Some elementary researches in the mathematics of life insurance [continued]

Some Elementary Researches in the Mathematics of Life Insurance

(II)

By Osamu Sumitsuji, Osaka (Japan)

Summary

This is a continuation of the article which appeared in Volume 1, 1959. The author shows numerous approximation formulae for premiums and mathematical reserves of endowment assurances with enhanced mortality. For variable additive and multiplicative extramortality, these formulae are based on different simple arrangements.

1. In my previous paper [1]), I showed the following relations:

$$\lim_{c\to 0}\frac{P^{c}_{x\overline{n}|}-P_{x\overline{n}|}}{c}-\lim_{\beta\to 0}\frac{P^{q(1+\beta)}_{x\overline{n}|}-P_{x\overline{n}|}}{\beta}=v+\frac{d}{d\delta}P_{x\overline{n}|}, \tag{1}$$

$$\lim_{c\to 0}\frac{{}_tV^{c}_{x\overline{n}|}-{}_tV_{x\overline{n}|}}{c}-\lim_{\beta\to 0}\frac{{}_tV^{q(1+\beta)}_{x\overline{n}|}-{}_tV_{x\overline{n}|}}{\beta}=\frac{d}{d\delta}\,{}_tV_{x\overline{n}|}, \tag{2}$$

where $P^{c}_{x\overline{n}|}$, ${}_tV^{c}_{x\overline{n}|}$ are calculated on the basis of the mortality $q_{x+t}+c$ $(t=0, 1, 2, \ldots)$ and $P^{q(1+\beta)}_{x\overline{n}|}$, ${}_tV^{q(1+\beta)}_{x\overline{n}|}$ are calculated on the basis of the mortality $q_{x+t}(1+\beta)$ $(t=0, 1, 2, \ldots)$.

When c is small, from (1) and (2) we may have

$$P^{c}_{x\overline{n}|}-P_{x\overline{n}|}\sim c\left(v+\frac{d}{d\delta}P_{x\overline{n}|}+P_{x\overline{n}|}-P_{\overline{n}|}\right), \tag{3}$$

$${}_tV^{c}_{x\overline{n}|}-{}_tV_{x\overline{n}|}\sim c\left(\frac{d}{d\delta}\,{}_tV_{x\overline{n}|}+\lim_{\beta\to 0}\frac{{}_tV^{q(1+\beta)}_{x\overline{n}|}-{}_tV_{x\overline{n}|}}{\beta}\right)\sim c\,\frac{d}{d\delta}\,{}_tV_{x\overline{n}|}. \tag{4}$$

[1]) Mitteilungen der Vereinigung schweizerischer Versicherungsmathematiker, Bd. 59 (1959).

When c is not small, the following modified formulas will give better results.

$$P^c_{\overline{x n}|} - P_{\overline{x n}|} \sim c\left(v + \frac{d}{d\delta} P_{x\overline{n}|} + P_{\overline{x n}|} - P_{\overline{n}|}\right) + \frac{1}{2} c^2 \left(\frac{d^2}{d\delta^2} P_{x\overline{n}|} + \frac{d}{d\delta} P_{x\overline{n}|}\right) \sim$$

$$\sim c\left(v + \frac{d}{d\delta} P_{x\overline{n}|} + P_{\overline{x n}|} - P_{\overline{n}|}\right) + c^2\left(\frac{n}{12} + \frac{d}{d\delta} P_{x\overline{n}|}\right), \qquad (5)$$

$${}_tV^c_{x\overline{n}|} - {}_tV_{\overline{x n}|} \sim c\left(\frac{d}{d\delta}\, {}_tV_{x\overline{n}|} + \underset{\beta\to 0}{\frac{{}_tV^{q(1+\beta)}_{x\overline{n}|} - {}_tV_{x\overline{n}|}}{\beta}}\right) +$$

$$+ \frac{1}{2} c^2 \left(\frac{d^2}{d\delta^2}\, {}_tV_{\overline{x n}|} + \frac{d}{d\delta}\, {}_tV_{x\overline{n}|}\right). \qquad (6)$$

Formula (5) is derived as follows:

$$P^c_{x\overline{n}|} - P_{x\overline{n}|} = c \underset{c\to 0}{\frac{P^c_{x\overline{n}|} - P_{x\overline{n}|}}{c}} + cv \frac{\sum_0^{n-1} D_{x+t}\left({}_{t+1}V_{\overline{x n}|} - {}_{t+1}V^c_{x\overline{n}|}\right)}{N_x - N_{x+n}} \sim$$

$$\sim c \underset{c\to 0}{\frac{P^c_{x\overline{n}|} - P_{x\overline{n}|}}{c}} + c^2 v \frac{\sum_0^{n-1} D_{x+t}\left(-\frac{d}{d\delta}\, {}_{t+1}V_{x\overline{n}|}\right)}{N_x - N_{x+n}}.$$

To the second term of the right side, carrying the following relations

$$\frac{\sum_0^{n-1} D_{x+t}\left(\frac{d}{d\delta}\, {}_tV_{x\overline{n}|} + \frac{d}{d\delta} P_{x\overline{n}|}\right)}{N_x - N_{x+n}} = -\frac{1}{2}\left(\frac{d^2}{d\delta^2} P_{x\overline{n}|} - \frac{d}{d\delta} P_{x\overline{n}|}\right) \sim -\frac{n}{12},$$

$$v \frac{d}{d\delta}\, {}_{t+1}V_{x\overline{n}|} = \frac{d}{d\delta}\, {}_tV_{x\overline{n}|} + \frac{d}{d\delta} P_{x\overline{n}|} + {}_tV_{x\overline{n}|} + P_{\overline{x n}|},$$

easily we have

$$P^c_{x\overline{n}|} - P_{x\overline{n}|} \sim c\left(v + \frac{d}{d\delta} P_{x\overline{n}|} + P_{\overline{x n}|} - P_{\overline{n}|}\right) + c^2\left(\frac{n}{12} + \frac{d}{d\delta} P_{x\overline{n}|}\right).$$

To obtain (6) we use the following relation.

