Introduction to a mathematical theory of the graded stationary population

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1. Fundamental relations

Consider a population divided into k grades. The grade $g (= 1, 2, \ldots, k)$ is assumed to consist of $l^g_{[x-s]+s}\, ds\, dx$ members of exact age x who entered the grade exactly s years earlier. The total number of members aged x in grade g ist thus

$$l^g_x\, dx = \int_0^x l^g_{[x-s]+s}\, ds\, dx \tag{1}$$

and the total number of members who entered the grade s years earlier, at different ages is

$$l^g(s)\, ds = \int_s^\infty l^g_{[x-s]+s}\, dx\, ds\,. \tag{2}$$

The total number of members of this grade will be

$$l^g = \int_0^\infty l^g_x\, dx = \int_0^\infty l^g(s)\, ds\,. \tag{3}$$

Let members of grades 1 to $k-1$ be subject to two independent decremental forces, a «force of mortality» $\mu(x)$ depending on x only and a «force of promotion» $\nu^g(s)$ depending on the grade and on s, whereas the members of grade k are only to be subject to $\mu(x)$. Hence

$$l^g_{[x-s]+s} = l^g_{[x-s]}\, {}_sp_{x-s}\, p^g_s \quad \text{for} \quad g = 1, 2, \ldots, k-1 \tag{4}$$

$$l^k_{[x-s]+s} = l^k_{[x-s]}\, {}_sp_{x-s} \tag{5}$$

where

$${}_sp_{x-s} = e^{-\int_0^s \mu_{x-s+t}\, dt} \quad \text{and} \quad p^g_s = e^{-\int_0^s \nu^g(t)\, dt}\,.$$

At each moment $l^g_x \mu(x)\, dx$ members of exact age x leave grade g and the population itself. But at the same time the

$$\int_0^x l^g_{[x-s]+s}\, \nu^g(s)\, ds\, dx \tag{6}$$

members of age x who disappear from their grade, are promoted into grade $g+1$.

Putting $g = 1, 2, 3, \ldots, k-1$ in succession, these are the only entries into grades $2, 3, \ldots, k$, therefore

$$l^{g+1}_{[x]} = \int_0^x l^g_{[x-s]+s}\, \nu^g(s)\, ds = \int_0^x l^g_{[x-s]}\, {}_sp_{x-s}\, h^g(s)\, ds \tag{7}$$

where

$$h^g(s) = p^g_s\, \nu^g(s)\,.$$

It will be noticed that ${}_sp_{x-s}$, p^g_s and $h^g(s)$ are only defined for positive or zero s and $x-s$.

We assume that entries into grade $g = 1$ occur only at $x = 0$.

Hence

$$l^1_{[x-s]+s} = 0 \qquad \text{for } s \neq x$$

and

$$= l^1(s) \qquad \text{for } s = x\,,$$

so that

$$l^1(s) = l^1_{[0]}\, {}_sp_0\, p^1_s \tag{8}$$

and

$$l^2_{[x]} = l^1_{[0]}\, {}_xp_0\, h^1(x)\,. \tag{9}$$

In future, $l^1_{[0]}\, {}_xp_0$ will be denoted by L_x.

We now proceed to calculate $l^g_{[x]}$.

Applying (7) to grades g and $g-1$ we obtain

$$l^g_{[x]} = \int_0^x l^{g-1}_{[x-y_1]}\, {}_{y_1}p_{x-y_1}\, h^{g-1}(y_1)\, dy_1\,. \tag{10}$$

A further application of (7) leads to

$$l^g_{[x]} = \int_0^x \int_0^{x-y_1} l^{g-2}_{[x-y_1-y_2]\, y_1+y_2} p_{x-y_1-y_2}\, h^{g-1}(y_1)\, h^{g-2}(y_2)\, dy_2\, dy_1 \qquad (11)$$

and so on, by induction, to

$$l^g_{[x]} = \int_0^x \cdots \int_0^{x-y_1-\ldots-y_{g-3}} l^2_{[x-y_1-\ldots-y_{g-2}]\, y_1+\ldots+y_{g-2}} p_{x-y_1-\ldots-y_{g-2}}\, h^{g-1}(y_1) \ldots h^2(y_{g-2})\, dy_{g-2} \ldots dy_1 . \qquad (12)$$

Because of (9) this is equal to

$$l^g_{[x]} = \int_0^x \cdots \int_0^{x-y_1-\ldots-y_{g-3}} L_x h^{g-1}(y_1) \ldots h^2(y_{g-2})\, h^1(x-y_1-\ldots-y_{g-2})\, dy_{g-2} \ldots dy_1 . \quad (13)$$

where L_x could be written before, instead of after, the integral signs. It should be noticed that $y_1, \ldots, y_{g-2}$ appear only as variables of integration and that the arguments of $h^1, \ldots, h^{g-1}$ add up to x, which is the upper limit of the first integral sign. The r. h. s. is therefore only dependent on x.

The special type of integral which appears here is called «convolution» or «Faltung». We shall not make here any further reference to the theory concerning this branch of analysis, but we introduce the usual notation by writing (13) as follows

$$l^g_{[x]} = L_x h^{g-1} * h^{g-2} * \ldots * h^1(x) . \qquad (13\,a)$$

The comparison of (13) and (13 a) supplies the definition of the *-symbol.
(For a systematic use of the Laplace transform in connection with this type of problem, see H. L. Seal, *Biometrika*, xxxiii, 1945.)

