, and $\tilde{\hat{f}}(-k)$ decreases at least in an algebraic way in the IR region. Lemmas 1 and 2 imply from eq. (2.23) that the mean number $\bar{N}[f]$ of created particles is very sensitive to the IR properties of $\tilde{f}(p)$ or to the asymptotic behavior of $f(y)$ at infinity. This is highlight in the two following theorems.

Theorem 1 *Let $\tilde{\chi} \in C^{\infty}(\mathbb{R})$ be a smooth function satisfying $\tilde{\chi}(0) \neq 0$, vanishing at infinity and such that all its derivatives vanish at the origin and at infinity. If $\tilde{f}_\alpha(p) \equiv \theta(p)\, p^\alpha\, \tilde{\chi}(p)$ $(\forall\, p \in \mathbb{R})$ is a normalised wave function where $\alpha > 0$, then one has the equivalences*

$$\bar{N}[\,f_\alpha\,] < \infty \iff \alpha > 1/2, \tag{3.14}$$

$$\bar{N}^{Th}_{2\pi}\,[\,f_\alpha\,] < \infty \iff \alpha > 1/2, \tag{3.15}$$

$$\Big|\bar{N}[\,f_\alpha\,] - \bar{N}^{Th}_{2\pi}\,[\,f_\alpha\,]\Big| < \infty \iff \alpha > 1/2. \tag{3.16}$$

If $\alpha \leq 1/2$, $\bar{N}[\,f_\alpha\,]$ is only UV divergent in the incoming momenta k, whereas $\bar{N}^{Th}_{2\pi}\,[\,f_\alpha\,]$ is both IR and UV divergent in k, and $\Big|\bar{N}[\,f_\alpha\,] - \bar{N}^{Th}_{2\pi}\,[\,f_\alpha\,]\Big|$ is only IR divergent in k.

Proof This theorem follows from lemma 1 and eq. (2.23). For example one has

$$\Big|\tilde{\bar{f}}_\alpha(-k)\Big|^2 - \Big|\tilde{\bar{f}}^{Th}_\alpha(-k)\Big|^2 = \begin{cases} \dfrac{C^2_{\alpha,-}\, A(k)^2}{\alpha^2\,(\log k)^{2\alpha}} + \mathcal{O}\left[\dfrac{1}{(\log k)^{1+2\alpha}}\right], & \text{if } k \approx 0^+, \\ \mathcal{O}\left[\dfrac{1}{(\log k)^{1+2\alpha}}\right], & \text{if } k \gg 1, \end{cases} \tag{3.17}$$

which implies that the difference $|\,\bar{N}[\,f_\alpha\,] - \bar{N}^{Th}_{2\pi}\,[\,f_\alpha\,]\,|$ is IR convergent in k if and only if $\alpha > 1/2$, and that it is UV convergent in k. □

Theorem 2 *Let $\tilde{f} \in L^2(\frac{dp}{2p}, \mathbb{R}_+)$ be a normalised wave function such that f and f' exist and are integrable, and such that its modulus $|f|$ satisfies*

$$|\,f(y)\,| \;\leq\; \begin{cases} \dfrac{C}{(-y)^{1+\varepsilon}}, & \text{if } y \leq -L, \\ \dfrac{C}{y^{1+\alpha}}, & \text{if } y \geq L, \end{cases} \tag{3.18}$$

where $C > 0$, $L > 1$, $\alpha > 1/2$ and $\varepsilon > 0$ are four constants. Then

$$\bar{N}[f] \;<\; \infty, \tag{3.19}$$

and if furthermore $\varepsilon > 1/2$,

$$\bar{N}^{Th}_{2\pi}[f] \;<\; \infty. \tag{3.20}$$

Proof This theorem is proved in a straightforward way from lemma 2. □

Theorem 1 shows that the numbers $\bar{N}[f]$ and $\bar{N}^{Th}_{2\pi}[f]$, and their difference diverge if $\tilde{f}\,(p)$ does not decrease sufficiently fast at the origin $p = 0$. Theorem 2 shows that the convergence of $\bar{N}[f]$ depends essentially on the asymptotic behavior of $f(y)$ *far* from the black hole, and that the convergence of $\bar{N}^{Th}_{2\pi}[f]$ depends on the behavior of $f(y)$ for both $y \gg 1$ and $-y \gg 1$.

3.2 A bound for the mean number $\bar{N}[f]$

A bound for the difference $|\,\bar{N}[f] - \frac{1}{2}\,\bar{N}^{Th}_{2\pi}[f]\,|$ is now given, as a first step towards obtaining a bound for the mean number $\bar{N}[f]$.

Lemma 3 *If $\tilde{f} \in L^2(\frac{dp}{2p}, \mathbb{R}_+)$ is a normalised wave function such that f exists and is integrable, then*

$$\left|\bar{N}[f] - \frac{1}{2}\,\bar{N}^{Th}_{2\pi}[f]\right| \;\leq\; \frac{1}{8\pi^2}\left|\int_0^\infty dp'\,\tilde{f}(p')^*\,t(p')^*\int_0^\infty dp\;P\,\frac{\tilde{f}(p)\,t(p)}{p-p'}\right|$$

$$+\;C\left[\int_0^\infty dp\,\frac{|\,\tilde{f}(p)\,|}{\sqrt{p\,(e^{2\pi p}-1)}}\left(1+p^2+\log\frac{1+p}{p}\right)\right]^2, \qquad (3.21)$$

where t is a complex function satisfying

$$|\,t(p)\,| \;=\; \sqrt{\frac{2\pi}{p\,(e^{2\pi p}-1)}}, \qquad (3.22)$$

and $C > 0$ is a constant.

Proof See appendix A.4.

Although this bound is quite complicated, it contains useful informations which will be exploited in section 5. The first term of this bound is the main UV contribution in momentum k and stems from the values of the wave function close to the horizon. The p^2 contribution in the second term is the UV correction to the first term, this is needed because the wave function f is not necessarily localised close to the horizon. The logarithmic expression in the second term is the IR contribution in k and stems from the values of the wave function far from the black hole. The first contribution in the second term is due to the finite values of k.

Some cruder but simpler bounds are given in the following theorem.

Theorem 3 *If $\tilde{f} \in L^2(\frac{dp}{2p}, \mathbb{R}_+)$ is a normalised wave function such that f exists and is integrable, then*

$$\bar{N}[f] \;\leq\; C\int_0^\infty \frac{dp}{2p}\,\frac{|\,\tilde{f}(p)\,|^2}{1-e^{-2\pi p}}, \qquad (3.23)$$

$$\left|\,\bar{N}[f] - \bar{N}^{Th}_{2\pi}[f]\right| \;\leq\; C\int_0^\infty \frac{dp}{2p}\,\frac{|\,\tilde{f}(p)\,|^2}{1-e^{-2\pi p}}, \qquad (3.24)$$

where $C > 0$ is a constant.

Proof This theorem follows from lemma 3. The first term on the r.h.s. of eq. (3.21) is bounded by applying a theorem on Hilbert transforms (see theorem A.2) which enables us to treat the principal value

$$\left| \int_0^\infty dp' \tilde{f}(p')^* t(p')^* \int_0^\infty dp \; P \frac{\tilde{f}(p)t(p)}{p-p'} \right| \;\; \leq \;\; 4\pi^2 \int_0^\infty \frac{dp}{2p} \frac{|\tilde{f}(p)|^2}{e^{2\pi p}-1}. \tag{3.25}$$

Using the Cauchy-Schwartz inequality for the second term, one gets eqs (3.23) and (3.24) from eq. (2.27). □

Theorems 1 and 3 are in agreement, both predict that the mean number $\bar{N}[f_\alpha]$ and the difference $|\bar{N}[f_\alpha] - \bar{N}^{Th}_{2\pi}[f_\alpha]|$ converge if $\alpha > 1/2$, where f_α is defined in theorem 1. From theorem 3, it is clear that if $\bar{N}[f]$ and $|\bar{N}[f] - \bar{N}^{Th}_{2\pi}[f]|$ are infinite, they may only be IR divergent in p.