When $i' = i + \varDelta i$ (viz. interest changes)

$$\varDelta\, {}_tV_{x\overline{n}|} = {}_tV'_{x\overline{n}|} - {}_tV_{x\overline{n}|}$$

$$= (P'_{x\overline{n}|} - P_{x\overline{n}|})\frac{\sum_0^{t-1} D_{x+t}}{D_{x+t}} +$$

$$+ v\varDelta i \left\{ \frac{\sum_0^{t-1} D_{x+t}({}_tV_{x\overline{n}|} + P_{x\overline{n}|})}{D_{x+t}} + \frac{\sum_0^{t-1} D_{x+t}(\varDelta\, {}_tV_{x\overline{n}|} + \varDelta P_{x\overline{n}|})}{D_{x+t}} \right\}$$

$$= \frac{d}{d\delta}\, {}_tV_{x\overline{n}|}\, v\varDelta i - v^2(\varDelta i)^2 \cdot$$

$$\cdot \left\{ \frac{\sum_0^{n-1} D_{x+t}\left(\frac{d}{d\delta}\, {}_tV_{x\overline{n}|} + \frac{d}{d\delta} P_{x\overline{n}|}\right)}{N_x - N_{x+n}} \frac{\sum_0^{t-1} D_{x+t}}{D_{x+t}} - \frac{\sum_0^{t-1} D_{x+t}\left(\frac{d}{d\delta}\, {}_tV_{x\overline{n}|} + \frac{d}{d\delta} P_{x\overline{n}|}\right)}{D_{x+t}} \right\} + \cdots.$$

On the other hand, clearly

$$\varDelta\, {}_tV_{x\overline{n}|} = \frac{d}{d\delta}\, {}_tV_{x\overline{n}|}\, v\varDelta i + v^2(\varDelta i)^2 \frac{1}{2}\left(\frac{d^2}{d\delta^2}\, {}_tV_{x\overline{n}|} - \frac{d}{d\delta}\, {}_tV_{x\overline{n}|}\right) + \cdots.$$

Therefore, we have

$$\frac{\sum_0^{n-1} D_{x+t} \frac{d}{d\delta}\, {}_tV_{x\overline{n}|}}{N_x - N_{x+n}} \frac{\sum_0^{t-1} D_{x+t}}{D_{x+t}} - \frac{\sum_0^{t-1} D_{x+t} \frac{d}{d\delta}\, {}_tV_{x\overline{n}|}}{D_{x+t}} = -\frac{1}{2}\left(\frac{d^2}{d\delta^2}\, {}_tV_{x\overline{n}|} + \frac{d}{d\delta}\, {}_tV_{x\overline{n}|}\right).$$

Using this relation, we have

$$\varDelta\, {}_tV_{x\overline{n}|} = c \lim_{c\to 0} \frac{{}_tV^c_{x\overline{n}|} - {}_tV_{x\overline{n}|}}{c} - cv \left\{ \frac{\sum_0^{n-1} D_{x+t}\, \varDelta\, {}_{t+1}V_{x\overline{n}|}}{N_x - N_{x+n}} \frac{\sum_0^{t-1} D_{x+t}}{D_{x+t}} - \frac{\sum_0^{t-1} D_{x+t}\, \varDelta\, {}_{t+1}V_{x\overline{n}|}}{D_{x+t}} \right\}$$

$$\sim c\left(\frac{d}{d\delta}\, {}_tV_{x\overline{n}|} + \lim_{\beta\to 0} \frac{{}_tV^{q(1+\beta)}_{x\overline{n}|} - {}_tV_{x\overline{n}|}}{\beta}\right) + \frac{1}{2}c^2\left(\frac{d^2}{d\delta^2}\, {}_tV_{x\overline{n}|} + \frac{d}{d\delta}\, {}_tV_{x\overline{n}|}\right).$$

2. We derive some approximation formulas to $\Delta P_{x\overline{n}|} = P'_{x\overline{n}|} - P_{x\overline{n}|}$ in case of linear increasing or linear decreasing extra mortality, viz.

$$q'_{x+t} = q_{x+t} + c\frac{t}{n}, \quad q'_{x+t} = q_{x+t}\left(1+\beta\frac{t}{n}\right), \quad q'_{x+t} = q_{x+t} + c\frac{n-t}{n}$$

and $q'_{x+t} = q_{x+t}\left(1+\beta\frac{n-t}{n}\right)$.

a) First of all, we assume the new mortality such as

$$\frac{n}{n}q_x, \quad \frac{n-1}{n}q_{x+1}, \quad \ldots, \quad \frac{n-t}{n}q_{x+t}, \quad \ldots$$

and that $P'_{x\overline{n}|}$, ${}_tV'_{x\overline{n}|}$ represent new net Premium and Premium Reserve respectively.

