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Hidden symmetries in non-Riemannian gravity

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Abstract. The peculiar role of the torsion in non-Riemannian gravity is elucidated in particular the fact it permits to obtain solutions for the field equations which have symmetries not obviously manifest in the action, the model then may have "Hidden symmetries" due to the fact the torsion acts as a compensating non symmetric term.

- Recently much effort has been devoted to the study of non standard gravitational theories in particular some models in which both non-metricity and torsion are different from zero.
  The Einstein's theory of gravity which was formulated more than eighty years ago provides an elegant and powerful formulation of gravitation in terms of a pseudo riemannian geometry. The Einstein equations are obtained by considering the variation with respect to the metric of the curvature scalar associated with the Levi Civita connection times the volume form of the spacetime. As assumption Einstein required that both non-metricity and torsion are vanishing, a position which is natural but not always convenient when we consider models with more degrees of freedom.
  In particular at the level of the so called string theories [1] there are hints that by using non-Riemannian geometry we may accomodate the several degrees of freedom coming from the low energy limit of string interactions.
  It is interesting to observe that in that case, since string theories are expected to produce effects which are at least in principle testable at low energies; there may be chances to obtain non-Riemannian models with predictions which can somehow be

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tested, moreover some models can have some effects on astronomical scales [2].
For instance recently models have been proposed that permit to account for the so called dark matter by invoking short range non-riemannian gravitational interactions [3].
There are several approaches to non-Riemannian gravity, perhaps one of the most popular is the one which uses the gauge approach applied to the affine group [4].
An appoach can be proposed in which the metric $\mathbf{g}$ and the connection ∇ are independent dynamical variables and instead of working with the affine group they rely on the definition of torsion and non-metricity in terms of $\mathbf{g}$ and ∇, [5]. In this approach we choose to freeze one of the gauge potentials of the affine group, that is we choose a particular gauge, for example selecting a particular coframe e^a or metric g_{ab}.
Recently using this approach an interesting relation has been found between certain models of non-Riemannian gravitation and Einstein-Proca theories [6], this relation has been also found in the context of a general metric affine gravity model [7].
In this letter the interesting role of the torsion in non-Riemannian gravity is discussed, in particular we show that the torsion can act as a compensating field in the field equations.
Though neither non-metricity nor torsion respect certain symmetries, the field equations may possess higher symmetries due to the properties of cancellation which exist in certain models of non-Riemannian gravity. As a result we can get a metric which has symmetries not shared by the torsion and the non-metricity.
We start from the very simple case in which the action does not contain the kinetic term for the non-metricity and then we move to the Proca-type theory where we show that although the invariants $Q \wedge \star Q$ and $T \wedge \star T$ may not be 'symmetric' , solutions for the field equations may possess higher symmetries. Finally we consider models which include scalar fields.
These properties of the torsion have been already used implicitly in [8] to discuss some Black-Hole dilaton solutions in non riemannian gravity.
These properties origins from the fact the torsion 1-form T defined below present non trivial variation with respect to the connection and the coframe, it follows that we get a sort of constraint. This constraint is the origin of the reduction properties which allows the metric to have symmetries not shared by the torsion and the nonmetricity.
These results are quite relevant because then we can look for solutions of the generalized Einstein equations using an ansatz for the metric satisfying certain symmetry properties while for the non-Riemannian fields in general we do not need to impose such an ansatz, in this sense and *only* in this sense we talk about *hidden symmetries.*
Before going to consider the different cases, let us briefly summarize the notation we will be using.