4 Asymptotic behaviors of the mean number $\bar{N}[f]$

4.1 Close to the horizon

The asymptotic behavior of the mean number $\bar{N}[f]$ close to the horizon is now investigated. The translation of the wave function f by a quantity y_o is first defined by

$$f_{y_o}(y) \;\; = \;\; f(y-y_o), \qquad \forall\, y \in \mathbb{R}. \tag{4.1}$$

Then one asks oneself the questions: Does the mean number of created particles in the state f_{y_o} tend to its thermal average if $y_o \to +\infty$? In another words, do we have

$$\lim_{y_0 \to +\infty} \left(\bar{N}[f_{y_o}] - \bar{N}^{Th}_{2\pi}[f_{y_o}] \right) \;\; \stackrel{?}{=} \;\; 0 \tag{4.2}$$

And if the answer to this question is positive, how does $\bar{N}[f_{y_o}]$ tend to $\bar{N}^{Th}_{2\pi}[f_{y_o}]$ in this limit? Notice that since a translation of f implies only a global change of the phase of $\tilde{f}$, the thermal average $\bar{N}^{Th}_{2\pi}[f_{y_o}]$ does not actually depend on y_o (see eq. (2.27)).

Theorem 1 tells us that the answer to question (4.2) may be negative, since from eq. (3.16) there are wave functions f such that

$$\left| \bar{N}[f_{y_o}] - \bar{N}^{Th}_{2\pi}[f_{y_o}] \right| \;\; = \;\; \infty, \qquad \forall\, y_o \in \mathbb{R}, \tag{4.3}$$

i.e., although the numbers $\bar{N}[f_{y_o}]$ and $\bar{N}^{Th}_{2\pi}[f_{y_o}]$ are both infinite in these cases, their differences are IR divergent. Thus, even if the wave function f is translated towards the horizon, the mean number $\bar{N}[f_{y_o}]$ may not tend to the thermal average $\bar{N}^{Th}_{2\pi}[f_{y_o}]$. One expects from theorem 1 and eqs (3.6) and (3.7) that such wave functions should decrease more slowly than $1/|y|^{3/2}$ at infinity. It turns out, however, that only the asymptotic behavior of $f(y)$ far from the black hole (i.e. for $y \to -\infty$) determines whether the mean number $\bar{N}[f_{y_o}]$ does tend or not to the thermal average $\bar{N}^{Th}_{2\pi}[\,f_{y_o}]$ close to the horizon, and if it does, how it does it. It is shown below that if $f(y)$ decreases strictly faster than $1/|y|^{3/2}$ far from the black hole, then the answer to question (4.2) is positive. Two bounds for the Fourier transform $\tilde{\hat{f}}_{y_o}(-k)$ are first given in the following lemmas.

Lemma 4 *If f is an integrable function such that $\tilde{f}\,(0) = 0$ and if its modulus $|f|$ satisfies*

$$|\,f(y)\,| \;\leq\; \frac{C}{|\,y\,|^{1+\alpha}}, \qquad \text{if } y \leq -L, \tag{4.4}$$

where C, L and α are three positive constants, then

$$\sqrt{2\pi}\;|\,\tilde{\hat{f}}_{y_o}(-k)\,| \;\leq\; \sqrt{k}\,e^{-y_o/2}\,\|\,f\,\|_{L^1} + \frac{4C}{\alpha}\,\frac{2^{\alpha}}{(\,y_o - \log k - 1\,)^{\alpha}}, \tag{4.5}$$

where $0 \leq k \leq e^{y_o - 2L}$. This result is also true in the thermal case, i.e. for $\tilde{\hat{f}}^{Th}_{y_o}(-k)$.

Proof See appendix A.5.

Lemma 5 *If the function $\delta\hat{f}$ is defined by the difference $\delta\hat{f}\,(x) \equiv \hat{f}\,(x) - \hat{f}^{Th}(x)$ $(x < 0)$ where f and its derivative are integrable, then*

$$\sqrt{2\pi}\;|\,\widetilde{\delta\hat{f}}_{y_o}(-k)\,| \;\leq\; \left(\frac{1}{\sqrt{k}} + \frac{1}{k}\right)\left(\|\,f\,\|_{L^1} + \|\,f'\,\|_{L^1}\right), \tag{4.6}$$

where $k > 0$.

Proof See appendix A.6.

A bound for the difference $\left|\,\bar{N}[\,f_{y_o}] - \bar{N}^{Th}_{2\pi}[\,f_{y_o}]\,\right|$ is now given in terms of $y_o > 0$.

Theorem 4 *If $\tilde{f} \in L^2(\frac{dp}{2p}, \mathrm{I\!R}_+)$ is a normalised wave function such that f and its derivative exist and are integrable, and if the modulus $|f|$ satisfies*

$$|\,f(y)\,| \;\leq\; \frac{C}{|\,y\,|^{1+\alpha}}, \qquad \text{if } y \leq -L, \tag{4.7}$$

where $C > 0$, $L \geq 1$ and $\alpha > 1/2$ are three constants, then

$$\left|\,\bar{N}[\,f_{y_o}] - \bar{N}^{Th}_{2\pi}[\,f_{y_o}]\,\right| \leq \frac{1}{\alpha^2(2\alpha - 1)}\,\frac{32\,C^2}{(y_o/4 + L - 1)^{2\alpha-1}} + e^{4L - y_o/4}\left(\|\,f\,\|_{L^1} + \|\,f'\,\|_{L^1}\right)^2 \tag{4.8}$$

where $y_o > 0$.

Proof Since the Fourier transforms $\tilde{\hat{f}}(-k)$ and $\tilde{\hat{f}}^{Th}(-k)$ behave similarly for $k \gg 1$ but differently for $k \approx 0^+$ (see lemmas 1 and 2), one writes

$$\left| \bar{N}\,[\,f_{y_o}\,] - \bar{N}_{2\pi}^{Th}\,[\,f_{y_o}\,] \right| \;\leq$$

$$\int_0^W \frac{dk}{2k}\left[\,|\,\tilde{\hat{f}}_{y_0}(-k)\,|^2 + |\,\tilde{\hat{f}}_{y_0}^{Th}(-k)\,|^2\right] + \int_W^\infty \frac{dk}{2k}\left|\,|\,\tilde{\hat{f}}_{y_0}(-k)\,|^2 - |\,\tilde{\hat{f}}_{y_0}^{Th}(-k)\,|^2\right|, \tag{4.9}$$

where $0 < W \leq e^{y_o - 2L}$. Notice that the bounds of both the integrals and integrands depends on y_o. Lemma 4 implies that

$$\int_0^W \frac{dk}{2k}\left[\,|\,\tilde{\hat{f}}_{y_0}(-k)\,|^2 + |\,\tilde{\hat{f}}_{y_0}^{Th}(-k)\,|^2\right] \;\leq$$
$$\frac{W}{2}\,e^{-y_o}\,\|\,f\,\|_{L^1}^2 + \frac{4^\alpha}{\alpha^2\,(2\alpha-1)}\,\frac{16\,C^2}{(\,y_o - \log W - 1\,)^{2\alpha-1}}. \tag{4.10}$$

If $\delta\hat{f}_{y_o}(x) \equiv \hat{f}_{y_o}(x) - \hat{f}_{y_o}^{Th}(x)$ one has

$$\left|\,|\,\tilde{\hat{f}}_{y_0}(-k)\,|^2 - |\,\tilde{\hat{f}}_{y_0}^{Th}(-k)\,|^2\right| \;\leq\; 2\,\|\,f\,\|_{L^1}\,|\,\widetilde{\delta\hat{f}_{y_o}}(-k)\,| + |\,\widetilde{\delta\hat{f}_{y_o}}(-k)\,|^2\,. \tag{4.11}$$

Applying lemma 5 one gets if $k > 0$,

$$\left|\,|\,\tilde{\hat{f}}_{y_0}(-k)\,|^2 - |\,\tilde{\hat{f}}_{y_0}^{Th}(-k)\,|^2\right| \;\leq\; \frac{1}{2}\left(\frac{1}{\sqrt{k}} + \frac{1}{k} + \frac{1}{k^2}\right)(\,\|\,f\,\|_{L^1} + \|\,f'\,\|_{L^1}\,)^2. \tag{4.12}$$

This last equation implies

$$\int_W^\infty \frac{dk}{2k}\left|\,|\,\tilde{\hat{f}}_{y_0}(-k)\,|^2 - |\,\tilde{\hat{f}}_{y_0}^{Th}(-k)\,|^2\right| \leq \frac{1}{4}\left(\frac{2}{\sqrt{W}} + \frac{1}{W} + \frac{1}{2W^2}\right)(\,\|\,f\,\|_{L^1} + \|\,f'\,\|_{L^1}\,)^2. \tag{4.13}$$

From eqs (4.10) and (4.13) one gets eq. (4.8) if $W = e^{y_o/2-2L}$ (which satisfies $W < e^{y_o-2L}$). □

This last theorem shows clearly that the mean number $\bar{N}[\,f_{y_o}\,]$ does tend to the thermal average $\bar{N}_{2\pi}^{Th}\,[\,f_{y_o}\,]$ when $y_o \to +\infty$ if f decreases sufficiently fast far from the black hole.