Then we have

$$P'_{x\overline{n}|} - P_{x\overline{n}|} = \frac{\sum_0^{n-1}\frac{t}{n}C_{x+t}(1 - {}_{t+1}V'_{x\overline{n}|})}{N_x - N_{x+n}}$$

$$\sim \frac{1}{n}\frac{\sum_0^{n-1} t\,C_{x+t}(1 - {}_{t+1}V_{x\overline{n}|})}{N_x - N_{x+n}}$$

$$\sim \frac{1}{n}\frac{\sum_0^{n-1} t\,C_{x+t}\left\{1 - \frac{t+1}{n}\left(1 - \frac{n-t-1}{2}i\right)\right\}}{N_x - N_{x+n}}$$

$$\left(\text{assuming } {}_tV_{x\overline{n}|} \sim \frac{t}{n}\left(1 - \frac{n-t}{2}i\right)\right)$$

$$\left.\begin{aligned} &\sim \frac{1}{6}P'_{x\overline{n}|}\left(1 - \frac{3}{n} + \frac{n}{4}i\right) \\ &\sim \frac{1}{3}(P_{x\overline{n}|} - P_{\overline{n}|}). \end{aligned}\right\} \qquad (7)$$

b) We assume

$$q'_{x+t} = q_{x+t} + c\frac{n-t}{n},$$

$$q'_{x+t} = q_{x+t}\left(1+\beta\frac{n-t}{n}\right).$$

Clearly we have

$$\frac{P_{x\overline{n}|}^{c\frac{n-t}{n}} - P_{x\overline{n}|}}{c}\Bigg|_{c\to 0} = \frac{\sum_{0}^{n-1} \frac{n-t}{n} D_{x+t}\, v\,(1 - {}_{t+1}V_{x\overline{n}|})}{N_x - N_{x+n}}$$

$$\left(q'_{x+t} = q_{x+t} + c\,\frac{n-t}{n}\right),$$

$$\frac{P_{x\overline{n}|}^{q_{x+t}\left(1+\beta\frac{n-t}{n}\right)} - P_{x\overline{n}|}}{\beta}\Bigg|_{\beta\to 0} = \frac{\sum_{0}^{n-1} \frac{n-t}{n} C_{x+t}(1 - {}_{t+1}V_{x\overline{n}|})}{N_x - N_{x+n}}$$

$$\left(q'_{x+t} = q_{x+t}\left(1 + \beta\,\frac{n-t}{n}\right)\right).$$

Therefore,

$$\frac{P_{x\overline{n}|}^{c\frac{n-t}{n}} - P_{x\overline{n}|}}{c}\Bigg|_{c\to 0} - \frac{P_{x\overline{n}|}^{q_{x+t}\left(1+\beta\frac{n-t}{n}\right)} - P_{x\overline{n}|}}{\beta}\Bigg|_{\beta\to 0} =$$

$$= \frac{\sum_{0}^{n-1} D_{x+t+1}(1 - {}_{t+1}V_{x\overline{n}|})}{N_x - N_{x+n}} - \frac{\sum_{0}^{n-1} t\, D_{x+t+1}(1 - {}_{t+1}V_{x\overline{n}|})}{n\,(N_x - N_{x+n})}$$

$$= v + \frac{d}{d\delta} P_{x\overline{n}|} - \frac{I^2 a_{x\overline{n-1}|} + I a_{x\overline{n-1}|}}{2n(\ddot{a}_{x\overline{n}|})^2}$$

$$= \left(v + \frac{d}{d\delta} P_{x\overline{n}|}\right)\left\{1 - \frac{1}{2n}\left(\frac{I^2 a_{x\overline{n-1}|}}{I a_{x\overline{n-1}|}} - 1\right)\right\}$$

$$\sim \left(v + \frac{d}{d\delta} P_{x\overline{n}|}\right)\left(\frac{2}{3} + \frac{5}{6n} + \frac{n(i + q_{x\overline{n}|})}{36}\right)$$

$$\sim \frac{P^c_{x\overline{n}|} - P_{x\overline{n}|}}{c}\Bigg|_{c\to 0}\left(\frac{2}{3} + \frac{5}{6n} + \frac{n(i + q_{x\overline{n}|})}{36}\right) - \frac{2}{3}\,(P_{x\overline{n}|} - P_{\overline{n}|}),$$

where $I^2 a_{x\overline{n-1}|} = \dfrac{d^2}{d\delta^2}\,\ddot{a}_{x\overline{n}|}$, $\quad I a_{x\overline{n-1}|} = -\dfrac{d}{d\delta}\,\ddot{a}_{x\overline{n}|}$

and $\dfrac{I^2 a_{x\overline{n-1}|}}{I a_{x\overline{n-1}|}} \sim \dfrac{2}{3}(n-1)\left\{1 - \dfrac{n+1}{12}\,(i + q_{x\overline{n}|})\right\}.$