It follows from (4) that

$$l^g_{[x-s]+s} = L_x p^g_s h^{g-1} * \ldots * h^1(x-s) \qquad (14)$$

and hence

$$l^g_x = L_x \int_0^x p^g_s h^{g-1} * \ldots * h^1(x-s)\, ds , \qquad (15)$$

$$l^g(s) = p^g_s \int_s^\infty L_x h^{g-1} * \ldots * h^1(x-s)\, dx . \qquad (16)$$

Now we alter our point of view and combine all grades from g upwards into one. The number of members in such an amalgamation will be denoted by a capital L, so that now

$L^g_{[x-s]+s}\, ds\, dx$ is the number of members of the population of exact age x who entered the grade g exactly s years earlier, and who are now in any of the grades $g, g+1, \ldots, k$.

$L^g_x\, dx$ is the number of members aged exactly x in grades g, $g+1, \ldots, k$ and

$L^g(s)\, ds$ is the number of members in these grades who entered grade g exactly s years ago, whatever their present age.

L^g ist the total number of members in grades $g, g+1, \ldots, k$.

The connections between these numbers are analogous to those existing between the l's given in (1), (2) and (3). All other formulae valid for the l's remain correct for the L's providing p^g_s is replaced by 1, because no promotion can take place out of the amalgamated grades $g, g+1, \ldots, k$. Furthermore, it is clear that for the highest grade, k, the expressions for capital L and for small l coincide.

We note that (1) can be written in the form

$$l^g_x = \int_0^x L^g_{[x-s]+s}\, p^g_s\, ds. \tag{17}$$

A further connection between the functions l and L can be found. It is obvious from general reasoning that $L^g_x = l^g_x + l^{g+1}_x + \ldots + l^k_x$ and we shall now prove this relation mathematically.

Consider the expression

$$L^g_x = L_x \int_0^x h^{g-1} * \ldots * h^1(x-s)\, ds \tag{18}$$

$$= L_x \int_0^x \int_0^{x-s} F(s+y)\, \nu^{g-1}(y)\, e^{-\int_0^y \nu^{g-1}(t)\, dt}\, dy\, ds$$

where

$$F(s+y) = h^{g-2} * \ldots * h^1(x-s-y).$$

Integration by parts with respect to y gives

$$L_x^g = L_x \int_0^x \int_0^{x-s} \frac{\partial}{\partial y} F(s+y)\, e^{-\int_0^y \nu^{g-1}(t)dt}\, dy\, ds - L_x \int_0^x \left[e^{-\int_0^{x-s} \nu^{g-1}(t)dt} F(x) - F(s) \right] ds.$$

But $F(x) = 0$ and $L_x \int_0^x F(s)\, ds$ is easily seen to be L_x^{g-1} by the definition above. Hence we obtain

$$L_x^g - L_x^{g-1} = L_x \int_0^x \int_0^{x-s} \frac{\partial}{\partial y} F(s+y)\, e^{-\int_0^y \nu^{g-1}(t)\, dt}\, dy\, ds.$$

Now

$$\frac{\partial}{\partial y} F(s+y) = \frac{\partial}{\partial s} F(s+y)$$

so that the right hand term reduces to

$$\int_0^x \int_0^{x-s} \frac{\partial}{\partial s} F(s+y)\, e^{-\int_0^y \nu^{g-1}(t)dt}\, dy\, ds = \int_0^x \int_0^{x-y} e^{-\int_0^y \nu^{g-1}(t)\, dt} \frac{\partial}{\partial s} F(s+y)\, ds\, dy$$

$$= \int_0^x e^{-\int_0^y \nu^{g-1}(t)\, dt} [F(x) - F(y)]\, dy = - \int_0^x e^{-\int_0^y \nu^{g-1}(t)\, dt} F(y)\, dy$$

$$= - l_x^{g-1},$$

by (15). We have finally $L_x^g = L_x^{g-1} - l_x^{g-1}$ and, since $L_x^k = l_x^k$,

$$L_x^g = l_x^g + l_x^{g+1} + \ldots + l_x^k = L_x^1 - l_x^1 - l_x^2 - \ldots - l_x^{g-1} \qquad (19)$$

for $g = 2, 3, \ldots, k$.

In particular, for $g = 2$

$$L_x^2 = L_x^1 - l_x^1 = L_x \int_0^x h^1(x-s)\, ds = L_x \int_0^x \nu^1(x-s)\, e^{-\int_0^{x-s} \nu^1(t)dt}\, ds$$

$$= L_c \left[1 - e^{-\int_0^x \nu^1(t)dt} \right] = L_x - l_x^1$$

so that

$$L_x^1 = L_x.$$

Comparing (18) with (13 a) we have

$$L_x^g = L_x \int_0^x \frac{l^g_{[x-s]}}{L_{x-s}} ds = \int_0^x l^g_{[x-s]} \, {}_s p_{x-s} \, ds \tag{20}$$

which is also obtainable from first principles.

