# Some elementary researches in the mathematics of life insurance [continued]

Autor(en): **Sumitsuji, Osamu**

Objekttyp: **Article**

Zeitschrift: **Mitteilungen / Vereinigung Schweizerischer Versicherungsmathematiker = Bulletin / Association des Actuaires Suisses = Bulletin / Association of Swiss Actuaries**

Band (Jahr): **61 (1961)**

PDF erstellt am: **23.05.2024**

Persistenter Link: https://doi.org/10.5169/seals-966744

# Some Elementary Researches in the Mathematics of Life Insurance (III)

*By Osamu Sumitsuji, Osaka (Japan)*

## Summary

The author gives further explications to his previous papers dealing with actuarial studies in the domain of substandard life risks which appeared in Volumes 1, 1959 and 2, 1960. He derives approximation formulae for premiums and mathematical reserves in case of endowment assurances with incresasing and decreasing extramortality. Several numerical examples show the good degree of approximation of these formulae.

§ 1. In my earlier paper [1], I showed the following approximation formula for Premium of Endowment with linear decreasing extra mortality.

$$\Delta P_{x\overline{n}|} = P_{x\overline{n}|}^{\left(1+\frac{n-t}{n}\beta\right) q_{x+t}} - P_{x\overline{n}|}$$

$$= \frac{\beta}{n} \frac{\sum_{t=0}^{n-1} (n-t)\, C_{x+t} \left(1 - {}_{t+1}V_{x\overline{n}|}^{\left(1+\frac{n-t}{n}\beta\right) q_{x+t}}\right)}{N_x - N_{x+n}}$$

$$\sim \frac{\beta}{n} \frac{\sum_{t=0}^{n-1} (n-t)\, C_{x+t} \left\{1 - \frac{t+1}{n}\left(1 - \frac{n-t-1}{2} i\right)\right\}}{N_x - N_{x+n}}$$

$$\sim \beta \left\{ P_{x\overline{n}|}^{(1+1)\, q_{x+t}} - P_{x\overline{n}|} - \frac{P^1_{x\overline{n}|}}{6}\left(1 - \frac{3}{n} + \frac{n}{4} i\right)\right\}$$

$$\sim \beta \frac{2}{3} (P_{x\overline{n}|} - P_{\overline{n}|}) \left(1 + \frac{1}{n} - \frac{n}{24} i\right) \tag{1}$$

---

[1]) Mitteilungen der Vereinigung schweizerischer Versicherungsmathematiker, Bd. 60, 2.

or roughly $$\sim \beta\, \tfrac{2}{3}\,(P_{x\overline{n}|} - P_{\overline{n}|})\,.$$

In the same way, we may have

$$\Delta P_{x+t\overline{n-t}|} = P_{x+t\overline{n-t}|}^{\left(1+\frac{n-t}{n}\beta\right) q_{x+t}} - P_{x+t\overline{n-t}|}$$

$$= \frac{\beta}{n} \frac{\sum_{s=0}^{n-t-1} (n-t-s)\, C_{x+t+s} \left(1 - {}_{s+1}V_{x+t\overline{n-t}|}^{\left(1+\frac{n-t}{n}\beta\right) q_{x+t}}\right)}{N_{x+t} - N_{x+n}}$$

$$\sim \beta\, \frac{2}{3}\, \frac{n-t}{n}\, (P_{x+t\overline{n-t}|} - P_{\overline{n-t}|}) \left(1 + \frac{1}{n-t} - \frac{n-t}{24}\, i\right). \quad (2)$$

Now we assume the new mortality such as

$$\left(1+\frac{k}{k}\beta\right) q_x, \quad \left(1+\frac{k-1}{k}\beta\right) q_{x+1}, \quad \ldots, \quad \left(1+\frac{k-t}{k}\beta\right) q_{x+t}, \quad \ldots,$$

$$q_{x+k}, \quad q_{x+k+1}, \quad \ldots,$$

where $k < n$ (in practice $k$ may be 5 or 10) and $\beta$ is a small constant.