On the other hand, using (7) we have

$$\begin{aligned}\lim_{\beta\to 0}\frac{P_{\overline{xn}|}^{q_{x+t}\left(1+\beta\frac{n-t}{n}\right)}-P_{\overline{xn}|}}{\beta}&=\frac{\sum\limits_{0}^{n-1}\frac{n-t}{n}C_{x+t}\left(1-{}_{t+1}V_{\overline{xn}|}\right)}{N_x-N_{x+n}}\\ &\sim P_{\overline{xn}|}-P_{\overline{n}|}-\frac{1}{6}P'_{\overline{xn}|}\left(1-\frac{3}{n}+\frac{n}{4}i\right)\\ &\sim\frac{2}{3}\left(P_{\overline{xn}|}-P_{\overline{n}|}\right).\end{aligned}\tag{8}$$

Therefore, we have

$$\begin{aligned}\lim_{c\to 0}\frac{P_{\overline{xn}|}^{c\frac{n-t}{n}}-P_{\overline{xn}|}}{c}&\sim\lim_{c\to 0}\frac{P_{\overline{xn}|}^{c}-P_{\overline{xn}|}}{c}\left(\frac{2}{3}+\frac{5}{6n}+\frac{n(i+q_{\overline{xn}|})}{36}\right)\\ &\sim 0{,}73\lim_{c\to 0}\frac{P_{\overline{xn}|}^{c}-P_{\overline{xn}|}}{c}.\end{aligned}\tag{9}$$

c) We assume

$$q'_{x+t}=q_{x+t}+\frac{t}{n}c,$$

$$q'_{x+t}=q_{x+t}\left(1+\frac{t}{n}\beta\right).$$

Clearly we have

$$\lim_{c\to 0}\frac{P_{\overline{xn}|}^{c\frac{n-t}{n}}-P_{\overline{xn}|}}{c}-\lim_{c\to 0}\frac{P_{\overline{xn}|}^{c\frac{t}{n}}-P_{\overline{xn}|}}{c}=\lim_{c\to 0}\frac{P_{\overline{xn}|}^{c}-P_{\overline{xn}|}}{c}$$

and

$$\lim_{\beta\to 0}\frac{P_{\overline{xn}|}^{q_{x+t}\left(1+\beta\frac{n-t}{n}\right)}-P_{\overline{xn}|}}{\beta}-\lim_{\beta\to 0}\frac{P_{\overline{xn}|}^{q_{x+t}\left(1+\beta\frac{t}{n}\right)}-P_{\overline{xn}|}}{\beta}=\lim_{\beta\to 0}\frac{P_{\overline{xn}|}^{q(1+\beta)}-P_{\overline{xn}|}}{\beta}.$$

Therefore, using (9) and (8) we have

$$\begin{aligned}\lim_{c\to 0}\frac{P_{\overline{xn}|}^{c\frac{t}{n}}-P_{\overline{xn}|}}{c}&\sim\lim_{c\to 0}\frac{P_{\overline{xn}|}^{c}-P_{\overline{xn}|}}{c}\left(\frac{1}{3}-\frac{5}{6n}-\frac{n(i+q_{\overline{xn}|})}{36}\right)\\ &\sim 0{,}27\lim_{c\to 0}\frac{P_{\overline{xn}|}^{c}-P_{\overline{xn}|}}{c}.\end{aligned}\tag{10}$$

$$\underset{\beta\to 0}{\frac{P_{x\overline{n}|}^{q_{x+t}\left(1+\frac{t}{n}\beta\right)}-P_{x\overline{n}|}}{\beta}} \sim \frac{1}{6}P'_{x\overline{n}|}\left(1-\frac{3}{n}+\frac{n}{4}i\right)$$

$$\sim \frac{1}{3}\left(P_{x\overline{n}|}-P_{\overline{n}|}\right). \qquad (11)$$

d) From *c)* we have

$$\underset{c\to 0}{\frac{P_{x\overline{n}|}^{c^{\frac{n-t}{n}}}-P_{x\overline{n}|}}{c}}-\underset{\beta\to 0}{\frac{P_{x\overline{n}|}^{q_{x+t}\left(1+\beta\frac{n-t}{n}\right)}-P_{x\overline{n}|}}{\beta}}+\underset{c\to 0}{\frac{P_{x\overline{n}|}^{c^{\frac{t}{n}}}-P_{x\overline{n}|}}{c}}-\underset{\beta\to 0}{\frac{P_{x\overline{n}|}^{q_{x+t}\left(1+\beta\frac{t}{n}\right)}-P_{x\overline{n}|}}{\beta}}$$

$$=v+\frac{d}{d\delta}P_{x\overline{n}|}.$$

Thus the relation (1) is divisible into two parts, viz. into the relations to linear increasing and to decreasing extra mortalities.