From the latter formula we can derive a relation which will be used in Chapter 3. We calculate

$$\frac{d L_x^g}{dx} = \frac{d L_x}{dx} \int_0^x \frac{l^g_{[x-s]}}{L_{x-s}} ds + L_x \frac{d}{dx} \int_0^x \frac{l^g_{[x-s]}}{L_{x-s}} ds .$$

Because of (20) the first integral $= \dfrac{L_x^g}{L_x}$. To the second term we apply the rule for differentiation of an integral with respect to an upper limit which appears also as a parameter of the integrand. Thus

$$\frac{d L_x^g}{dx} = \frac{d L_x}{dx} \frac{L_x^g}{L_x} + L_x \left[\frac{l^g_{[0]}}{L_0} + \int_0^x \frac{d}{dx} \frac{l^g_{[x-s]}}{L_{x-s}} ds \right] .$$

Now

$$\frac{d}{dx} \frac{l^g_{[x-s]}}{L_{x-s}} = - \frac{d}{ds} \frac{l^g_{[x-s]}}{L_{x-s}} ,$$

hence

$$\frac{d L_x^g}{dx} = \frac{d L_x}{dx} \frac{L_x^g}{L_x} + L_x \left[\frac{l^g_{[0]}}{L_0} - \frac{l^g_{[0]}}{L_0} + \frac{l^g_{[x]}}{L_x} \right]$$

and

$$\frac{d L_x^g}{L_x^g dx} = \frac{d L_x}{L_x dx} + \frac{l^g_{[x]}}{L_x^g} = -\mu_x + \frac{l^g_{[x]}}{L_x^g} . \tag{21}$$

Of course, if $l^g_{[x]} = 0$, then

$$\frac{d L_x^g}{L_x^g dx} = \frac{d L_x}{L_x dx} ,$$

i. e. the force of mortality μ_x is the only decremental force affecting the function L_x^g.

2. Limited ranges for promotions

The functions ν and hence p and h of the preceding paragraphs may have any form. But in order to relate them more closely to realistic assumptions we assume now that promotions from one grade into the next can only take place within a certain range of values of s. More precisely assume that the values $\nu^g(s)$ only differ from zero inside the range from α_g to β_g, the limits excluded; then formula (13), which is the basis of all the other relations, can be given an explicit form.

This assumption leads to

$$p_s^g = 1 \text{ for } s \leqslant \alpha_g, \; p_s^g = p_{\beta_g}^g \text{ for } s \geqslant \beta_g$$
$$h^g(s) = 0 \text{ for } s \leqslant \alpha_g \quad \text{and for} \quad s \geqslant \beta_g .$$

First only take account of the fact that $\nu^g(s) = 0$ for $s < \alpha_g$. This gives $h^g(s) = 0$ and $p_s^g = 1$ for $s \leqslant \alpha_g$. We accordingly examine (13) and in particular the limits of the integrations. In view of the ralation just obtained, we can replace the lower limits of the integrations for $y_1, \ldots, y_{g-2}$ by $\alpha_{g-1}, \ldots, \alpha_2$ respectively.

We then observe that

$$h^1(x - y_1 - \ldots - y_{g-2})$$

is only different from zero if $x - y_1 - \ldots - y_{g-2} > \alpha_1$, where the values y_i exceed α_{g-1}. It follows that we must have

$$\begin{aligned}
y_{g-2} &< x - y_1 - y_2 - \ldots - y_{g-3} - \alpha_1 \\
y_{g-3} &< x - y_1 - y_2 - \ldots - y_{g-4} - \alpha_2 - \alpha_1 \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
y_2 &< x - y_1 - \alpha_{g-3} - \alpha_{g-4} - \ldots - \alpha_2 - \alpha_1 \\
y_1 &< x - \alpha_{g-2} - \alpha_{g-3} - \ldots - \alpha_2 - \alpha_1 .
\end{aligned} \tag{22}$$

This shows that the upper limits of integration will be the right hand terms of (22) and the complete integration will have the following ranges:

$$\int_{\alpha_{g-1}}^{x-\alpha_{g-2}-\ldots-\alpha_1}, \; \int_{\alpha_{g-2}}^{x-y_1-\alpha_{g-3}-\ldots-\alpha_1}, \ldots, \int_{\alpha_3}^{x-y_1-\ldots-y_{g-4}-\alpha_2-\alpha_1}, \; \int_{\alpha_2}^{x-y_1-\ldots-y_{g-3}-\alpha_1} . \tag{23}$$

Substituting new variables, viz.

$$z_{g-2} = y_{g-2} - \alpha_2\,,\; z_{g-3} = y_{g-3} - \alpha_3\,,\; \ldots\,,\; z_1 = y_1 - \alpha_{g-1}$$

and

$$\bar{x} = x - \alpha_{g-1} - \ldots - \alpha_2 - \alpha_1$$

(13) is transformed into

$$L_x \int_0^{\bar{x}} \int_0^{\bar{x}-z_1} \ldots \int_0^{\bar{x}-z_1-\ldots-z_{g-3}} h^{g-1}(z_1+\alpha_{g-1})\, h^{g-2}(z_2+\alpha_{g-2}) \ldots h^2(z_{g-2}+\alpha_2)\, h^1(\bar{x}-z_1-\ldots-z_{g-2}+\alpha_1)\, dz_{g-2} \ldots dz_1 . \tag{24}$$

We write for this expression $L_x H_x(\alpha_1, \ldots, \alpha_{g-1})$.

The r. h. s. of (13), can, of course, be written

$$L_x H_x(0, \ldots 0) .$$

If there is also an upper limit to the values of y for which $h^g(y) \neq 0$, so that $y \leqslant \beta_g$, then that part of the $(g-1)$ dimensional space for which one or more y's fall outside these limits must be substracted from the part of the space considered in (24).