An approximation to $\Delta P_{x\overline{n}|}$ is derived as follows:

$$\Delta P_{x\overline{n}|} = P_{x\overline{n}|}^{\left(1+\frac{k-t}{k}\beta\right) q_{x+t}} - P_{x\overline{n}|}$$

$$= \frac{\beta}{k} \frac{\sum_{t=0}^{k-1} (k-t)\, C_{x+t} \left(1 - {}_{t+1}V_{x\overline{n}|}^{\left(1+\frac{k-t}{k}\beta\right) q_{x+t}}\right)}{N_x - N_{x+n}}$$

$$= \frac{\beta}{k} \frac{\sum_{t=0}^{k-1} (k-t)\, C_{x+t} \left\{1 - {}_{t+1}V_{x\overline{k}|}^{\left(1+\frac{k-t}{k}\beta\right) q_{x+t}} + \left({}_{t+1}V_{x\overline{k}|}^{\left(1+\frac{k-t}{k}\beta\right) q_{x+t}} - {}_{t+1}V_{x\overline{n}|}^{\left(1+\frac{k-t}{k}\beta\right) q_{x+t}}\right)\right\}}{N_x - N_{x+n}}$$

using the approximation

$${}_{t+1}V_{x\overline{n}|}^{\left(1+\frac{k-t}{k}\beta\right) q_{x+t}} \sim \frac{t+1}{n} \left(1 - \frac{n-t-1}{2}\, i\right),$$

we have

$$P_{x\overline{n}|}^{\left(1+\frac{k-t}{k}\beta\right) q_{x+t}} - P_{x\overline{n}|} \sim \beta\, \frac{\ddot{a}_{x\overline{k}|}}{\ddot{a}_{x\overline{n}|}} \left\{\frac{2}{3} (P_{x\overline{k}|} - P_{\overline{k}|}) \left(1 + \frac{1}{k} - \frac{k}{24}\, i\right)\right.$$

$$\left. + \frac{P^1_{x\overline{k}|}}{6}\, \frac{n-k}{nk}\, \frac{(k+1)(k+2)}{k} \left(1 + \frac{k+1}{4}\, i\right)\right\}. \quad (3)$$

In the same way, we may write for $t < k$

$$\Delta P_{x+t\overline{n-t|}} = P_{x+t\overline{n-t|}}^{\left(1+\frac{k-t}{k}\beta\right)q_{x+t}} - P_{x+t\overline{n-t|}}$$

$$\sim \beta\frac{k-t}{k}\frac{\ddot{a}_{x+t\overline{k-t|}}}{\ddot{a}_{x+t\overline{n-t|}}}\left\{\frac{2}{3}(P_{x+t\overline{k-t|}} - P_{\overline{k-t|}})\left(1+\frac{1}{k-t}-\frac{k-t}{24}i\right)\right.$$

$$\left.+\frac{P^1_{x+t\overline{k-t|}}}{6}\frac{n-k}{(n-t)(k-t)}\frac{(k-t+1)(k-t+2)}{k-t}\left(1+\frac{k-t+1}{4}i\right)\right\}. \tag{4}$$

and for $t \geqq k$ clearly

$$\Delta P_{x+t\overline{n-t|}} = 0. \tag{5}$$

§ 2. In case of the change of mortality, easily we have

$$\Delta\, {}_tV_{x\overline{n|}} = {}_tV'_{x\overline{n|}} - {}_tV_{x\overline{n|}}$$

$$= (1-{}_tV_{x\overline{n|}})\left(1-\frac{\dfrac{\ddot{a}_{x\overline{n|}}}{\ddot{a}'_{x\overline{n|}}}}{\dfrac{\ddot{a}_{x+t\overline{n-t|}}}{\ddot{a}'_{x+t\overline{n-t|}}}}\right)$$

$$= (1-{}_tV_{x\overline{n|}})\left(1-\frac{1+\Delta P_{x\overline{n|}}\,\ddot{a}_{x\overline{n|}}}{1+\Delta P_{x+t\overline{n-t|}}\,\ddot{a}_{x+t\overline{n-t|}}}\right), \tag{6}$$

where

$$\Delta P_{x+t\overline{n-t|}} = P'_{x+t\overline{n-t|}} - P_{x+t\overline{n-t|}}, \quad (t = 0, 1, 2, \ldots).$$