3. *a)* As to Premium Reserve we have

$$\underset{c\to 0}{\frac{{}_tV_{x\overline{n}|}^{c^{\frac{n-t}{n}}}-{}_tV_{x\overline{n}|}}{c}}=\underset{c\to 0}{\frac{P_{x\overline{n}|}^{c^{\frac{n-t}{n}}}-P_{x\overline{n}|}}{c}}\,\frac{\sum\limits_0^{t-1}D_{x+t}}{D_{x+t}}-\frac{\sum\limits_0^{t-1}\frac{n-t}{n}D_{x+t}v\left(1-{}_{t+1}V_{x\overline{n}|}\right)}{D_{x+t}}, \qquad \text{(A)}$$

$$\underset{c\to 0}{\frac{{}_tV_{x\overline{n}|}^{c^{\frac{t}{n}}}-{}_tV_{x\overline{n}|}}{c}}=\underset{c\to 0}{\frac{P_{x\overline{n}|}^{c^{\frac{t}{n}}}-P_{x\overline{n}|}}{c}}\,\frac{\sum\limits_0^{t-1}D_{x+t}}{D_{x+t}}-\frac{\sum\limits_0^{t-1}\frac{t}{n}D_{x+t}v\left(1-{}_{t+1}V_{x\overline{n}|}\right)}{D_{x+t}}, \qquad \text{(B)}$$

$$\underset{\beta\to 0}{\frac{{}_tV_{x\overline{n}|}^{q_{x+t}\left(1+\beta\frac{n-t}{n}\right)}-{}_tV_{x\overline{n}|}}{\beta}}=\underset{\beta\to 0}{\frac{P^{q_{x+t}\left(1+\beta\frac{n-t}{n}\right)}-P_{x\overline{n}|}}{\beta}}\,\frac{\sum\limits_0^{t-1}D_{x+t}}{D_{x+t}}-\frac{\sum\limits_0^{t-1}\frac{n-t}{n}C_{x+t}(1-{}_{t+1}V_{x\overline{n}|}}{D_{x+t}}, \qquad \text{(C)}$$

$$\underset{\beta\to 0}{\frac{{}_tV_{x\overline{n}|}^{q_{x+t}\left(1+\beta\frac{t}{n}\right)}-{}_tV_{x\overline{n}|}}{\beta}}=\underset{\beta\to 0}{\frac{P_{x\overline{n}|}^{q_{x+t}\left(1+\beta\frac{t}{n}\right)}-P_{x\overline{n}|}}{\beta}}\,\frac{\sum\limits_0^{t-1}D_{x+t}}{D_{x+t}}-\frac{\sum\limits_0^{t-1}\frac{t}{n}C_{x+t}\left(1-{}_{t+1}V_{x\overline{n}|}\right)}{D_{x+t}}. \qquad \text{(D)}$$

Therefore, we have

$$(A)+(B)-\{(C)+(D)\}=\frac{d}{d\delta}\,{}_tV_{x\overline{n}|}$$

or

$$(A)+(B)=\frac{d}{d\delta}\,{}_tV_{x\overline{n}|}+\underset{\beta\to 0}{\frac{{}_tV_{x\overline{n}|}^{q(1+\beta)}-{}_tV_{x\overline{n}|}}{\beta}},$$

This is nothing else but the already mentioned relation (2).

b) Now we show some approximations to $\Delta_t V_{x\overline{n}|}$ in case of linear decreasing or linear increasing extra mortality.

1. Easily we have

$$\lim_{c\to 0}\frac{{}_tV^c_{x\overline{n}|}-{}_tV_{x\overline{n}|}}{c}=\lim_{c\to 0}\frac{P^c_{x\overline{n}|}-P_{x\overline{n}|}}{c}\,\frac{\sum\limits_{0}^{t-1}D_{x+t}}{D_{x+t}}-\frac{\sum\limits_{0}^{t-1}D_{x+t}\,v(1-{}_{t+1}V_{x\overline{n}|})}{D_{x+t}}$$

$$\sim-\lim_{c\to 0}\frac{P^c_{x\overline{n}|}-P_{x\overline{n}|}}{c}\,\ddot{a}_{x\overline{t}|}\,\frac{D_x}{D_{x+t}}\left(1-\frac{t}{n}\right).$$

In the same way we may write

$$\lim_{c\to 0}\frac{{}_tV^{c\frac{n-t}{n}}_{x\overline{n}|}-{}_tV_{x\overline{n}|}}{c}\sim 2\lim_{c\to 0}\frac{P^{c\frac{n-t}{n}}_{x\overline{n}|}-P_{x\overline{n}|}}{c}\,\ddot{a}_{x\overline{t}|}\,\frac{D_x}{D_{x+t}}\left(1-\frac{t}{n}\right)\left(1-\frac{t}{2n}\right)$$

$$\sim\lim_{c\to 0}\frac{{}_tV^c_{x\overline{n}|}-{}_tV_{x\overline{n}|}}{c}\,0{,}73\cdot 2\left(1-\frac{t}{2n}\right)$$

$$\sim 1{,}46\left(1-\frac{t}{2n}\right)\lim_{c\to 0}\frac{{}_tV^c_{x\overline{n}|}-{}_tV_{x\overline{n}|}}{c}, \tag{12}$$

$$\lim_{c\to 0}\frac{{}_tV^{c\frac{t}{n}}_{x\overline{n}|}-{}_tV_{x\overline{n}|}}{c}\sim-0{,}73\left(0{,}63-\frac{t}{n}\right)\lim_{c\to 0}\frac{{}_tV^c_{x\overline{n}|}-{}_tV_{x\overline{n}|}}{c}, \tag{13}$$

where $\frac{2}{3}+\frac{5}{6n}+\frac{n(i+q_{x\overline{n}|})}{36}\sim 0{,}73$ (see (9)).

When t is small, particularly when $t=1$ and $t=2$, in (12) and in (13) it is better to replace $\frac{t}{2n}$ and $\frac{t}{n}$ by $\frac{t-1}{2n}$ and $\frac{t-1}{n}$ respectively.