A general non-Riemannian geometry is specified by a metric tensor $\mathbf{g}$ and a linear connection ∇. Using a local coframe e^a with its dual frame X_b such that $e^a(X_b) = \delta^a_b$. The connection 1-forms satisfy $\omega^c{}_b(X_a) = e^c(\nabla_{X_a} X_b)$. The tensor $\mathbf{S} = \nabla \mathbf{g}$ defines the non metricity of the theory. We can choose the local orthonormal frame in which the metric tensor is $\mathbf{g} = \eta_{ab} e^a \otimes e^b$, ($\eta_{ab} = diag(-1, 1, 1, 1, ...)$). The non-metricity 1-forms are defined by $Q_{ab} = \mathbf{S}(-, X_a, X_b)$ and the torsion 2-forms are $T^a = de^a + \omega^a{}_b \wedge e^b$ the curvature two forms are $R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$ and the curvature scalar R is defined by $R\star 1 = R^a{}_b \wedge \star(e_a \wedge e^b)$ in terms of the Hodge operator $\star$ of the metric. The torsion 1-form is defined by $T = i_a T^a$ where i_a is the contraction operator, the 1-form Q is defined as $Q = Q^a{}_a$.

1 Pure Einstein-Vacuum from Non-Riemannian gravity

To begin with, let us consider the action:

$$\Lambda[e,\omega] = kR\star 1 + \frac{\beta}{2}(Q \wedge \star Q) + \frac{\gamma}{2}(T \wedge \star T) \tag{1.1}$$

Where R is the scalar curvature associated with a general non-Riemannian connection. The connection variation gives:

$$kD\star(e_a \wedge e^b) = 2\beta\, \delta^b{}_a \star Q + \gamma e^b \wedge i_a \star T \tag{1.2}$$

Which can be split into:

$$\beta \star Q = \frac{\gamma(1-n)}{2n} \star T \tag{1.3}$$

and

$$kD\star(e_a \wedge e^b) = \delta^b{}_a \frac{1-n}{n}\gamma \star T + \gamma e^b \wedge i_a \star T \tag{1.4}$$

The coframe variation gives the Einstein equations:

$$k\, R^a{}_b \wedge \star(e_a \wedge e^b \wedge e_c) + \tau_c[\beta] + \tau_c[\gamma] = 0 \tag{1.5}$$

where:

$$\tau_c[\beta] = \frac{\beta}{2}(Q \wedge i_c \star Q + i_c Q \wedge \star Q) \tag{1.6}$$

$$\tau_c[\gamma] = \gamma[i_k(T^k \wedge \star(T \wedge e_c)) - D i_c \star T - \frac{1}{2}(T \wedge i_c \star T + i_c T \wedge \star T)]$$

If we choose the coupling constants to satisfy the relation:

$$4n^2(n-2)\beta k + (n-1)^2(n-2)\gamma k + 8(1-n)\gamma\beta = 0 \tag{1.7}$$

Then the Einstein equations can be reduced to:

$$k\overset{o}{R}{}^a{}_b \wedge \star(e_a \wedge e^b \wedge e_c) = 0 \tag{1.8}$$

where the superscript o refers to the Levi-Civita part of the curvature 2-forms. This simplification properties can be proved with the following steps:

1] We solve eq. (4) for the non-Riemannian part of the connection $\lambda^a{}_b$ we use that expression to calculate the terms which appear in (5-6).

2] If the constraint (7) is verified then eq. (5) is reduced to (8).

For further details see [6, 10].

In conclusion of this paragraph we summarise that for the action (1) the relation between Q and T is:

$$Q = \frac{\gamma(1-n)}{2n\beta}T \tag{1.9}$$

And the Einstein equations reduce to the Levi-Civita form:

$$\overset{o}{R}{}^a{}_b \wedge \star(e_a \wedge e^b \wedge e_c) = 0 \tag{1.10}$$

Which do not contain either T or Q, it follows that the symmetries of Q and T in general are not related to the symmetries of the solutions of (10) which are nothing but the Einstein-Vacuum gravitational field equations.
This is not a real surprise since now the non-metricity and torsion are gravitationally decoupled so they do not contribute to the curvature tensor.
However this is not obvious from the action (1).