We must, therefore, when calculating $l^g_{[x]}$, subtract from (24) $L_x H_x(\beta_1, \alpha_2, \ldots, \alpha_{g-1})$, $L_x H_x(\alpha_1, \beta_2, \alpha_3, \ldots, \alpha_{g-1})$, and so on up to $L_x H_x(\alpha_1, \ldots, \alpha_{g-2}, \beta_{g-1})$. But in this way the area where two or more y's are larger than their respective β's has been subtracted too often. Thus the final formula to replace (13) is

$$l^g_{[x]} = L_x [H_x(\alpha_1, \ldots, \alpha_{g-1}) + (-1)^{\{\alpha_1, \ldots, \alpha_{g-2}, \beta_{g-1}\}} H_x(\alpha_1, \ldots, \beta_{g-1}) + \ldots \tag{25}$$
$$\ldots + (-1)^{\{\alpha_1, \ldots, \beta_{g-2}, \beta_{g-1}\}} H_x(\alpha_1, \ldots, \alpha_{g-3}, \beta_{g-2}, \beta_{g-1}) + \ldots + (-1)^{\{\beta_1, \ldots, \beta_{g-1}\}} H_x(\beta_1, \ldots, \beta_{g-1})]$$

where $\{\}$ stands for the number of β appearing as arguments.

If the total of the α's and β's appearing as argument in any H exceeds the subscript x, the expression is to be replaced by zero. If all H expressions are used, i. e. if x exceeds $\beta_1 + \ldots + \beta_{g-1}$, then $l^g_{[x]}$ must, of course, reduce to zero, i. e. no promotion takes place at age x.

From (25) there follows immediately the formula for

$$l^g_{[x-s]+s} = l^g_{[x-s]}\, {}_sp_{x-s}\, p^g_s$$

and

$$L^g_{[x-s]+s} = l^g_{[x-s]}\, {}_sp_{x-s}\,.$$

If we calculate now $l^g_x = \int\limits_0^x l^g_{[x-s]+s}\, ds$ we find that x occurs only as subscript in the H's and in L_x, and that consequently integration can be applied to every term in (25) with the result

$$l^g_x = L_x \left[\int\limits_0^{x-\alpha_1-\ldots-\alpha_{g-1}} H_{x-s}(\alpha_1, \ldots, \alpha_{g-1})\, p^g_s\, ds + (-1) \int\limits_0^{x-\alpha_1-\ldots-\beta_{g-1}} H_{x-s}(\alpha_1, \ldots, \beta_{g-1})\, p^g_s\, ds + \ldots \text{ etc.} \right]. \tag{26}$$

The upper limits of the integrals are fixed by the rule attached to formula (25).

We find, further,

$$L^g_x = L_x \left[\int\limits_0^{x-\alpha_1-\ldots-\alpha_{g-1}} H_{x-s}(\alpha_1, \ldots, \alpha_{g-1})\, ds + (-1) \int\limits_0^{x-\alpha_1-\ldots-\beta_{g-1}} H_{x-s}(\alpha_1, \ldots, \beta_{g-1})\, ds + \ldots \text{ etc.} \right]. \tag{27}$$

It may be worth pointing out that although the expression in (25) within square brackets reduces to zero if all $H > 0$, no analogous reduction occurs in (26) or (27), because here the various integrals have different upper limits.

Finally, we habe

$$L^g(s) = \int\limits_{s+\alpha_1+\ldots+\alpha_{g-1}}^{\infty} L_x \left[H_{x-s}(\alpha_1, \ldots, \alpha_{g-1}) + (-1)^{|\alpha_1, \ldots, \beta_{g-1}|} H_{x-s}(\alpha_1, \ldots, \beta_{g-1}) + \ldots \text{ etc.} \right] dx \tag{28}$$

and $l^g(s)$ is then found by means of the relation $l^g(s) = L^g(s)\, p^g_s$.

3. Special assumptions concerning $\nu^g(s)$

The simplest case appears when we assume that $\nu^g(s)$ has a constant value c_g for all values of s. This leads to $p_s^g = e^{-c_g s}$ and $h^g(s) = c_g e^{-c_g s}$. We give here a summary of the results which derive from the first 20 formulae of Chapter 1 under two different assumptions:

(I) that all c_g are different for different g, and

(II) that $c_1 = c_2 = \ldots = c_k$.

We obtain then

(I)

$$l^g_{[x-s]+s} = L_x c_1 \ldots c_{g-1} e^{-c_g s} \sum_{i=1}^{g-1} e^{-c_i(x-s)} \prod_{j \neq i} \frac{1}{c_j - c_i}$$

$$l^g_x = L_x c_1 \ldots c_{g-1} \sum_{i=1}^{g} e^{-c_i x} \prod_{j \neq i} \frac{1}{c_j - c_i}$$

$$l^g(s) = c_1 \ldots c_{g-1} e^{-c_g s} \sum_{i=1}^{g-1} e^{c_i s} \prod_{j \neq i} \frac{1}{c_j - c_i} \int_s^\infty e^{-c_i x} L_x dx$$

$$l^g = c_1 \ldots c_{g-1} \sum_{i=1}^{g} \prod_{j \neq i} \frac{1}{c_j - c_i} \int_0^\infty e^{-c_i x} L_x dx .$$

(II)

$$l^g_{[x-s]+s} = L_x c^{g-1} e^{-cx} \frac{(x-s)^{g-2}}{(g-2)!}$$

$$l^g_x = L_x c^{g-1} e^{-cx} \frac{x^{g-1}}{(g-1)!}$$

$$l^g(s) = \frac{c^{g-1}}{(g-2)!} \int_0^\infty (x-s)^{g-2} e^{-cx} L_x dx$$

$$l^g = \frac{c^{g-1}}{(g-1)!} \int_0^\infty (x-s)^{g-1} e^{-cx} L_x dx .$$

(Actuaries will recognise in these integrals the functions which they denote by $\overline{N}$, $\overline{S}$ etc. The function which, in actuarial practice, is called l_x, has here been denoted by L_x.)