2. Another rough but fancy approximation to

$$\lim_{c\to 0}\frac{{}_tV^{c\frac{n-t}{n}}_{x\overline{n}|}-{}_tV_{x\overline{n}|}}{c}$$

is obtained as follows:

$$\lim_{c\to0}\frac{{}_tV_{x\overline{n}|}^{c^{\frac{t}{n}}} - {}_tV_{x\overline{n}|}}{c} = \lim_{c\to0}\frac{P_{x\overline{n}|}^{c^{\frac{t}{n}}} - P_{x\overline{n}|}}{c}\,\frac{\sum_0^{t-1} D_{x+t}}{D_{x+t}} - \frac{\sum_0^{t-1} t\, D_{x+t}\, v(1-{}_{t+1}V_{x\overline{n}|})}{n D_{x+t}}$$

$$= \frac{I^2 a_{x\overline{n-1}|} - I a_{x\overline{n-1}|}}{2n(\ddot{a}_{x\overline{n}|})^2}\,\ddot{a}_{x\overline{t}|}\,\frac{D_x}{D_{x+t}} -$$

$$-\left\{\frac{I^2 a_{x\overline{n-1}|} - I a_{x\overline{n-1}|}}{2n(\ddot{a}_{x\overline{n}|})}\,\frac{D_x}{D_{x+t}} - t\,\frac{I a_{x+t\,\overline{n-t}|}}{n\,\ddot{a}_{x\overline{n}|}} - \frac{I^2 a_{x+t\,\overline{n-t-1}|} - I a_{x+t\,\overline{n-t-1}|}}{2n\,\ddot{a}_{x\overline{n}|}}\right\} +$$

$$+ \lim_{\beta\to0}\frac{{}_tV_{x\overline{n}|}^{q_{x+t}\left(1+\beta\frac{t}{n}\right)} - {}_tV_{x\overline{n}|}}{\beta}$$

$$= -\frac{\ddot{a}_{x+t\,\overline{n-t}|}}{\ddot{a}_{x\overline{n}|}}\left(\frac{I^2 a_{x\overline{n-1}|} - I a_{x\overline{n-1}|}}{2n\,\ddot{a}_{x\overline{n}|}} - \frac{I^2 a_{x+t\,\overline{n-t-1}|} + (2t-1)\, I a_{x+t\,\overline{n-t-1}|}}{2n\,\ddot{a}_{x+t\,\overline{n-t}|}}\right) +$$

$$+ \lim_{\beta\to0}\frac{{}_tV_{x\overline{n}|}^{q_{x+t}\left(1+\frac{t}{n}\beta\right)} - {}_tV_{x\overline{n}|}}{\beta},$$

roughly (see Appendix), when n is not large

$$\sim \frac{1}{n}\,\frac{d^2}{d\delta^2}\,{}_tV_{x\overline{n}|} + \lim_{\beta\to0}\frac{{}_tV_{x\overline{n}|}^{q_{x+t}\left(1+\frac{t}{n}\beta\right)} - {}_tV_{x\overline{n}|}}{\beta}$$

$$\sim \frac{1}{n}\,\frac{d^2}{d\delta^2}\,{}_tV_{x\overline{n}|}\,. \tag{14}$$

Therefore, we may have, though roughly, when n is not large

$$\lim_{c\to0}\frac{{}_tV_{x\overline{n}|}^{c^{\frac{n-t}{n}}} - {}_tV_{x\overline{n}|}}{c} \sim \frac{d}{d\delta}\,{}_tV_{x\overline{n}|} - \frac{1}{n}\,\frac{d^2}{d\delta^2}\,{}_tV_{x\overline{n}|}\,. \tag{15}$$

(14) and (15) are of no practical use, because they are rough and troublesome.

But from them we can easily see the outlines of the map of

$$\lim_{c\to0}\frac{{}_tV_{x\overline{n}|}^{c^{\frac{t}{n}}} - {}_tV_{x\overline{n}|}}{c} \quad\text{and}\quad \lim_{c\to0}\frac{{}_tV_{x\overline{n}|}^{c^{\frac{n-t}{n}}} - {}_tV_{x\overline{n}|}}{c} \quad (t = 1, 2, \ldots, n),$$

at least, of their signs–plus or minus (see Appendix).

3. It is not easy to derive the practicable approximation to

$$\frac{{}_tV_{\overline{xn}|}^{q_{x+t}\left(1+\beta\frac{t}{n}\right)} - {}_tV_{\overline{xn}|}}{\beta}\Bigg|_{\beta\to 0}.$$

$$\frac{{}_tV_{\overline{xn}|}^{q_{x+t}\left(1+\frac{t}{n}\beta\right)} - {}_tV_{\overline{xn}|}}{\beta}\Bigg|_{\beta\to 0} = \frac{\sum_0^{n-1} t\,C_{x+t}(1-{}_{t+1}V_{\overline{xn}|})}{n(N_x-N_{x+n})}\,\frac{\sum_0^{t-1} D_{x+t}}{D_{x+t}} - \frac{\sum_0^{t-1} t\,C_{x+t}(1-{}_{t+1}V_{\overline{xn}|})}{n\,D_{x+t}}$$

$$\sim\left\{\frac{P'_{\overline{xn}|}}{6}\left(1-\frac{3}{n}+\frac{n}{4}i\right)-P'_{\overline{xt}|}\left[\frac{t}{2n}-\frac{t^2}{3n^2}-\frac{1}{2n}+\frac{t^2 i}{24n}\left(4-3\frac{t}{n}\right)\right]\right\}\frac{\sum_0^{t-1} D_{x+t}}{D_{x+t}}$$