4.2 Far from the horizon

The asymptotic behavior of the mean number $\bar{N}[f]$ far from the horizon is now investigated. In this region, i.e. in the limit $y \to -\infty$, the metric in y coordinates tends to the Minkowski metric, and thus there is no local creation of particles there. However, this does not imply that the mean number $\bar{N}[f]$ is arbitrary small if the test function f is translated towards that region, because one expects that the queue of $f(y)$ close to the horizon may have a significant contribution to $\bar{N}[f]$ even in that limit. Theorem 1 tells us that there are indeed wave functions f such that

$$\bar{N}\,[\,f_{y_o}\,] \;=\; \infty, \qquad \forall\, y_o \in \mathbb{R}, \tag{4.14}$$

and that these wave functions decrease more slowly than $1/|y|^{3/2}$ at infinity. One ask thus oneself the questions: If f_{y_o} is defined by eq. (4.1), under what conditions does one have

$$\lim_{y_0 \to -\infty} \bar{N}[f_{y_o}] \stackrel{?}{=} 0, \tag{4.15}$$

and if the answer to this question is positive, how does $\bar{N}[f_{y_o}]$ vanish in this limit? It is shown below that if $f(y)$ decreases strictly faster than $1/y^{3/2}$ close to the horizon, then the answer to question (4.15) is positive. Three bounds for the Fourier transform $\tilde{\hat{f}}_{y_o}(-k)$ are first given in the following lemma.

Lemma 6 *Let f be an integrable function such that its modulus $|f|$ satisfies*

$$|f(y)| \leq \frac{C}{y^{1+\alpha}}, \qquad \text{if } y \geq L, \tag{4.16}$$

where $C > 0$, $L \geq 1$ and $\alpha > 0$ are three constants, and assume that $y_o < 0$.

a) If $\tilde{f}(0) = 0$ and $\alpha > 1/2$, then

$$\sqrt{2\pi}\, |\tilde{\hat{f}}_{y_o}(-k)| \leq 22\, C\sqrt{k} + k\, e^L \| f \|_{L^1}, \tag{4.17}$$

where $e^{-L} \geq k \geq 0$.

b) If f is such that $\tilde{f}(p) = \theta(p)\, \tilde{f}(p)$, $\forall\, p \in \mathbb{R}$, then

$$\sqrt{2\pi}\, |\tilde{\hat{f}}_{y_o}(-k)| \leq k \| f \|_{L^1} e^{L+y_o/2} + \frac{2C}{\alpha} \frac{1}{(L - y_o/2)^\alpha}, \tag{4.18}$$

where $k \geq 0$.

c) If the derivative of f is integrable, then

$$\sqrt{2\pi}\, |\tilde{\hat{f}}_{y_o}(-k)| \leq \frac{2^{1+\alpha} C}{\alpha(\log k - 2y_o - 2)^\alpha} + \frac{2}{\sqrt{k}} \left(2^{1+\alpha} C + \| f \|_{L^1} + \| f' \|_{L^1} \right) \tag{4.19}$$

where $k \geq 2\, e^{2L}$.

Proof See appendix A.7.

A bound for $\bar{N}[f_{y_o}]$ is now given in terms of $y_o < 0$.

Theorem 5 *If $\tilde{f} \in L^2(\frac{dp}{2p}, \mathbb{R}_+)$ is a normalised wave function such that f exists and is integrable, and if the modulus $|f|$ satisfies*

$$|f(y)| \leq \frac{C}{y^{1+\alpha}}, \qquad \text{if } y \geq L, \tag{4.20}$$

where $C > 0$, $L \geq 1$ *and* $\alpha > 1/2$ *are three constants, then*

$$\bar{N}\,[\,f_{y_o}\,] \;\leq\; \frac{1}{\alpha\,(2\alpha-1)}\,\frac{8\,C^2}{(-y_0/4+L/2-1)^{2\alpha-1}}$$
$$+\;20\,e^{4L+y_o/2}\left(2^{1+\alpha}\,C+\|\,f\,\|_{L^1}+\|\,f'\,\|_{L^1}\right)^2, \tag{4.21}$$

where $y_0 \leq -4\,(1+L)$.

Proof Lemma 6 is applied. The bounds (4.17), (4.18) and (4.19) are useful for small, finite and large values of k respectively. If w and W are two constants such that $0 < w \leq e^{-L}$ and $2e^{2L} \leq W$, then

$$\int_0^w \frac{dk}{2k}\,|\,\tilde{\hat{f}}_{y_o}(-k)\,|^2 \;\leq\; \frac{242\,C^2}{\pi}\,w+\frac{1}{4\pi}\,e^{2L}\,w^2\,\|\,f\,\|_{L^1}^2, \tag{4.22}$$

$$\int_w^W \frac{dk}{2k}\,|\,\tilde{\hat{f}}_{y_o}(-k)\,|^2 \;\leq\; \frac{1}{4\pi}\,\|\,f\,\|_{L^1}^2\,e^{2L+y_o}\,W^2+\frac{2\,C^2}{\pi}\,\frac{1}{\alpha^2\,(L-y_o/2)^{2\alpha}}\,\log\frac{W}{w}, \tag{4.23}$$

$$\int_W^\infty \frac{dk}{2k}\,|\,\tilde{\hat{f}}_{y_o}(-k)\,|^2 \;\leq\; \frac{2\,C^2}{\pi}\,\frac{4^\alpha}{\alpha^2\,(2\,\alpha-1)}\,\frac{1}{(\log W-2y_o-2)^{2\alpha-1}}$$
$$+\frac{2}{\pi}\,\frac{1}{W}\left(2^{1+\alpha}\,C+\|\,f\,\|_{L^1}+\|\,f'\,\|_{L^1}\right)^2. \tag{4.24}$$

These bounds imply eq. (4.21) if $w = e^{y_o/4}$ and $W = e^{L-y_o/4}$ under the constraint $y_o \leq -4\,(1+L)$, so that the assumptions stated on w and W are satisfied. □

This last theorem shows clearly that $\bar{N}[\,f_{y_o}\,]$ does vanish when $y_o \to -\infty$ if f decreases sufficiently fast close to the horizon.

5 A wave packet

The physics of a wave packet in the CGHS black-hole background is now considered. This wave packet depends on two parameters δ and p_o and is defined by

$$\tilde{f}^{\,\delta}_{p_o}\,(p) \;=\; \theta(p)\,\sqrt{2\,|p|}\;\Delta_\delta(p-p_o), \qquad \forall\, p \in \mathbb{R}, \tag{5.1}$$

where $p_o \geq \delta > 0$ and

i) Δ_δ is a normalised function in $L^2(dp, \mathbb{R})$: $\int_{-\infty}^{+\infty} dp\;\,|\,\Delta_\delta(p)\,|^2 = 1$;

ii) the support of Δ_δ is included in the interval $(-\delta, \delta)$;

iii) $\tilde{f}^{\,\delta}_{p_o} \in C^r(\mathbb{R})$ where $r \geq 0$; $\left(\tilde{f}^{\,\delta}_{p_o}\right)^{(n)}$ vanish at the origin and at infinity for $n = 0, 1, .., r$; the derivatives $\left(\tilde{f}^{\,\delta}_{p_o}\right)^{(r+1)}$ and $\left(\tilde{f}^{\,\delta}_{p_o}\right)^{(r+2)}$ exist almost everywhere and are bounded and integrable respectively;

iv) the function Δ_δ is real and positive;

v) the function $f^{\delta}_{p_o}$ and its derivative are integrable.