The expressions for $L^g_{[x-s]+s}$, $L^{g}(s)$ and L^g can be found from those for the corresponding l by multiplication by $e^{c_g s}$. For L^g_x the following formula holds:

(I)

$$L^g_x = L_x \left[1 - \sum_{i=1}^{g-1} e^{-c_i x} \prod_{j \neq i} \frac{c_j}{c_j - c_i} \right].$$

(II)

$$L^g_x = L_x \left[1 - \sum_{i=1}^{g-1} c^{i-1} e^{-cx} \frac{x^{i-1}}{(i-1)!} \right].$$

Formulae referring to a limited range of promotions [see (21) to (28)] are given by Seal, l. c.

It is clear that if the c_g tend to a common limit c, then the formulae under assumption (I) will tend to coincide with those under (II). If they are different but fairly close together, then it will be possible to find a value c so that a formula under assumption (II) gives a satisfactory approximation to its counterpart under (I).

The value of c which satisfies this latter condition depends on the particular formula under consideration: we here consider first L^g_x, and then l^g_x. The method could equally well be applied to any other functions.

Some algebraic theorems will be needed in what follows and they are set out here for convenience to avoid repeated interruptions to the main argument at later stages.

The equation

$$\sum_{i=1}^{g} \prod_{j \neq i} \frac{c_j - \xi}{c_j - c_i} = 1$$

is of order $g-1$ in ξ and has the g solutions $c_1, \ldots, c_g$. Hence it is an identity and

$$\sum_{i=1}^{g} \prod_{j \neq i} \frac{c_j}{c_j - c_i} = 1. \tag{a}$$

Also

$$\sum_{i=1}^{g} \prod_{j \neq i} \frac{(c_j - \xi)\, c_i^t}{c_j - c_i} = \xi^t$$

is satisfied by $c_1, \ldots, c_g$. Hence it is an identity for $t = 1, 2, \ldots, g-1$, and it follows that

$$\sum_{i=1}^{g} \prod_{j \neq i} \frac{c_j c_i^t}{c_j - c_i} = c_1 \ldots c_g \sum_{i=1}^{g} \prod_{j \neq i} \frac{c_i^{t-1}}{c_j - c_i} = 0, \text{ for } t = 1, 2, \ldots, g-1. \qquad \text{(b)}$$

(cf. The Theory of Equations, Burnside & Panton, 3rd Ed. p. 319.)

We shall also use (l. c. p. 320)

$$\sum_{i=1}^{g} \prod_{j \neq i} \frac{c_i^{g-1}}{c_j - c_i} = (-1)^{g-1} \qquad \text{(c)}$$

and

$$\sum_{i=1}^{g} \prod_{j \neq i} \frac{c_i^{g}}{c_j - c_i} = (-1)^{g} (c_1 + \ldots + c_g). \qquad \text{(d)}$$

Let us now turn to a comparison of the formulae for L_x^g under assumptions (I) and (II).

We want to find c so that

$$\sum_{i=1}^{g-1} e^{-cx} \frac{(cx)^{i-1}}{(i-1)!} \sim \sum_{i=1}^{g-1} e^{-c_i x} \prod_{j \neq i} \frac{c_j}{c_j - c_i}. \qquad (2.1)$$

Expanding the expotentials we have for the l. h. s.

$$\sum_{i=0}^{g-2} \sum_{s=0}^{\infty} \frac{(-cx)^s}{s!} \frac{(cx)^i}{i!} = \sum_{i=0}^{g-2} \sum_{t=i}^{g-2} (-1)^{t-i} \frac{(cx)^t}{i!\,(t-i)!} + \sum_{i=0}^{g-2} \sum_{t=g-1}^{\infty} (-1)^{t-i} \frac{(cx)^t}{i!\,(t-i)!} =$$

$$= \sum_{t=0}^{g-2} (cx)^t \sum_{i=0}^{t} \frac{(-1)^{t-i}}{i!\,(t-i)!} + \sum_{t=g-1}^{\infty} \sum_{i=0}^{g-2} (-1)^{t-i} \frac{(cx)^t}{i!\,(t-i)!}. \qquad (2.2)$$

Now in the first sum the term for $t = 0$ is unity whilst the other terms disappear, because they are

$$\frac{(-1)^t}{t!} (1-1)^t = 0, \quad \text{for } t \neq 0.$$

The lowest term in the second sum is

$$(cx)^{g-1}\left[\sum_{i=0}^{g-1}\frac{(-1)^{g-1-i}}{i!\,(g-1-i)!}-\frac{1}{(g-1)!}\right]=-\frac{(cx)^{g-1}}{(g-1)!}. \quad (2.3)$$

Thus we have as a first approximation for the l. h. s.

$$1-\frac{(cx)^{g-1}}{(g-1)!}.$$

Consider now the r. h. s. Expanding the exponential once again we get

$$\sum_{i=1}^{g-1}\sum_{s=0}^{\infty}\frac{(-c_i x)^s}{s!}\prod_{j\neq i}\frac{c_j}{c_j-c_i}=\sum_{s=0}^{\infty}\frac{(-x)^s}{s!}\sum_{i=1}^{g-1}\prod_{j\neq i}\frac{c_i^s c_j}{c_j-c_i}. \quad (2.4)$$

The first term is $\sum_{i=1}^{g-1}\prod_{j\neq 1}\frac{c_j}{c_j-c_i}=1$, because of (a).