2 Massless Weyl-field from Proca theories

Let us consider now the model obtained from the action density:

$$\Lambda[e,\omega] = kR\star 1 + \frac{\alpha}{2}(dQ\wedge\star dQ) + \frac{\beta}{2}(Q\wedge\star Q) + \frac{\gamma}{2}(T\wedge\star T) \tag{2.1}$$

By considering the connection variation we get the two equations:

$$\begin{aligned} \alpha d\star dQ + \beta\star Q &= \frac{\gamma(1-n)}{2n}\star T \\ kD\star(e_a\wedge e^b) &= \delta^b{}_a\frac{1-n}{2n}\gamma\star T + \gamma e^b\wedge i_a\star T \end{aligned} \tag{2.2}$$

By solving the 2nd for the non-Riemannian part of the connection we can easily get that:

$$\alpha d\star dQ + \beta_o\star Q = 0 \tag{2.3}$$

where:

$$\beta_o = \beta - \frac{1}{4}\frac{\gamma k(n-1)^2(n-2)}{kn^2(n-2) - 2\gamma(n-1)} \tag{2.4}$$

by choosing β, γ, k in a proper way we can satisfy the condition:

$$\beta_0 = 0 \tag{2.5}$$

so that eq (13) reduces to:

$$\alpha d \star dQ = 0 \tag{2.6}$$

while the Einstein equations reduce to:

$$k\overset{0}{G}_c + \tau_c[\alpha] = 0 \tag{2.7}$$

Where $\tau_c[\alpha] = \frac{\alpha}{2}(dQ \wedge i_c \star dQ - i_c dQ \wedge \star dQ)$.
The Weyl field then is massless although the action contains the mass term $Q \wedge \star Q$.
Equations (16,17) are still satisfied if we consider the transformation:

$$Q \rightarrow Q + df \tag{2.8}$$

where f is any scalar function (0-form). The terms $Q \wedge \star Q$ and $T \wedge \star T$ are not invariant under the above mentioned transformation, however the presence of both terms in the action allows for a cancellation with the term originating from the full non-Riemannian Einstein-Hilbert term $R \star 1$ in such a way only the gauge invariant term $\frac{\alpha}{2}(dQ \wedge \star dQ)$ contribuites.
It is important to stress that this cancellation occurs only because there is an explicit torsion term in the action, if we do not put this term we cannot "increase" the degree of symmetry of the solutions.

3 Massless-Black-Hole-Dilaton solutions in Non-Riemannian gravity

As we know it is very difficult if not impossible to find Black-Hole solutions starting from a Proca-type actions or similar, and in general the no-hair theorem does not allow for solutions containing scalar fields unless we consider Dilaton gravity models.
However the presence of the torsion in the action may give the possibility of eliminating non-symmetric terms, so that, even if we start from massive terms for the Weyl field, the solutions may be massless and then not difficult to discuss.
Consider for example the action density:

$$\Lambda[e, \omega, \psi] = kR \star 1 + \frac{\alpha}{2} e^{-2\psi}(dQ \wedge \star dQ) + \frac{\gamma}{2}(T \wedge \star T) + \frac{\beta}{2}(Q \wedge \star Q) + \frac{\delta}{2}(d\psi \wedge \star d\psi) \tag{3.1}$$

The connection variation gives the equation:

$$\alpha\, d(e^{-2\psi} \star dQ) + \beta \star Q = \frac{\gamma(1-n)}{2n} \star T \tag{3.2}$$

$$kD \star (e_a \wedge e^b) = \delta^b{}_a \frac{1-n}{n} \gamma \star T + \gamma e^b \wedge i_a \star T$$

If we satisfy the relation between the coupling constants:

$$4n^2(n-2)\beta k + (n-1)^2(n-2)\gamma k + 8(1-n)\gamma\beta = 0 \tag{3.3}$$

the equation for Q becomes a Maxwell-Dilaton type:

$$\alpha\, d(e^{-2\psi} \star dQ) = 0 \tag{3.4}$$

and the generalised Einstein equations reduce to:

$$k\overset{o}{G}_c + \frac{\alpha}{2}e^{-2\psi}(dQ \wedge i_c \star dQ - i_c dQ \wedge \star dQ) - \frac{\delta}{2}(d\psi \wedge i_c \star d\psi + i_c d\psi \wedge \star d\psi) = 0 \tag{3.5}$$