(see (7)), (16)

or

$$\sim\left\{P'_{\overline{xt}|}\frac{1}{n}\frac{d^2}{d\delta^2}\,{}_tV_{\overline{xn}|}+t\,\frac{P'_{\overline{xn}|}-P'_{\overline{xt}|}}{6}\left(1-\frac{3}{n}\right)+\frac{ti}{24}\left[n\,P'_{\overline{xn}|}-t\,P'_{\overline{xt}|}\left(6-8\frac{t}{n}+3\frac{t^2}{n^2}\right)\right]\right\}\cdot\frac{\ddot{a}_{\overline{xt}|}}{t}\,\frac{D_x}{D_{x+t}}$$

$$\sim\left(P'_{\overline{xt}|}\frac{1}{n}\frac{d^2}{d\delta^2}\,{}_tV_{\overline{xn}|}+z(x,n,t,i)\right)\frac{\ddot{a}_{\overline{xt}|}}{t}\,\frac{D_x}{D_{x+t}}, \qquad (17)$$

where usually $z(x,n,t,i) \geqq 0$.

(16) may be of practical use when t is not large, about when

$$t < \frac{2}{3}n.$$

(17) is of no practical use, but from it we can easily see the outlines of the map of

$$\frac{{}_tV_{\overline{xn}|}^{q_{x+t}\left(1+\frac{t}{n}\beta\right)} - {}_tV_{\overline{xn}|}}{\beta}\Bigg|_{\beta\to 0}$$

($t = 1, 2, \ldots, n$), at least, of their Signs–almost all of them are plus (see Appendix), with the exception of $t \sim n-1$.

When n is small, the following approximation (20) is simpler than the formula (16).

From (9) we may write roughly, when c is small

$$\varDelta P_{x\overline{n}|}^{\frac{n-t}{n}c} = P_{x\overline{n}|}^{\frac{n-t}{n}c} - P_{x\overline{n}|} \sim \frac{2}{3}\varDelta P_{x\overline{n}|}^{c}.$$

In the same way we may have

$$\varDelta P_{x\overline{n}|}^{\frac{n^2-t^2}{n^2}c} \sim \frac{5}{6}\varDelta P_{x\overline{n}|}^{c},$$

$$\varDelta P_{x\overline{n}|}^{\frac{n^2-t^2}{n^2}cf} \sim \frac{5}{6}f\varDelta P_{x\overline{n}|}^{c} \quad \text{(when } cf \text{ is small).}$$

From (12) we may write roughly, when c is small

$$\varDelta_t V_{x\overline{n}|}^{\frac{n-t}{n}c} \sim \frac{4}{3}\left(1-\frac{t}{2n}\right)\varDelta_t V_{x\overline{n}|}^{c} \sim \frac{8}{6}\left(1-\frac{t}{2n}\right)c\frac{d}{d\delta}\,{}_tV_{x\overline{n}|}.$$

In the same way we may have,

$$\varDelta_t V_{x\overline{n}|}^{\frac{n^2-t^2}{n^2}c} \sim \frac{7}{6}\left(1+\frac{1}{7}\frac{t}{n}-\frac{3}{7}\frac{t^2}{n^2}\right)c\frac{d}{d\delta}\,{}_tV_{x\overline{n}|},$$

$$\varDelta_t V_{x\overline{n}|}^{\frac{n^2-t^2}{n^2}cf} \sim \frac{7}{6}\left(1+\frac{1}{7}\frac{t}{n}-\frac{3}{7}\frac{t^2}{n^2}\right)cf\frac{d}{d\delta}\,{}_tV_{x\overline{n}|}$$

$$\sim \frac{7}{6}cf\frac{d}{d\delta}\,{}_tV_{x\overline{n}|}$$

$$\left(\text{when } cf \text{ is small and } \frac{t}{n} \lesssim \frac{2}{3}\right).$$

Therefore

$$\varDelta_t V_{x\overline{n}|}^{\frac{n-t}{n}c+\frac{n^2-t^2}{n^2}cf} \sim \frac{c}{6}\left(8-\frac{4t}{n}+7f\right)\frac{d}{d\delta}\,{}_tV_{x\overline{n}|}, \quad \left(\frac{t}{n} \lesssim \frac{2}{3}\right).$$

Consequently, when c and cf are small

$$\varDelta_t V_{x\overline{n}|}^{\frac{t}{n}c+\frac{t^2}{n^2}cf} \sim c(1+f)\frac{d}{d\delta}\,{}_tV_{x\overline{n}|} - \frac{c}{6}\left(8-\frac{4t}{n}+7f\right)\frac{d}{d\delta}\,{}_tV_{x\overline{n}|}$$

$$\sim \frac{c}{3}\left(1-\frac{2t}{n}+\frac{f}{2}\right)\left|\frac{d}{d\delta}\,{}_tV_{x\overline{n}|}\right|, \qquad \left(\frac{t}{n} \lesssim \frac{2}{3}\right). \quad (18)$$