The next $g-2$ terms disappear according to (b).

The next term is thus

$$\frac{(-x)^{g-1}}{(g-1)!}\sum_{i=1}^{g-1}\prod_{j\neq i}\frac{c_i^{g-1}c^j}{c_j-c_i}=-c_1\ldots c_{g-1}\frac{x^{g-1}}{(g-1)!}.$$

The r. h. s. is therefore as a first approximation

$$1-c_1\ldots c_{g-1}\frac{x^{g-1}}{(g-1)!}.$$

We thus conclude that the two approximations coincide if

$$c^{g-1}=c_1c_2\ldots c_{g-1}. \quad (2.5)$$

Now consider l_x^g in lieu of L_x^g. It is our problem to find c so that

$$\frac{(cx)^{g-1}}{(g-1)!}e^{-cx}\sim c_1\ldots c_{g-1}\sum_{i=1}^{g}e^{-c_i x}\prod_{j\neq i}\frac{1}{c_j-c_i}.$$

In the expansion of the r. h. s. the first $(g-1)$ terms disappear because of (b), and the next two terms are

$$c_1 \dots c_{g-1} \left[\frac{x^{g-1}}{(g-1)!} + \frac{x^g}{g!} (c_1 + \dots + c_g) \right] \sim \frac{c_1 \dots c_{g-1} x^{g-1}}{(g-1)!} e^{-\frac{c_1 + \dots + c_g}{g} x} .$$

Hence if we take

$$c = \frac{c_1 + \dots + c_g}{g} \tag{2.6}$$

and if $c_1, \dots, c_{g-1}$ can each be taken as approximately c, then it is this value (2.6) which makes l_x^g under assumption (II) a good approximation for the exact value given under (I).

The use of constant forces of promotion $\nu^g(s)$ has the advantage of great simplicity, but it suffers from a disadvantage which is serious in practical application. They cannot be used in a case in which all members of a certain grade have been promoted after a maximum length of time s spent in the grades. To cope with this problem, it is necessary to let $\nu^g(s)$ tend to infinity as s approaches its limit and the form $\nu^g(s) = \dfrac{c}{b-s}$ has been found useful (b and c may have different values for different grades).

We have, of course, $\lim\limits_{s=b} \nu^g(s) = \infty$. This expression leads to a simple form for p_s^g, viz.

$$p_s^g = e^{-\int_0^s \frac{c}{b-t} dt} = \left(\frac{b-s}{b}\right)^c, \text{ with } p_b^g = 0. \tag{2.7}$$

We have also

$$h_s^g = \frac{c}{b-s} \left(\frac{b-s}{b}\right)^c = d\,(b-s)^{c-1}, \text{ say} \tag{2.8}$$

4. Short outline of computation

Practical computation starts off with the values $L_x = L_s$. From these values l_x^1 is found for every x, using $l_x^1 = L_x p_x^1$. The difference between L_x and l_x^1 is $L_x - l_x^1 = L_x^2$.

Having thus obtained the values in the grades 2 and above for every x, we can calculate $l^2_{[x]}$ from (21) and then

$$L^2_{[x-s]+s} = l^2_{[x-s]}\, {}_sp_{x-s}\,.$$

We calculate now $l^2_{[x-s]+s} = L^2_{[x-s]+s}\, p^2_s$ which leads to

$$l_x = \int_0^x l^2_{[x-s]+s}\, ds\,.$$

From now on all steps are periodically repeated: first $L^2_x - l^2_x = L^3_x$ and then, again from (21), we find $l^3_{[x]}$. The succeeding steps produce $L^3_{[x-s]+s}$, $l^3_{[x-s]+s}$, l^3_x, L^4_x, ... etc. through all the grades.

If a check is desired, (7) can be used to recalculate $l^g_{[x]}$ for every grade $g = 2, 3, \ldots, k$.

The procedure outlined is based on the assumption that the form of ν (or of p or h) and all parameters involved are known. But this is not the case which arises most frequently in practice. There we are usually faced with the problem of finding the parameters, if only the form of $\nu^g(s)$ is known and a «hierarchy» is given. By this expression we mean the set of values $l^1, l^2, \ldots, l^k$, or the equivalent set, $L^1, L^2, \ldots, L^k$. We could proceed by trial and error, but for $\nu^g(s) = \dfrac{c}{b-s}$ a more satisfactory procedure has been developed.

Let us consider $l^g = \int_0^\infty L^g(s)\, p^g_s\, ds$, which follows at once from (3).