The equation for ψ is:

$$\delta\, d \star d\psi + \alpha\, e^{-2\psi}(dQ \wedge \star dQ) = 0 \tag{3.6}$$

If we choose the coframe to be:

$$\begin{aligned} e^0 &= f(r)dt \\ e^1 &= 1/f(r)dr \\ e^2 &= R(r)d\theta \\ e^3 &= R(r)\sin\theta d\phi \end{aligned} \tag{3.7}$$

we can set the solutions for Q and T^a to be:

$$\begin{aligned} Q &= -q\cos\theta d\phi \\ T^a &= \frac{8\beta}{9\gamma} q\cos\theta(e^a \wedge d\phi) \end{aligned} \tag{3.8}$$

These solutions are not spherically symmetric, however a solution exists for ψ and for the metric which can be written as:

$$\begin{aligned} \psi &= -\frac{1}{2}ln(b_1 - \frac{b_2}{r}) \\ R(r) &= \sqrt{(r-r_1)r} \\ f &= \sqrt{1 + \frac{\alpha b_1 q^2}{2kr_1 r}} \\ \delta &= -4k \\ r_1 &= \frac{b_2}{b_1} \end{aligned} \tag{3.9}$$

which are spherically symmetric.
If we satisfy the conditions:

$$\begin{aligned} \frac{\alpha b_1}{2kr_1} &< 0 \\ r_1 &\neq -\frac{\alpha b_1 q^2}{2kr_1} \end{aligned} \tag{3.10}$$

then we have a black-hole type solution.
Although Q and T are not spherically symmetric, the invariant:

$$dQ \wedge \star dQ = \frac{q^2}{R(r)^2} \star 1 \tag{3.11}$$

is spherically symmetric and the generalised reduced Einstein-equations allows for the spherically symmetric solution. The same can be said for equations (22,24) for Q and ψ.
What has been said for the action (19) can be generalised to a more general action. Indeed consider an action density of the type:

$$\Lambda[e, \omega, \psi] = k\, R \star 1 + \frac{\alpha}{2} e^{-2\psi}(dQ \wedge \star dQ) + \frac{\beta}{2}(Q \wedge \star Q) + \frac{\delta}{2}(d\psi \wedge \star d\psi) + F[e, \omega] \tag{3.12}$$

where $F[e, \omega]$ is a term depending on the variables e, ω.
Suppose that in the limit $e^{-2\psi} \to 1$ there is isomorphism with the theory described by the action density:

$$\Lambda[e, A, \psi] = k \overset{o}{R} \star 1 + \frac{\alpha}{2}(dA \wedge \star dA) + \frac{\delta}{2}(d\psi \wedge \star d\psi) \tag{3.13}$$

and

$$\alpha\, d \star dA = 0 \tag{3.14}$$

It is easy to conclude that the equations (22,23) are still satisfied [9]. We can still use the ansatz (25,27), the torsion two forms are:

$$T^a = \frac{8\beta}{9\gamma} q \cos\theta (e^a \wedge d\phi) \tag{3.15}$$

The non-metricity 1-forms can be calculated using the formula:

$$Q_{ab} = e_a i_b A_1 + e_b i_a A_1 - \frac{1}{2} g_{ab} A_1 + \frac{1}{4} g_{ab} Q \tag{3.16}$$

where the expression for A_1 in general depends on which term $F[e, \omega]$ we consider [10].

In conclusion the presence of the torsion term and its compensating properties which is the origin of the Obukhov theorem [7] permit us to find a massless spherically symmetric solution starting from a massive model.
These interesting properties are not really surprising if we think that one of the motivations of non-Riemannian gravity and in general Metric-Affine-Gravity is to introduce extra degree of freedoms to solve problems like unitarity and renormalizability in quantum gravity [4]. It is clear then the torsion could be used to "increase" the symmetry properties of the solutions for the metric and other fields not directly related to non-metricity and torsion.
It has been however relevant to stress this caracteristic of the torsion in non-Riemannian models of gravity explicitly.

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