From property *i)* the wave function $\tilde{f}^{\,\delta}_{p_o}$ is normalised in $L^2(\frac{dp}{2p}, \mathbb{R}_+)$, and property *ii)* implies that $\tilde{f}^{\,\delta}_{p_o}$ is centred about p_o in momentum space. From property *iii)* one shows that the function $f^{\delta}_{p_o}$ decreases in configuration space at least as $1/|y|^{r+2}$ if $|y| \gg 1$, and property *iv)* implies that $f^{\delta}_{p_o}$ is centred about $y = 0$. Property *v)* will be useful below, when theorems 4 and 5 are applied to this wave function. The function $f^{\delta}_{p_o,y_o}$ is defined to be the translation of $f^{\delta}_{p_o}$ as in eq. (4.1), and its Fourier transform is given by

$$\tilde{f}^{\,\delta}_{p_o,y_o}(p) \;=\; \theta(p)\,\sqrt{2\,|p|}\;\Delta_\delta(p - p_o)\;e^{-ipy_o}, \qquad \forall\, p \in \mathbb{R}. \tag{5.2}$$

For example, if Δ_δ is the triangle-shaped function

$$\Delta_\delta(p) \;=\; \sqrt{\frac{3}{2\,\delta^3}}\;\left[\theta(p)\,\theta(\delta - p)\,(\delta - p) + \theta(-p)\,\theta(\delta + p)\,(\delta + p)\right], \tag{5.3}$$

conditions *i)* to *v)* with $r = 0$ are satisfied. In this case, the wave function in configuration space is given by

$$f^{\delta}_{p_o,y_o}(y) \;=\; 4\sqrt{\frac{3p_o}{2\pi\,\delta^3}}\;e^{ip_o(y-y_o)}\,\sin^2\left[\frac{\delta\,(y - y_o)}{2}\right]\frac{1}{(y - y_o)^2} + \delta\,\mathcal{O}\left(\sqrt{\frac{\delta}{p_o}}\right), \tag{5.4}$$

and takes its maximum value at $y = y_o$ where

$$f^{\delta}_{p_o,y_o}(y_o) \;=\; \sqrt{\frac{3\,\delta\,p_o}{2\pi}} + \delta\,\mathcal{O}\left(\sqrt{\frac{\delta}{p_o}}\right). \tag{5.5}$$

The wave length of the generic wave packet (5.1) is given approximately by

$$\Delta y \;=\; \frac{2\pi}{p_o}. \tag{5.6}$$

If Δx is the wave length of the incoming wave packet $\hat{f}^{\,\delta}_{p_o,y_o}(x)$, one has

$$\Delta y\,(x) \;=\; y\,(x + \Delta x) - y\,(x)\,. \tag{5.7}$$

The incoming momentum is approximately given by $k_o = 2\pi/\Delta x$. If $k_o \gg 1$, the incoming and outgoing momenta are related by [12]

$$p_o \;\approx\; (1 - e^x)\,k_o, \qquad \textit{if}\;\; x < 0. \tag{5.8}$$

Close to the horizon, one has $p_o \approx (-x)\, k_o$ and thus the outgoing momenta p_o is strongly shifted towards the IR region when k_o is kept fixed. In the limit $x \to -\infty$, the incoming and outgoing momenta are asymptotically equal. One also has if $p_o \gg 1$,

$$k_o \;\approx\; (1 + e^{y})\, p_o, \tag{5.9}$$

and thus close to the horizon the incoming momenta k_o is strongly shifted towards the UV region when p_o is kept fixed.

The limits $\delta \to 0^+$ and $y_o \to \pm\infty$ of the mean number of particles $\bar{N}\,[\, f^{\delta}_{p_o,y_o}\,]$ are now considered. As it is shown below, these limits do not commute. The limits $y_o \to \pm\infty$ are first evaluated in the following theorem.

Theorem 6 *If the wave function $f^{\delta}_{p_o}$ is defined by eq. (5.1), then for all $\delta > 0$*

$$\lim_{y_o \to +\infty} \left(\bar{N}\,[\, f^{\delta}_{p_o,y_o}\,] \;-\; \bar{N}^{Th}_{2\pi}[\, f^{\delta}_{p_o,y_o}\,] \right) \;=\; 0, \tag{5.10}$$

$$\lim_{y_o \to -\infty} \bar{N}[\, f^{\delta}_{p_o,y_o}\,] \;=\; 0, \tag{5.11}$$

and thus

$$\lim_{\delta \to 0}\; \lim_{y_o \to +\infty} \left(\bar{N}\,[\, f^{\delta}_{p_o,y_o}\,] \;-\; \bar{N}^{Th}_{2\pi}[\, f^{\delta}_{p_o,y_o}\,] \right) \;=\; 0, \tag{5.12}$$

$$\lim_{\delta \to 0}\; \lim_{y_o \to -\infty} \bar{N}[\, f^{\delta}_{p_o,y_o}\,] \;=\; 0. \tag{5.13}$$

Proof Since the function $f^{\delta}_{p_o}(y)$ decreases at least as $1/|y|^2$ at infinity and property v) is satisfied by assumption, theorems 4 and 5 can be applied to this wave function and one concludes immediately. □

The generalised function $\tilde{f}^{\,o}_{p_o}$ is now defined by

$$\tilde{f}^{\,o}_{p_o}\,(p) \;=\; \lim_{\delta \to 0^+} \tilde{f}^{\,\delta}_{p_o}\,(p), \qquad \forall\, p \in \mathbb{R}. \tag{5.14}$$

The expectation value of the momentum operator in the corresponding state equals p_o and its variance vanishes in this state. Notice that, in configuration space, the function $f^{\delta}_{p_o,y_o}$ becomes more and more extended in the limit $\delta \to 0^+$, although, if $\tilde{f}^{\,\delta}_{p_o} \in C^{\infty}(\mathbb{R})$, $f^{\delta}_{p_o,y_o}$ decreases at infinity faster than the inverse of any algebraic function of y. Theorems 4 and 5 may thus not be applied to $f = f^{o}_{p_o}$, because the quantity L defined in eqs (4.7) or (4.20) tends to infinity when $\delta \to 0^+$, and consequently the bounds (4.8) and (4.21) diverge exponentially in this limit. The following lemma will be needed to consider the physics of the delocalised wave function $f^{o}_{p_o}$.

Lemma 7 *Let δ and ε be two positive constants. If δ is small enough, one has for all $y_o \in (-\varepsilon/\delta, \varepsilon/\delta)$*

$$\left| \bar{N}\,[\, f^{\delta}_{p_o,y_o}\,] - \frac{1}{2}\bar{N}^{Th}_{2\pi}\,[\, f^{\delta}_{p_o,y_o}\,] \right| \;\le\; C \left[n(p_o)\,\delta + \frac{\varepsilon(\varepsilon + 1)}{e^{2\pi p_o} - 1} \right], \tag{5.15}$$

where $C > 0$ is a constant and where the function n satisfies

$$|n(p_o)| \;\leq\; \begin{cases} \dfrac{\log^2 p_o}{p_o}, & \text{if } p_o \approx 0^+, \\ p_o^4\, e^{-4\pi p_o}, & \text{if } p_o \gg 1, \end{cases} \tag{5.16}$$

and is bounded except in the neighborhood of $p_o = 0$.

Proof See appendix A.8.

This lemma implies that the mean number $\bar{N}\,[\,f^{\delta}_{p_o,y_o}\,]$ is approximately equal to the *half* of the thermal average $\bar{N}^{Th}_{2\pi}\,[\,f^{\delta}_{p_o,y_o}\,]$ if δ and ε are small enough. It may be applied to the generalised function $f^{o}_{p_o}$ to calculate $\bar{N}\,[\,f^{o}_{p_o,y_o}\,]$.