Now when n is small, we may assume roughly for small β

$$q'_{x+t} = q_{x+t}\left(1+\frac{t}{n}\beta\right)$$

$$\sim q_{x+t}+\frac{t}{n}\beta q_x+\frac{t^2}{n^2}\beta(q_{x+n}-q_x)\,. \tag{19}$$

Therefore using (18) easily we have, when β is small

$$\varDelta_t V_{x\overline{n}|}^{q_{x+t}\left(1+\frac{t}{n}\beta\right)} \sim \frac{\beta}{3}q_x\left(1-\frac{2t}{n}+\frac{q_{x+n}-q_x}{2q_x}\right)\left|\frac{d}{d\delta}\,{}_tV_{x\overline{n}|}\right|,\quad \left(\frac{t}{n}\lesssim\frac{2}{3}\right). \tag{20}$$

The approximation (20) may be useful in case of small n, and also useful for the rough estimation of $\varDelta_t V_{x\overline{n}|}^{q_{x+t}\left(1+\frac{t}{n}\beta\right)}$ for small β, where n is moderate and $\frac{t}{n}\,n$ is small–about $\frac{t}{n}\lesssim\frac{2}{3}$.

Appendix

1. $$\frac{d}{d\delta}\,{}_tV_{x\overline{n}|} = -\frac{\ddot{a}_{x+t\,\overline{n-t}|}}{\ddot{a}_{x\overline{n}|}}\left(\frac{Ia_{x\overline{n-1}|}}{\ddot{a}_{x\overline{n}|}}-\frac{Ia_{x+t\,\overline{n-t-1}|}}{\ddot{a}_{x+t\,\overline{n-t}|}}\right)\sim-\frac{t}{2}\left(1-i\frac{2n-t}{6}\right)(1-{}_tV_{x\overline{n}|}),$$

2. $$\frac{d^2}{d\delta^2}\,{}_tV_{x\overline{n}|} = -\frac{\ddot{a}_{x+t\,\overline{n-t}|}}{\ddot{a}_{x\overline{n}|}}\cdot$$

$$\cdot\left\{\frac{I^2a_{x+t\,\overline{n-t-1}|}}{\ddot{a}_{x+t\,\overline{n-t}|}}-2\frac{Ia_{x+t\,\overline{n-t-1}|}}{\ddot{a}_{x+t\,\overline{n-t}|}}\frac{Ia_{x\overline{n-1}|}}{\ddot{a}_{x\overline{n}|}}+2\left(\frac{Ia_{x\overline{n-1}|}}{\ddot{a}_{x\overline{n}|}}\right)^2-\frac{I^2a_{x\overline{n-1}|}}{\ddot{a}_{x\overline{n}|}}\right\}\sim$$

$$\sim\frac{t}{6}\left\{n-2t+\frac{t(2n-t)}{2}i\left(1-\frac{2n-t}{12}i\right)\right\}(1-{}_tV_{x\overline{n}|}),$$

where

$$Ia_{x\overline{n-1}|} = -\frac{d}{d\delta}\ddot{a}_{x\overline{n}|} = \frac{1}{D_x}(S_{x+1}-S_{x+n}-(n-1)N_{x+n}),$$

$$I^2a_{x\overline{n-1}|} = \frac{d^2}{d\delta^2}\ddot{a}_{x\overline{n}|} = \frac{1}{D_x}\left\{\sum_1^{n-1}S_{x+t}+\sum_1^{n-2}S_{x+t+1}-(2n-3)S_{x+n}-(n-1)^2N_{x+n}\right\}.$$

3. Usually $\frac{d}{d\delta}\,{}_tV_{x\overline{n}|} < 0$, and as to $\frac{d^2}{d\delta^2}\,{}_tV_{x\overline{n}|}$ we may have

$$\frac{d^2}{d\delta^2}\,{}_tV_{x\overline{n}|} > 0 \text{ when } T > t > 0,$$

$$\frac{d^2}{d\delta^2}\,{}_tV_{x\overline{n}|} < 0 \text{ when } n > t > T, \quad \text{where } T \sim 0{,}8n$$

in other words, ${}_tV_{x\overline{n}|}$ is convex in respect to i when $T > t > 0$ and after them in $[T < t < n]$ becomes concave in respect to i.

Zusammenfassung

Es handelt sich um eine Fortsetzung der in Heft 1, 1959, erschienenen Abhandlung. Verfasser gibt zahlreiche Näherungsformeln für Prämie und Reserve der gemischten Versicherung bei erhöhter Sterblichkeit, wobei für veränderliche additive und multiplikative Übersterblichkeit verschiedene gesetzmässig einfache Ansätze behandelt werden.

Résumé

Il s'agit de la suite de l'étude parue dans le cahier nº 1, 1959. L'auteur donne plusieurs formules approximatives de primes et de réserves mathématiques pour les assurances mixtes à mortalité élevée. Pour les surmortalités variables additives et multiplicatives, ces formules reposent sur diverses hypothèses fonctionnelles simples.

Riassunto

Si tratta della continuazione del lavoro apparso nel fascicolo 1, 1959. L'autore dà numerose formole approssimative per premi e riserve matematiche di assicurazioni miste con mortalità elevata. Per le supermortalità variabili additive e multiplicative, queste formole si basano su diverse impostazioni funzionali semplici.