Carrying (1) and (2) to (6) we have an approximation formule to

$$\Delta\, {}_tV_{x\overline{n|}} = {}_tV_{x\overline{n|}}^{\left(1+\frac{n-t}{n}\beta\right)q_{x+t}} - {}_tV_{x\overline{n|}}.$$

$$\Delta\, {}_tV_{x\overline{n|}} \sim (1-{}_tV_{x\overline{n|}})\left[1-\frac{1+\beta\frac{2}{3}(P_{x\overline{n|}}-P_{\overline{n|}})(1+G(n))\,\ddot{a}_{x\overline{n|}}}{1+\beta\frac{2}{3}\frac{n-t}{n}(P_{x+t\overline{n-t|}}-P_{\overline{n-t|}})(1+G(n-t))\,\ddot{a}_{x+t\overline{n-t|}}}\right], \tag{7}$$

where $G(n-t) = \dfrac{1}{n-t} - \dfrac{n-t}{24}i$, $(t = 0, 1, 2, \ldots)$.

Using (3) and (4) or (5), we have from (6)

(i) for $t < k$

$$
{}_tV_{x\overline{n}|}^{\left(1+\frac{k-t}{k}\beta\right)q_{x+t}} - {}_tV_{x\overline{n}|} \sim (1 - {}_tV_{x\overline{n}|})
$$

$$
\cdot\left[1-\frac{1+\beta\left\{\frac{2}{3}(P_{x\overline{k}|}-P_{\overline{k}|})(1+G(k))+\frac{P^1_{x\overline{k}|}}{6}H(n,k)\right\}\ddot{a}_{x\overline{k}|}}{1+\beta\frac{k-t}{k}\left\{\frac{2}{3}(P_{x+t\overline{k-t}|}-P_{\overline{k-t}|})(1+G(k-t))+\frac{P^1_{x+t\overline{k-t}|}}{6}H(n-t,k-t)\right\}\ddot{a}_{x+t\overline{k-t}|}}\right], \tag{8}
$$

where

$$
H(n-t,k-t) = \frac{n-k}{(n-t)(k-t)}\frac{(k-t+1)(k-t+2)}{k-t}\left(1+\frac{k-t+1}{4}i\right),
$$

$$(t = 0,\ 1,\ 2,\ \ldots);$$

and

(ii) for $t \geqq k$

$$
\begin{aligned}
\varDelta\,{}_tV_{x\overline{n}|} &= -(1-{}_tV_{x\overline{n}|})\,\varDelta P_{x\overline{n}|}\,\ddot{a}_{x\overline{n}|} \\
&\sim -\beta(1-{}_tV_{x\overline{n}|})\left\{\frac{2}{3}(P_{x\overline{k}|}-P_{\overline{k}|})(1+G(k))+\frac{P^1_{x\overline{k}|}}{6}H(n,k)\right\}\ddot{a}_{x\overline{k}|}.
\end{aligned} \tag{9}
$$

From (9) we see clearly that when $t \geqq k$, the value of $\varDelta\,{}_tV_{x\overline{n}|}/1-{}_tV_{x\overline{n}|}$ is in-dependent of $t$.

§ 3. In case of linear increasing extra mortality viz.

$$
q'_{x+t} = \left(1+\frac{t}{n}\beta\right)q_{x+t},\ \ (t = 0, 1, 2, \ldots),
$$

I showed in my earlier paper an approximation

$$
\begin{aligned}
P_{x\overline{n}|}^{\left(1+\frac{t}{n}\beta\right)q_{x+t}} &- P_{x\overline{n}|} \\
&\sim \frac{\beta}{6}P^1_{x\overline{n}|}\left(1-\frac{3}{n}+\frac{n}{4}i\right) \\
&\sim \beta\frac{1}{3}(P_{x\overline{n}|}-P_{\overline{n}|}).
\end{aligned} \tag{10}
$$

In the same way, we may write

$$P_{x+t\overline{n-t}|}^{\left(1+\frac{t}{n}\beta\right)q_{x+t}} - P_{x+t\overline{n-t}|}$$
$$\sim \beta\left\{\frac{t}{n}(P_{x+t\overline{n-t}|} - P_{\overline{n-t}|}) + \frac{P^1_{x+t\overline{n-t}|}}{6}\left(1-\frac{3}{n-t}+\frac{n-t}{4}i\right)\frac{n-t}{n}\right\}.$$