Theorem 7 *If the generalised function $f^{o}_{p_o}$ is defined by eq. (5.14), then*

$$\bar{N}\,[\,f^{o}_{p_o,y_o}\,] \;=\; \frac{1}{2}\,\frac{1}{e^{2\pi p_o}-1}, \qquad \forall\, y_o \in \mathbb{R}, \tag{5.17}$$

where $p_o > 0$, and in consequence

$$\lim_{y_o\to\pm\infty}\;\lim_{\delta\to 0}\left(\bar{N}\,[\,f^{\delta}_{p_o,y_o}\,] - \frac{1}{2}\,\bar{N}^{Th}_{2\pi}\,[\,f^{\delta}_{p_o,y_o}\,]\right) \;=\; 0, \tag{5.18}$$

where the wave function $f^{\delta}_{p_o}$ is defined in eq. (5.1).

Proof The limit $\delta \to 0^+$ is first evaluated in eq. (5.15) of lemma 7 and the result obtained is then true for all $y_o \in \mathbb{R}$. The limit $\varepsilon \to 0^+$ is next evaluated and eq. (5.17) is obtained. Equation (5.18) follows then from eq. (2.27). □

This theorem shows again that the mean number of particles created in a state may not be thermal close to the horizon and may not vanish far from the black hole. Theorems 6 and 7 imply that the limits $\delta \to 0^+$ and $y_o \to \pm\infty$ of the mean number $\bar{N}\,[\,f^{\delta}_{p_o,y_o}\,]$ do not commute.

6 Conclusions

In the present paper, exact calculations of the mean number $\bar{N}[f]$ of massless scalar particles created spontaneously in a given state f have been performed in the CGHS black-hole background. Since our approach do not rely on the approximation of the horizon, one was able to draw some rigorous conclusions on the issues related to the convergence of the mean number $\bar{N}[f]$ and to the approach to the thermal equilibrium as well, and to calculate exactly the mean number of particles created in a given mode.

The main conclusion of this paper is that the physics close - or asymptotically close - to the horizon depends on the global properties of the considered state. For example, the approach to the thermal equilibrium of the mean number $\bar{N}[f]$ close to the horizon depends on the queue of the state far from the black hole, if this decreases sufficiently fast, otherwise $\bar{N}[f]$ may be not thermal in that limit. Similarly, the mean number of particles created in a given mode is not thermal even close to the horizon, because the corresponding state is unlocalised, and the contribution of the part of the state which is far from the black hole must also be taken into account. Since $\bar{N}[f]$ is essentially not a local quantity, and because a state without negative momentum components cannot be localised, the mean number of particles $\bar{N}[f]$ created close to the horizon depends to some extent on the spacetime properties far from the black hole.

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A Appendices

A.1 Two useful theorems

Theorem A.1 (Paley-Wiener [13]) *Assume that $g \in L^2(dy, \mathbb{R})$. Then there is a function $\omega : \mathbb{R} \to \mathbb{R}$ such that the Fourier transform of $f(y) \equiv g(y)\, e^{i\omega(y)}$ satisfies*

$$\tilde{f}(p) = \theta(p)\, \tilde{f}(p), \qquad \forall\, p \in \mathbb{R}, \tag{A.1}$$

if and and only if

$$\left| \int_{-\infty}^{+\infty} dy\, \frac{\log|g(y)|}{1+y^2} \right| < \infty. \tag{A.2}$$

In particular, eq. (A.1) implies that the modulus $|f|$ is strictly bounded at infinity from below by a decreasing exponential function.

Theorem A.2 (Hilbert transform [13]) *If $\tilde{g} \in L^2(dp, \mathbb{R})$, then*

$$\tilde{f}(p) \equiv \frac{1}{\pi} P \int_{-\infty}^{+\infty} dp'\, \frac{\tilde{g}(p')}{p'-p} \tag{A.3}$$

converges almost everywhere if $p \in \mathbb{R}$. Furthermore, one has $\tilde{f} \in L^2(dp, \mathbb{R})$ and

$$\tilde{g}(p) = -\frac{1}{\pi} P \int_{-\infty}^{+\infty} dp'\, \frac{\tilde{f}(p')}{p'-p}, \qquad \forall\, p \in \mathbb{R}, \tag{A.4}$$

$$\int_{-\infty}^{+\infty} dp\, |\tilde{g}(p)|^2 = \int_{-\infty}^{+\infty} dp\, |\tilde{f}(p)|^2. \tag{A.5}$$

A.2 Proof of lemma 1

Lemma A.1 *Let be a differentiable function $g \in L^1(dx, \mathbb{R}_+)$ such that $\tilde{g}(0) = 0$, and define the function h by*

$$g(x) \;=\; \frac{h(x)}{x\,(-\log x)^{1+\alpha}}, \qquad \text{if } x > 0, \tag{A.6}$$

where $\alpha > 0$. If the limit $C \in \mathbb{C}$ of h exists when $x \to +\infty$ (or $x \to 0^+$), and if one has

$$h'(x) \;=\; \mathcal{O}\left(\frac{1}{x \log x}\right) \tag{A.7}$$

when $x \gg 1$ (or $x \approx 0^+$), then

$$\tilde{g}(-k) \;=\; \frac{C}{\sqrt{2\pi}}\,\frac{A_g(k)}{\alpha\,(\log k)^{\alpha}} + \mathcal{O}\left[\frac{1}{(\log k)^{\alpha+1}}\right], \tag{A.8}$$

where A_g is a function depending on g and satisfying

$$1/2 \;\leq\; |A_g(k)| \;\leq\; 3/2 \tag{A.9}$$

when $k \approx 0^+$ (or $k \gg 1$).

Proof The proof will be only sketched here (see ref. [11] for a complete proof). The idea is to split the Fourier transform $\tilde{g}(-k)$ into two terms:

$$\sqrt{2\pi}\,\tilde{g}(-k) \;=\; \int_0^{\frac{1}{2k}} dx\, g(x)\, e^{ikx} + \int_{\frac{1}{2k}}^{\infty} dx\, g(x)\, e^{ikx}. \tag{A.10}$$

Under assumption (A.7), one shows that the second term on the r.h.s. of this last equation is of higher order than the first one by integrating by part. One next defines the function $A_g(k)$ by

$$\int_0^{\frac{1}{2k}} dx\, g(x)\, e^{ikx} \;=\; A_g(k) \int_0^{\frac{1}{2k}} dx\, g(x), \tag{A.11}$$

from which inequalities (A.9) are deduced from the L'Hospital rule. Approximating and integrating the r.h.s. of eq. (A.11) yields eq. (A.8) from eqs (A.6) and (A.7). □

Lemma A.2 *If $g \in L^1(dx, \mathbb{R}_+)$ is such that $\tilde{g}(0) = 0$, and if its modulus $|g|$ satisfies*

$$|g(x)| \;\leq\; \frac{2\,C}{(x - \log 2)^{1+\varepsilon}}, \qquad \text{if } x \geq l, \tag{A.12}$$

where $C > 0$, $l > \log 2$, and $\varepsilon > 0$ are three constants, then

$$\sqrt{2\pi}\;|\tilde{g}(-k)| \;\leq\; \frac{4\,C}{q(\varepsilon)}\,k^{q(\varepsilon)}\left[e^{q(\varepsilon)} + \frac{1}{2\,[1 - q(\varepsilon)]}\right] + k\,(e^{l} - 1)\,\|g\|_{L^1(dx,\mathbb{R}_+)}, \tag{A.13}$$

when $0 \leq k \leq l^{-1}$, and where the function q is defined in eq. (3.5).