Let us further assume that $\nu^g(s) = \dfrac{c}{b-s}$ for $a \leqslant s \leqslant b$ and $= 0$ outside this range. Then

$$\begin{aligned} p^g_s &= 1 && \text{for} \quad s \leqslant a \\ &= \left(\frac{b-s}{b-a}\right)^c && \text{for} \quad a \leqslant s \leqslant b \\ &= 0 && \text{for} \quad s \geqslant b\,. \end{aligned}$$

It follows that

$$l^g = \int_0^\infty L^g(s)\, p^g_s\, ds = \int_0^a L^g(s)\, ds + \int_a^b L^g(s) \left(\frac{b-s}{b-a}\right)^c ds\,. \qquad (3.1)$$

We are concerned with finding an approximation for the second integral on the r. h. s. By one of the mean value theorems of the integral calculus we have

$$\int_a^b L^g(s)\left(\frac{b-s}{b-a}\right)^c ds = L^g(s_1)\int_a^b \left(\frac{b-s}{b-a}\right)^c ds$$

where s_1 is a value between a and b.

The last integral is equal to

$$\frac{1}{(b-a)^c}\int_a^b (b-s)^c ds = \frac{b-a}{c+1}. \qquad (3.2)$$

We have thus, from (3.1)

$$l^g = \int_0^a L^g(s)\,ds + \frac{L^g(s_1)(b-a)}{c+1}. \qquad (3.3)$$

Now if $L^g(s)$ does not vary much with s (and this is the case in many applications) $L^g(s_1)(b-a). \sim \int_a^b L^g(s)\,ds$ and formula (3.3) can be written

$$l^g \sim \int_0^a L^g(s)\,ds + \frac{1}{c+1}\int_a^b L^g(s)\,ds. \qquad (3.4)$$

Hence, if $L^g(s)$ is known, and the required l^g is given, c can be found approximately from (3.4) and all functions can be calculated as described at the beginning of this Chapter.

For the applications of the theory we are actually not so much concerned with values like $l^g_{[x]}$, $L^g_{[x-s]+s}$ and so on, but rather with the integrals of these values between certain limits of the arguments, such as $\int_y^{y+1} l^g_{[x]}\,dx$, $\int_t^{t+1}\int_y^{y+1} L^g_{[x-s]+s}\,dx\,ds$ and others. The computation, however, still proceeds on the lines described above and a numerical illustration will make the whole process clear.

5. Example

We assume that the values $\int\limits_y^{y+1} L_x dx$ are known for every integral y; and that they are as follows:

y	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\int\limits_y^{y+1} L_x dx$	921	915	909	902	894	886	877	866	855	842	828	811	791	767	741	—

The total of these numbers is $\int\limits_0^{15} L_x dx = 12{,}805$.

Then let us assume that the following hierarchy has been fixed:

Grade	1	3523
»	2	6282
»	3	3000

It is further required that promotions shall start in grade 1 after $s = 3$. By $s = 6$ all members are to have been promoted to grade 2 at least and the form of the force of promotion is to be $\nu^1(s) = \frac{c}{6-s}$ which is, in grade 1, also $= \frac{c}{6-x}$. The force of promotion from grade 2 into grade 3 is supposed to be constant, commencing at $s = 2$ and ceasing at $s = 7$. This means that all those who have not reached grade 3 after having spent 7 years in grade 2 will never be promoted.

We have first to fix the value of c in $\nu^1(s) = \frac{c}{6-s}$. This will be done by the aid of (3.4). With the present assumption this formula reads

$$3{,}523 \sim \int\limits_0^3 L^g(s)\,ds + \frac{1}{c+1} \int\limits_3^b L^g(s)\,ds$$

$$\sim 2745 + \frac{2682}{c+1}.$$

Therefore $\qquad c \sim 2.45$.

Our first step consists of calculating, for $y = 3, 4, 5$

$$\int_y^{y+1} l_x^1 dx = \int_y^{y+1} L_x p_x^1 dx = \int_y^{y+1} L_x \left(\frac{6-x}{3}\right)^{2.45} dx. \qquad (4.1)$$

This is, with sufficient accuracy,

$$\int_y^{y+1} L_x dx \int_y^{y+1} \left(\frac{6-x}{3}\right)^{2.45} dx = \int_y^{y+1} L_x dx \frac{1}{3.45} \frac{1}{3^{2.45}} \left[(6-y)^{3.45} - (5-y)^{3.45}\right].$$

Thus we have

y	0	1	2	3	4	5	6	etc.
$\int_y^{y+1} L_x dx$	921	915	909	902	894	886	877	etc.
Factor	1	1	1	.653	.194	.0196	0	
$\int_y^{y+1} l_x^1 dx$	921	915	909	589	173	17	0	
$\int_y^{y+1} L_x^2 dx$	0	0	0	313	721	869	877	etc.

The total of the third line is 3524 which is near enough to 3523. The last line is found from

$$\int_y^{y+1} L_x^2 dx = \int_y^{y+1} L_x dx - \int_y^{y+1} l_x^1 dx.$$

The numbers in grades 2 and above must now be split up according to seniorities.