Therefore, we may write

(i) for small $t$,

$$\Delta P_{x+t\overline{n-t}|} \sim \beta\frac{P^1_{x+t\overline{n-t}|}}{6}\left(1+\frac{2t-3}{n}-\frac{3t}{n(n-t)}+\frac{n^2-t^2}{4n}i\right). \quad (11)$$

(ii) for large $t$,

$$\Delta P_{x+t\overline{n-t}|} \sim \beta\frac{n+2t}{3n}(P_{x+t\overline{n-t}|} - P_{\overline{n-t}|}). \quad (12)$$

Using (10) and (11) or (12), from (6) we have

(i) for small $t$,

$$\Delta\, {}_tV_{x\overline{n}|} \sim (1-{}_tV_{x\overline{n}|})\left[1-\frac{1+\beta\frac{A^1_{x\overline{n}|}}{6}\left(1-\frac{3}{n}+\frac{n}{4}i\right)}{1+\beta\frac{A^1_{x+t\overline{n-t}|}}{6}\left(1+\frac{2t-3}{n}-\frac{3t}{n(n-t)}+\frac{n^2-t^2}{4n}i\right)}\right]. \quad (13)$$

(ii) for large $t$,

$$\Delta\, {}_tV_{x\overline{n}|} \sim (1-{}_tV_{x\overline{n}|})\left[1-\frac{1+\beta\frac{A^1_{x\overline{n}|}}{6}\left(1-\frac{3}{n}+\frac{n}{4}i\right)}{1+\beta\frac{n+2t}{3n}(P_{x+t\overline{n-t}|}-P_{\overline{n-t}|})\,\ddot{a}_{x+t\overline{n-t}|}}\right]. \quad (14)$$

In my earlier paper [1]), I showed an approximation formula to

$$\Delta\, {}_tV_{x\overline{n}|} = {}_tV_{x\overline{n}|}^{\left(1+\frac{t}{n}\beta\right)q_{x+t}} - {}_tV_{x\overline{n}|},$$

but it is useful only when $t$ is small ($t < \frac{2}{3}n$).

On the other hand formula (14) is useful when $t$ is large, thought it may be poor when $t$ is small ($t < \frac{2}{3}n$).

---

[1]) Mitteilungen der Vereinigung schweizerischer Versicherungsmathematiker, Bd. 60, 2.

§ 4. I will show some numerical examples.

Mortality Table: Deutsche Sterbetafel 1924/26, Männer.

Interest: $i = 3\%$; $\beta = 100\%$.

(i) $q'_{x+t} = \left(1 + \frac{n-t}{n}\right) q_{x+t}$

$n = 10$

| | | $x = 30$ | $x = 50$ |
|---|---|---|---|
| $t = 2$ | true value | — 0.00373 | — 0.00893 |
| | by (7) | — 0.00361 | — 0.00916 |
| $t = 5$ | true value | — 0.00516 | — 0.01398 |
| | by (7) | — 0.00500 | — 0.01373 |
| $t = 8$ | true value | — 0.00285 | — 0.00861 |
| | by (7) | — 0.00278 | — 0.00863 |

(ii) $q'_{x+t} = \left(1 + \frac{n-t}{n}\right) q_{x+t}$

$n = 20$

| | | $x = 30$ | $x = 50$ |
|---|---|---|---|
| $t = 4$ | true value | — 0.00662 | — 0.01206 |
| | by (7) | — 0.00627 | — 0.01367 |
| | | | (— 0.01216)* |
| $t = 8$ | true value | — 0.00967 | — 0.02237 |
| | by (7) | — 0.00944 | — 0.02428 |
| | | | (— 0.02312)* |
| $t = 10$ | true value | — 0.01007 | — 0.02584 |
| | by (7) | — 0.00989 | — 0.02756 |
| | | | (— 0.02655)* |
| $t = 14$ | true value | — 0.00850 | — 0.02609 |
| | by (7) | — 0.00839 | — 0.02704 |
| | | | (— 0.02636)* |
| $t = 18$ | true value | — 0.00354 | — 0.01273 |
| | by (7) | — 0.00351 | — 0.01300 |
| | | | (— 0.01277)* |

* The error by (7) is partly due to the use of the approximation

$$\Delta P_{x\overline{n}|} \sim \frac{2}{3}(P_{x\overline{n}|} - P_{\overline{n}|})\left(1 + \frac{1}{n} - \frac{n}{24}i\right)$$

If we use the true value of $\Delta P_{x\overline{n}|}$ we shall have the value bracked.