Proof The primitive G of g is defined by $G(x) = \int_0^x dx'\, g(x')$. Since $\tilde{g}\,(0) = G(0) = 0$, one gets by splitting the integral and by integrating by parts,

$$\sqrt{2\pi}\,\tilde{g}\,(-k) \;=\; -\mathrm{i}\,k\int_0^{1/k} dx\, G(x)\; e^{\mathrm{i}kx} + \int_{1/k}^{\infty} dx\, g(x)\,\left(e^{\mathrm{i}kx} - e^{\mathrm{i}}\right). \qquad \text{(A.14)}$$

From assumption (A.12), a bound for the first term on the r.h.s. of this last equation is obtained by writing $\int_0^{1/k} dx = \int_0^{l} dx + \int_l^{1/k} dx$, where $0 \leq k \leq l^{-1}$, and by noting that the behavior of the bound for $\int_l^{1/k} dx\, G(x)\; e^{\mathrm{i}kx}$ depends on ε. A bound for the second term is easily obtained from assumption (A.12) as well and one gets finally eq. (A.13). □

Proof of lemma 1 From eqs (3.6) and (3.7), one has

$$|\,\hat{f}_\alpha(-x)\,| \;\leq\; \frac{2\,|C_{\alpha,-}|}{(x-\log 2)^{1+\alpha}} + \mathcal{O}\left(\frac{1}{x^{2+\alpha}}\right), \qquad \textit{if } x \gg 1, \qquad \text{(A.15)}$$

$$\hat{f}_\alpha(-x) \;=\; \frac{C_{\alpha,+}}{x\,(-\log x)^{1+\alpha}} + \mathcal{O}\left[\frac{1}{x\,(-\log x)^{2+\alpha}}\right], \qquad \textit{if } x \approx 0^+, \qquad \text{(A.16)}$$

and

$$\hat{f}_\alpha^{Th}(-x) \;=\; \frac{C_{\alpha,\mp}}{x\,(-\log x)^{1+\alpha}} + \mathcal{O}\left[\frac{1}{x\,(-\log x)^{2+\alpha}}\right], \quad \textit{if} \begin{cases} x \gg 1, \\ x \approx 0^+. \end{cases} \qquad \text{(A.17)}$$

To obtain eq. (A.15), one made use of $|y(-x)| \geq x - \log 2$ and $y'(-x) \leq 2$ if $x \geq \log 2$. Equations (3.2), (3.3) and (3.4) follow then from lemma A.1 if $g(x) = \hat{f}_\alpha(-x)$. Equation (3.1) follows from lemma A.2. □

A.3 Proof of lemma 2

Lemma A.3 *If $g \in L^1(dx, \mathbb{R}_+)$ is a function such that $\tilde{g}\,(0) = 0$, and if its modulus $|g|$ satisfies*

$$|\,g(x)\,| \;\leq\; \frac{C_-}{x\,(\log x)^{1+\alpha}}, \qquad \textit{if } x \geq l, \qquad \text{(A.18)}$$

where $C_- > 0$, $l > 1$, and $\alpha > 0$ are three constants, then

$$\sqrt{2\pi}\;\,|\tilde{g}\,(-k)\,| \;\leq\; \frac{3\,C_-}{\alpha\,(-\log k)^{\alpha}} + k\,l\,\|\,g\,\|_{L^1(dx,\mathbb{R}_+)} \qquad \text{(A.19)}$$

where $0 \leq k \leq l^{-1}$.

Proof The proof is similar to the one of lemma A.2. In this case, however, the behavior of the bound for $\int_l^{1/k} dx\, G(x)\, e^{ikx}$ one obtains does not depend on the parameter α. □

Lemma A.4 *If $g \in L^1(dx, \mathbb{R}_+)$ is a differentiable function such that its modulus satisfies*

$$|g(x)| \;\leq\; \frac{C_+}{x\,(-\log x)^{1+\alpha}}, \qquad \text{if } 0 < x \leq l, \tag{A.20}$$

where $C_+ > 0$, $0 < l < 1$ and $\alpha > 0$ are three constants, then

$$\sqrt{2\pi}\;|\tilde{g}(-k)| \;\leq\; \frac{2^\alpha\, C_+}{\alpha\,(\log k)^\alpha} + \frac{1}{\sqrt{k}}\,\frac{2^{1+\alpha}\, C_+}{(\log k)^{1+\alpha}} + \frac{1}{k}\int_{\frac{1}{\sqrt{k}}}^{\infty} dx\; |g'(x)| \tag{A.21}$$

where $k \geq l^{-2}$.

Proof The Fourier transform $\tilde{g}(k)$ is split into two terms,

$$\sqrt{2\pi}\,\tilde{g}(-k) \;=\; \int_0^{\frac{1}{\sqrt{k}}} dx\, g(x)\, e^{ikx} + \int_{\frac{1}{\sqrt{k}}}^{\infty} dx\, g(x)\, e^{ikx}. \tag{A.22}$$

A bound for the first term on the r.h.s. of this last equation is obtained directly from assumption (A.20) if $k \geq l^{-2}$, and a bound for the second term is deduced by integrating it by parts. Equation (A.21) then follows. □

Proof of lemma 2 From eqs (3.8) and (3.9), one has

$$|\hat{f}(-x)| \;\leq\; \begin{cases} \dfrac{C_-}{x^{1+\epsilon}}, & \text{if } x \gg 1, \\[2ex] \dfrac{C_+}{x\,(-\log x)^{1+\alpha}}, & \text{if } x \approx 0^+, \end{cases} \tag{A.23}$$

and

$$|\hat{f}^{Th}(-x)| \;\leq\; \begin{cases} \dfrac{C_-}{x\,(-\log x)^{1+\epsilon}}, & \text{if } x \gg 1, \\[2ex] \dfrac{C_+}{x\,(-\log x)^{1+\alpha}}, & \text{if } x \approx 0^+. \end{cases} \tag{A.24}$$

Lemmas A.2 and A.3 imply eqs (3.10) and (3.12) respectively. Lemma A.4 is next applied to $g(x) = \hat{f}(-x)$. Since

$$x'(y)\,\frac{d}{dx}\hat{f}(x)\bigg|_{x(y)} \;=\; -\frac{x''(y)}{x'(y)^2}\, f(y) + \frac{1}{x'(y)}\, f'(y), \tag{A.25}$$

one has

$$\int_{\frac{1}{\sqrt{k}}}^{\infty} dx\, \left|\frac{d}{dx}\hat{f}(-x)\right| \;\leq\; \int_{-\infty}^{\log\sqrt{k}} dy\; (1 + e^y)\,(|f(y)| + |f'(y)|). \tag{A.26}$$

This result is also true for $g(x) = \hat{f}^{Th}(-x)$, and eqs (3.11) and (3.13) are then obtained. □

A.4 Proof of lemma 3

From eqs (2.13) and (2.23) one has

$$\bar{N}[f] = \frac{1}{8\pi^3}\int_0^\infty dp'\,\tilde{f}(p')^*\int_0^\infty dp\,\tilde{f}(p) \times \Gamma(\mathrm{i}p')\,\Gamma(-\mathrm{i}p)\int_0^\infty dk\,\sinh(\pi k)\,\Gamma(-\mathrm{i}p'-\mathrm{i}k+0^+)\,\Gamma(\mathrm{i}p+\mathrm{i}k+0^+), \tag{A.27}$$

where eq. (2.15) has been used. Stirling's formula [9],

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}}\,e^{-z}\,z^z\left[1+\mathcal{O}\left(\frac{1}{z}\right)\right], \tag{A.28}$$

where $|\arg z| < \pi$, implies that

$$\sinh(\pi k)\,\Gamma(-\mathrm{i}p'-\mathrm{i}k+0^+)\,\Gamma(\mathrm{i}p+\mathrm{i}k+0^+) = \pi\,e^{-\frac{\pi}{2}(p+p')}\,\frac{e^{\mathrm{i}(p-p')\log k}}{k}\left[1+\mathcal{O}\left(\frac{1+p^2+p'^2}{k}\right)\right]. \tag{A.29}$$

The integral over k in eq. (A.27) is split into three terms according to the partition $\mathbb{R}_+ = [0,w]\cup(w,W)\cup[W,\infty)$, where w and W are two positive constants which are small and large enough respectively. The contribution of the non-compact interval $[W,\infty)$ is given from eq. (A.29) by

$$\int_W^\infty \frac{dk}{2k}\,U(-k,p')^*\,U(-k,p) = \frac{\delta(p-p')}{4p\,(e^{2\pi p}-1)} + \frac{\mathrm{i}}{8\pi^2}\,t(p)\,t(p')^*\,P\,\frac{1}{p-p'} + \frac{1}{\sqrt{p\,(e^{2\pi p}-1)\,p'\,(e^{2\pi p'}-1)}}\,\mathcal{O}\left(\frac{1+p^2+p'^2}{W}\right), \tag{A.30}$$

where the function t has been defined by

$$t(p) = \Gamma(-\mathrm{i}p)\,e^{-\frac{\pi}{2}p}\,e^{\mathrm{i}p\log W}, \tag{A.31}$$

and satisfies eq. (3.22) (see eq. (2.15)). The first term on the r.h.s. of eq. (A.30) is the half of the kernel of $\bar{N}^{Th}_{2\pi}[f]$ (see eq. (2.27)), and thus

$$\int_0^\infty dp'\,\tilde{f}(p')^*\int_0^\infty dp\,\tilde{f}(p)\int_W^\infty\frac{dk}{2k}\,U(-k,p')^*\,U(-k,p) = \frac{1}{2}\bar{N}^{Th}_{2\pi}[f] + \frac{\mathrm{i}}{8\pi^2}\int_0^\infty dp'\tilde{f}(p')^*\,t(p')^*\int_0^\infty dp\,P\frac{\tilde{f}(p)\,t(p)}{p-p'} + \mathcal{O}\left(\frac{1}{W}\right)\left|\int_0^\infty dp\,\frac{\tilde{f}(p)\,(1+p^2)}{\sqrt{p(e^{2\pi p}-1)}}\right|^2. \tag{A.32}$$