Namely, we first want to find

$$\int_y^{y+1} \int_0^1 L^2_{[x-s]+s} \, ds \, dx.$$

As no promotions from grade 2 occur within the first year, this is also

$$\int_y^{y+1} \int_0^1 l^2_{[x-s]+s} \, dx \, ds.$$

We can use an argument which is analogous to the derivation of formula (21). Out of 902 members of the total population between ages 3 and 4 there will be 894 survivors after one year. Therefore out of 313 members in grade 2, in the same year group, $313 \cdot \frac{894}{902} = 310$ can be expected to survive, so that $721 - 310 = 411$ is the number of the survivors of those who have entered the grade during the last year and are now aged 4—5. In this way the following numbers of surviving members with a seniority of not more than one year («New Entrants») are found:

y	3	4	5	6
$\int_y^{y+1} L_x^2 dx$	313	721	869	877
Probability of surviving one year	$\frac{894}{902}$	$\frac{886}{894}$	$\frac{877}{886}$	
Survivors at age $y+1$ to $y+2$	310	715	860	
New Entrants into grade at age y to $y+1$	313*	411	154	17

* In this group all members are, of course, «New Entrants».

These numbers can now be carried forward by multiplying them again by their probabilities of survivorship, taken as

$$\frac{\int_{y+1}^{y+2} L_x dx}{\int_y^{y+1} L_x dx}$$

and we thus obtain the following complete table of the distribution of

Grades 2 and above

Seniority	Ages last birthday												Totals $\int_s^{s+1} L^2(s)\,ds$
	3	4	5	6	7	8	9	10	11	12	13	14	
0—1	313	411	154	17									895
1—2		310	408	152	17								887
2—3			307	404	150	16							877
3—4				304	398	149	16						867
4—5					301	393	146	16					856
5—6						297	387	144	16				844
6—7							293	381	141	15			830
7—8								287	373	137	15		812
8—9									281	365	133	14	793
9—10										274	353	129	756
10—11											266	341	607
11—12												257	257
Total $\int_y^{y+1} L_x^2\,dx$	313	721	869	877	866	855	842	828	811	791	767	741	9281

The totals in the last line are already known and thus provide a check on our computations.

We must now obtain from these figures those which relate to grade 2 alone. Consider equation

$$l^2 = \int_0^2 L^2(s)\,ds + \int_2^7 L^2(s)\,p_s^2\,ds + \int_7^{12} L^2(s)\,p_7^2\,ds$$

where $p_s^2 = e^{-(s-2)c}$ and $p_7^2 = e^{-5c}$.

By trial and error we find that $l^2 = 6282$ can be obtained by putting $c = .153$. Then

$$\int_0^2 L^2(s)\,ds = 1782$$

$$\int_7^{12} L^2(s)\,p_7^2\,ds = 3225 \cdot .46533 = 1501$$

and

$$\int_2^7 L^2(s)\,p_s^2\,ds \sim \sum_{t=2}^{6} \int_t^{t+1} L^2(s)\,ds \int_t^{t+1} e^{-(s-2)c}\,ds = \sum_{t=2}^{6} \int_t^{t+1} L^2(s)\,ds \,\frac{e^{-(t-2)c} - e^{-(t-1)c}}{.153}$$

which can be calculated as follows:

t	2	3	4	5	6	
$\int_t^{t+1} L^2(s)\,ds$	877	867	856	844	830	
$\dfrac{e^{-.153(t-2)} - e^{-.153(t-1)}}{.153}$	.92725	.79569	.68281	.58595	.50288	
$\int_t^{t+1} l^2(s)\,ds$	813	690	584	494	417	Total = 2998

The total in grade 2 alone is therefore

$$\begin{array}{r} 1782 \\ +\,2998 \\ +\,1501 \\ \hline 6281 \end{array}$$

The reducing factors shown in the penultimate line of the above table and the further factor $e^{-.153 \times 5} = .46533$ must now be applied to the table for grades 2 and above given on the previous page in order to obtain

Grade 2 only

Seniority	Ages last birthday												Totals $\int_s^{s+1} l^2(s)\,ds$
	3	4	5	6	7	8	9	10	11	12	13	14	
0—1	313	411	154	17									895
1—2		310	408	152	17								887
2—3			284	375	139	15							813
3—4				242	317	119	12						690
4—5					205	268	100	11					584
5—6						174	227	84	9				494
6—7							147	191	71	8			417
7—8								133	174	64	7		378
8—9									131	170	62	6	369
9—10										127	164	60	351
10—11											124	159	283
11—12												120	120
Total $\int_y^{y+1} l_x^2\,dx$	313	721	846	786	678	576	486	419	385	369	357	345	6281

The differences between the last lines of this table and of the previous one give the values of $\int_y^{y+1} l_x^3\,dx$ as follows:

	Ages last birthday:												Total
	3	4	5	6	7	8	9	10	11	12	13	14	
Members in Grade 3	—	→	23	91	188	279	356	409	426	422	410	396	3000

The analysis of this grade according to seniority can be done in the same way as that shown for grades 2 and above. The result is

Grade 3 (highest)

Seniority	Ages last birthday										Totals $\int_s^{s+1} l^3(s)\,ds$
	5	6	7	8	9	10	11	12	13	14	
0—1	23	68	98	93	81	59	25	6	1	—	454
1—2		23	67	97	92	80	58	24	6	1	448
2—3			23	66	95	90	78	57	24	6	439
3—4				23	65	94	88	76	55	23	424
4—5					23	64	92	86	74	53	392
5—6						22	63	90	83	71	329
6—7							22	61	87	81	251
7—8								22	59	84	165
8—9									21	57	78
9—10										20	20
Total $\int_y^{y+1} l_x^3 d_x$	23	91	188	279	356	409	426	422	410	396	3000