(iii) $q'_{x+t} = \left(1 + \dfrac{k-t}{k}\right) q_{x+t}$

$k = 5$ $n = 20$

| | | $x = 30$ | $x = 50$ |
|---|---|---|---|
| $t = 2$ | true value | —0.00564 | —0.01467 |
| | by (8) | —0.00550 | —0.01441 |
| $t = 4$ | true value | —0.00822 | —0.02227 |
| | by (8) | —0.00828 | —0.02234 |
| $t = 10$ | true value | —0.00607 | —0.01696 |
| | by (9) | —0.00598 | —0.01700 |

(iv) $q'_{x+t} = \left(1 + \dfrac{k-t}{k}\right) q_{x+t}$

$k = 10$ $n = 20$

| | | $x = 30$ | $x = 50$ |
|---|---|---|---|
| $t = 2$ | true value | —0.00502 | —0.01186 |
| | by (8) | —0.00483 | —0.01190 |
| $t = 4$ | true value | —0.00839 | —0.02146 |
| | by (8) | —0.00814 | —0.02164 |
| $t = 8$ | true value | —0.01082 | —0.03208 |
| | by (8) | —0.01066 | —0.03191 |
| $t = 10$ | true value | —0.01007 | —0.03104 |
| | by (9) | —0.00987 | —0.03085 |
| $t = 14$ | true value | —0.00641 | —0.02039 |
| | by (9) | —0.00632 | —0.02029 |
| $t = 18$ | true value | —0.00231 | —0.00771 |
| | by (9) | —0.00226 | —0.00766 |

(v) $q'_{x+t} = \left(1 + \frac{t}{n}\right) q_{x+t}$

$n = 20$

| | | $x = 30$ | $x = 50$ |
|---|---|---|---|
| $t = 2$ | true value | 0.00162 | 0.00732 |
| | by (13) | 0.00164 | 0.00789 |
| | by (14) | 0.00159 | 0.00545 |
| $t = 4$ | true value | 0.00267 | 0.01304 |
| | by (13) | 0.00274 | 0.01378 |
| | by (14) | 0.00263 | 0.01191 |
| $t = 8$ | true value | 0.00314 | 0.01848 |
| | by (13) | 0.00325 | 0.01838 |
| | by (14) | 0.00314 | 0.01780 |
| $t = 10$ | true value | 0.00260 | 0.01754 |
| | by (13) | 0.00267 | 0.01697 |
| | by (14) | 0.00264 | 0.01691 |
| $t = 14$ | true value | 0.00040 | 0.00832 |
| | by (14) | 0.00043 | 0.00801 |
| $t = 18$ | true value | — 0.00117 | — 0.00373 |
| | by (14) | — 0.00119 | — 0.00361 |

## Zusammenfassung – Résumé – Riassunto

Verfasser gibt ergänzende Ausführungen zu seiner in den Heften 59, 1 und 60, 2 erschienenen Abhandlung zur Technik der erhöhten Risiken in der Lebensversicherung. Es werden Näherungsformeln für Prämie und Reserve der gemischten Versicherung bei steigender und bei fallender Übersterblichkeit hergeleitet. Zahlreiche Rechenbeispiele belegen die Güte der angeführten Approximationen.

L'auteur apporte un complément à ses précédentes études sur la technique des risques aggravés dans l'assurance vie, étude parues dans les cahiers nº 1, 1959 et 2, 1960. Il développe différentes formules approximatives de primes et de réserves mathématiques pour les assurances mixtes à surmortalité croissante et décroissante. Plusieurs exemples numériques attestent de la bonne approximation de ces formules.

L'autore dà schiarimenti complementari ai suoi precedenti lavori sulla tecnica di rischi tarati nell'assicurazione vita, apparsi nei fascicoli 1, 1959 e 2, 1960. Deduce diverse formole approssimative per premi e riserve matematiche di assicurazioni miste con sopramortalità crescente e decrescente. Numerosi esempi numerici provano la buone approssimazione di tali formole.