If $\varepsilon > 0$ is small enough one has furthermore

$$\left|\int_0^w dk\,\sinh(\pi k)\,\Gamma(-\mathrm{i}p'-\mathrm{i}k+0^+)\,\Gamma(\mathrm{i}p+\mathrm{i}k+0^+)\right| \le C\,e^{-\frac{\pi}{2}(p+p')}\times\begin{cases}\log\left(1+\frac{w}{\varepsilon}\right), & \textit{if } p,p'\ge\varepsilon>0,\\ \log\left(1+\frac{w}{p}\right), & \textit{otherwise,}\end{cases} \tag{A.33}$$

$$\left| \int_w^W dk \, \sinh(\pi k) \, \Gamma(-\mathrm{i}p' - \mathrm{i}k + 0^+) \, \Gamma(\mathrm{i}p + \mathrm{i}k + 0^+) \right| \;\leq\; C \, \frac{W}{w} \, e^{-\frac{\pi}{2}(p+p')}, \tag{A.34}$$

where $C > 0$ is a constant. From the last three equations one gets

$$\left| \bar{N}[f] - \frac{1}{2} \bar{N}^{Th}_{2\pi}[f] \right| \;\leq\; \frac{1}{8\pi^2} \left| \int_0^\infty dp' \, \tilde{f}\,(p')^* \, t(p')^* \int_0^\infty dp \; P \, \frac{\tilde{f}\,(p) \, t(p)}{p - p'} \right| \tag{A.35}$$

$$+ \, C \left[\int_0^\infty dp \, \frac{|\, \tilde{f}\,(p) \,|}{\sqrt{p(e^{2\pi p} - 1)}} \, (1 + p^2) \right]^2 + \, C \left[1 + \log\left(1 + \frac{w}{\varepsilon}\right) \right] \left[\int_\varepsilon^\infty dp \, \frac{|\, \tilde{f}\,(p) \,|}{\sqrt{p(e^{2\pi p} - 1)}} \right]^2$$

$$+ \, C \int_0^\varepsilon dp \, \frac{|\, \tilde{f}\,(p) \,|}{\sqrt{p(e^{2\pi p} - 1)}} \, \log\left(1 + \frac{w}{p}\right) \int_0^\infty dp \, \frac{|\, \tilde{f}\,(p) \,|}{\sqrt{p(e^{2\pi p} - 1)}}$$

which implies eq. (3.21). □

A.5 Proof of lemma 4

The Fourier transform of $\hat{f}_{y_o}(x)$ is split into two terms,

$$\int_{-\infty}^0 dx \, \hat{f}_{y_o}(x) \, e^{ikx} \;=\; \int_0^{\frac{1}{\sqrt{kx_o}}} dx \, \hat{f}_{y_o}(-x) \, \left(e^{-\mathrm{i}kx} - 1\right) + \int_{\frac{1}{\sqrt{kx_o}}}^\infty dx \, \hat{f}_{y_o}(-x) \, \left(e^{-\mathrm{i}kx} - 1\right) \tag{A.36}$$

where $x_o \equiv e^{y_o}$. A bound for the first term on the r.h.s. of this last equation is given for both the CGHS black-hole and thermal cases by

$$\left| \int_0^{\frac{1}{\sqrt{kx_o}}} dx \, \hat{f}_{y_o}(-x) \, \left(e^{-\mathrm{i}kx} - 1\right) \right| \;\leq\; \sqrt{k} \, e^{-y_o/2} \, \|\, f \,\|_{L^1} . \tag{A.37}$$

To obtained a bound for the second term, the CGHS black-hole and thermal cases must be treated separately, although similar bounds will be found for these two cases. One assumes in both cases that $e^{L - y_o} \leq x$, so that use of assumption (4.7) can be made. In the second term one has $(kx_o)^{-0.5} \leq x$, one must thus have $e^{L-y_o} \leq (kx_o)^{-0.5}$. This means that the bounds obtained below are true for $0 \leq k \leq e^{y_o - 2L}$.

The thermal case is first considered. Assumption (4.4) implies

$$|\, \hat{f}^{Th}_{y_o}(-x) \,| \;\leq\; \frac{C}{x \, [\log(xx_o)]^{1+\alpha}}, \qquad \textit{if } \; x \geq e^{L - y_o}, \tag{A.38}$$

from which one deduces, if $0 \leq k \leq e^{y_o - 2L}$, that

$$\left| \int_{\frac{1}{\sqrt{kx_o}}}^\infty dx \, \hat{f}^{Th}_{y_o}(-x) \, \left(e^{-\mathrm{i}kx} - 1\right) \right| \;\leq\; \frac{2C}{\alpha} \, \frac{2^\alpha}{(\, y_o - \log k \,)^\alpha} . \tag{A.39}$$

Equations (A.37) and (A.39) imply eq. (4.5) for the thermal case.

The case of the CGHS black hole is now considered. An auxiliary transformation $y_A(x)$ is defined by

$$y_A(x) = \begin{cases} -\log x, & \textit{if } 0 < x < \log 2, \\ -x + \log 2, & \textit{if } \log 2 \le x < \infty. \end{cases} \tag{A.40}$$

Since

$$y'(x) = \frac{1}{1 - e^{-x}} \le 2 \times \begin{cases} 1/x, & \textit{if } 0 < x < \log 2, \\ 1, & \textit{if } \log 2 \le x < \infty, \end{cases} \tag{A.41}$$

assumption (4.4) implies when $x \ge e^{L - y_o}$ that

$$|\,\hat{f}_{y_o}(-x)\,| \le \frac{2\,C}{[\,y_o - y_A(x)\,]^{1+\alpha}} \times \begin{cases} 1/x, & \textit{if } 0 < x < \log 2, \\ 1, & \textit{if } \log 2 \le x < \infty. \end{cases} \tag{A.42}$$

The cases $(kx_o)^{-0.5} \le \log 2$ and $(kx_o)^{-0.5} \ge \log 2$ have to be considered separately. If $(kx_o)^{-0.5} \le \log 2$, one has from eq. (A.42)

$$\begin{aligned} \int_{\frac{1}{\sqrt{kx_o}}}^{\infty} dx \;|\,\hat{f}_{y_o}(-x)\,| &\le \int_{\frac{1}{\sqrt{kx_o}}}^{\log 2} dx\, \frac{2\,C}{x\,[\log(xx_o)\,]^{1+\alpha}} + \int_{\log 2}^{\infty} dx\, \frac{2\,C}{(\,x + y_o - \log 2\,)^{1+\alpha}} \\ &\le \frac{2\,C}{\alpha}\, \frac{2^\alpha}{(\,y_o - \log k\,)^\alpha}. \end{aligned} \tag{A.43}$$

If $(kx_o)^{-0.5} \ge \log 2$, one has also from eq. (A.42)

$$\int_{\frac{1}{\sqrt{kx_o}}}^{\infty} dx \;|\,\hat{f}_{y_o}(-x)\,| \le \int_{\frac{1}{\sqrt{kx_o}}}^{\infty} dx\, \frac{2\,C}{(\,x + y_o - \log 2\,)^{1+\alpha}} \le \frac{2\,C}{\alpha}\, \frac{2^\alpha}{(\,y_o - \log k - 1\,)^\alpha}, \tag{A.44}$$

since $0 < -2\log z \le 1/z$ (if $0 < z < 1$) implies for $z = \log 2\sqrt{kx_o}$

$$0 < -\log 2\,(\,2 \log\log 2 + \log k + y_0) \le \frac{1}{\sqrt{kx_o}}. \tag{A.45}$$

Equation (4.5) for the black hole case is then deduced from eqs (A.37), (A.43) and (A.44). □

A.6 Proof of lemma 5

The subscript y_o is dropped in this subsection for clarity. Defining the transformation $x_{Th} = x_{Th}(y)$ by eq. (2.9), one gets by integrating by parts

$$\int_{-\infty}^{0} dx\, \delta\hat{f}\,(x)\, e^{ikx} = \frac{i}{k} \int_{-\infty}^{+\infty} dy\, \left[x'(y)\, \hat{f}'\,(x(y))\, e^{ikx(y)} - x'_{Th}(y)\, \hat{f}'^{Th}(x_{Th}(y))\, e^{ikx_{Th}(y)} \right] \tag{A.46}$$

Equation (A.25) implies

$$\int_{-\infty}^{0} dx\, \delta\hat{f}(x)\, e^{ikx} = \frac{i}{k}\Big\{ -\int_{-\infty}^{+\infty} dy\, f'(y) \left[\frac{1}{x'(y)}\, e^{ikx(y)} - \frac{1}{x'_{Th}(y)}\, e^{ikx_{Th}(y)} \right] + \int_{-\infty}^{+\infty} dy\, f(y) \left[\frac{x''(y)}{x'(y)^2}\, e^{ikx(y)} - \frac{x''_{Th}(y)}{x'_{Th}(y)^2}\, e^{ikx_{Th}(y)} \right] \Big\}$$

$$= \frac{i}{k}\Big\{ \int_{-\infty}^{+\infty} dy\, e^{y}\, [f(y) + f'(y)] \left[e^{ikx(y)} - e^{ikx_{Th}(y)} \right] + \int_{-\infty}^{+\infty} dy\, f'(y)\, e^{ikx(y)} \Big\}. \tag{A.47}$$

Now if one defines $z = e^{-y}$ one gets

$$e^{y} \left| e^{-ikx(y)} - e^{-ikx_{Th}(y)} \right| \leq \begin{cases} \frac{k}{z}\, [\, z - \log(1+z)\,], \\ \frac{2}{z}, \end{cases} \tag{A.48}$$

and in particular

$$e^{y} \left| e^{-ikx(y)} - e^{-ikx_{Th}(y)} \right| \leq \frac{2}{z_m}, \tag{A.49}$$

where z_m is defined by $k\,[\, z_m - \log(1+z_m)\,] = 2$. Since $0 \leq z - \log(1+z) \leq \frac{z^2}{2}$, one obtains $\frac{2}{z_m} \leq \sqrt{k}$ and thus

$$e^{y} \left| e^{ikx(y)} - e^{ikx_{Th}(y)} \right| \leq \sqrt{k}. \tag{A.50}$$

Equations (A.47) and (A.50) imply finally eq. (4.6). □

A.7 Proof of lemma 6

a) Equation (4.16) implies eq. (A.12) with $l = \log(1+e^L) \leq e^L$ since $|y(-x)| \geq x - \log 2$ and $y'(-x) \leq 2$ if $x \geq \log 2$. Lemma A.2 is then applied with $\alpha > 1/2$ and eq. (4.17) is obtained.

b) Since $\tilde{f}\,(-k) = 0$ if $k \geq 0$, one has

$$\sqrt{2\pi}\, \tilde{\hat{f}}\,(-k) = \int_{-\infty}^{+\infty} dy\, f(y) \left[e^{ikx(y)} - e^{iky} \right]. \tag{A.51}$$

One writes then

$$\sqrt{2\pi}\; \tilde{\hat{f}}_{y_o}(-k) = \int_{-\infty}^{L+y_o/2} dy\, f_{y_o}(y)\, e^{iky} \left\{ e^{ik[x(y)-y]} - 1 \right\} + \int_{L+y_o/2}^{+\infty} dy\, f_{y_o}(y) \left[e^{ikx(y)} - e^{iky} \right]. \tag{A.52}$$

A bound for the first integral on the r.h.s. of this last equation is obtained from

$$|\, x(y) - y\,| \leq e^{y}, \qquad \forall\, y \in \mathbb{R}. \tag{A.53}$$

Assumption (4.16) is used in the second integral to deduce eq. (4.18).

c) Equation (4.19) is obtained in the same way as eq. (3.11) of lemma 2. □

A.8 Proof of lemma 7

This lemma is deduced from eq. (3.21) of lemma 3. The second term on the r.h.s. of this equation gives a contribution of order δ if $f = f^{\delta}_{p_o,y_o}$. To treat the first term with the principal value, one writes in eq. (5.2),

$$e^{-\mathrm{i}py_o} \;=\; e^{-\mathrm{i}p_oy_o} + e^{-\mathrm{i}p_oy_o}\left[e^{-\mathrm{i}(p-p_o)\,y_o} - 1\right]. \qquad \text{(A.54)}$$

The real contribution of $e^{-\mathrm{i}p_oy_o}$ in this first term vanishes because the function Δ_δ is real by assumption and the integrand is anti-hermitian. The imaginary contribution of $e^{-\mathrm{i}p_oy_o}$ in the first term is proportional to

$$\int_0^\infty dp \int_0^\infty dp'\,|\,\tilde{f}^{\delta}_{p_o,y_o}(p)\,t(p)\,\tilde{f}^{\delta}_{p_o,y_o}(p')\,t(p')\,|\;\frac{\sin\{\arg[t(p)] - \arg[t(p')]\}}{p-p'}, \qquad \text{(A.55)}$$

and this expression may be bounded by a term of order δ. Finally, the contributions of $e^{-\mathrm{i}p_oy_o}\left[e^{-\mathrm{i}(p-p_o)\,y_o} - 1\right]$ in this term are of order $\varepsilon(1+\varepsilon)$ from eq. (3.25) and the assumption on $|y_o|$ (since $|\,(p-p_o)\,y_o\,| \le \varepsilon$). □

References

[1] S. W. Hawking, *Particle creation by black holes*, Commun. Math. Phys. 43 (1975) 199.

[2] C.G. Callan, S.B. Giddings, J.A. Harvey, A. Strominger, *Evanescent black holes*, Phys. Rev. D45 (1992) R1005; S.B. Giddings, W.M. Nelson, *Quantum emission from two-dimensional black holes*, Phys. Rev. D46 (1992) 2486.

[3] G. Wanders, *Fermion emission in a two-dimensional black hole space-time*, Phys. Rev. D50 (1994) 2731.

[4] G. Morchio, D. Pierotti, F. Strocchi, *Infrared and vacuum structure in two-dimensional local quantum field theory models. The massless scalar field*, International School for Advanced Studies, Trieste, 1987; G. Morchio, D. Pierotti, F. Strocchi, *Infrared and vacuum structure in two-dimensional local quantum field theory models. The massless scalar field*, J. Math. Phys. 31 (1990) 1467.

[5] S.A. Fulling, S.N.M. Ruijsenaars, *Temperature, periodicity and horizons*, Phys. Rep. 152 (1987) 135.

[6] F. Vendrell, *A semi-classical relativistic black-hole*, Helv. Phys. Acta 70 (1997) 598.

[7] F. Vendrell, *Stimulated emission of particles by 1+1 dimensional black holes*, Helv. Phys. Acta 70 (1997) 637.

[8] F. Vendrell, *A black hole in two-dimensional space-time*, hep-th/9705135.

[9] D.R. Guo, Z.X. Wang, *Special functions* (World Scientific, Singapore, 1989).

[10] Y.-X. Gui, *Quantum field in* $\eta-\xi$ *spacetime*, Phys. Rev. D42 (1990) 1988; $\eta-\xi$ *spacetime and thermo fields*, Phys. Rev. D46 (1992) 1869.

[11] F. Vendrell, *Création de bosons de masse nulle dans des espace-temps bidimensionnels courbes* (University of Lausanne, Switzerland, 1996).

[12] J. Audretsch, R. Müller, *Amplification of the black-hole Hawking radiation by stimulated emission*, Phys. Rev. D45 (1992) 513.

[13] D.C. Champeney, *A handbook of Fourier theorems* (Cambridge University Press, Cambridge, 1987).