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Recursive Evaluation of the Terms in the Development of the Adjustment Coefficient as a Power Series in the Safety Loading

1. Introduction

Let Z_t denote the surplus process of an insurance company. Specifically, assume that

$$Z_t = w + ct - S_t \tag{1}$$

where $Z_0 = w$ is the initial reserve, c is the premium per unit time such that ct is the total premium income received in $(0, t)$, and S_t is the aggregate claims process.
The common assumption is made that S_t is a compound Poisson process with Poisson parameter λ and distribution function F_X of claim amounts. We assume that $F_X(0) = 0$ such that negative claim amounts are not possible. Furthermore it is assumed that

$$c > \lambda q_1$$

where q_1 is the mean claim size, which means that the premium income in $(0, t)$ exceeds the expected claim amount in $(0, t)$. Thus the relative safety loading, which is defined as

$$\eta = \frac{c}{\lambda q_1} - 1$$

is supposed to be positive.
Let $\psi(w, t)$ denotes the finite time probability of ruin, i.e.

$$\psi(w, t) = \text{Prob}\{Z_t \leq 0 \quad \text{for some } t, \quad 0 < t < \infty\}.$$

The infinite time probability of ruin is then defined by

$$\psi(w) = \lim_{t \to \infty} \psi(w, t).$$

The difficulties in the calculation of the ruin probilities are discussed by many authors and have led to the derivation of several approximation formulas. For an overview of these problems we refer to *Taylor* (1988).
A classical upper bound for the probability of ultimate ruin is given by Lundberg's inequality

$$\psi(w) \leq e^{-Rw} \tag{2}$$

where R, the so-called adjustment coefficient, is defined as the unique positive solution of the equation in r

$$\lambda + rc = \lambda \int_0^\infty e^{rx}\, dF_X(x) \tag{3}$$

(*Gerber*, 1977, p. 118).
Though the adjustment coefficient seems to be a useful tool in risk theory – see e.g *Gerber* (1984), *Van Wouwe* et al. (1984) and *Centeno* (1986), who apply the adjustment coefficient for the measurement of the effects of reinsurance – its practical application still requires the solution of the equation (3).
Necessary and sufficient conditions for the existence of the adjustment coefficient are given by *Mammitzch* (1986). *Herkenrath* (1986) discussed the estimation of the adjustment coefficient by means of stochastic approximation procedures. More recently *Taylor* (1988, p. 69) provided a power series of the adjustment coefficient in terms of the security loading η:

$$R = \left(\frac{2\eta q_1}{q_2}\right) - \frac{1}{3}\left(\frac{q_3}{q_2}\right)\left(\frac{2\eta q_1}{q_2}\right)^2 + \left(\frac{2q_3^2}{9q_2^2} - \frac{q_4}{12q_2}\right)\left(\frac{2\eta q_1}{q_2}\right)^3 + O\left(\frac{2\eta q_1}{q_2}\right)^4 \tag{4}$$

where q_n is the n-th moment of an individual claim size:

$$q_n = \int_0^\infty x^n\, dF_X(x).$$

The aim of the present paper is to facilitate the use of this power series by deriving a recurrence relation for the evaluation of its coefficients.

2. The recurrence relations

Theorem The adjustment coefficient can be developed as

$$R = \sum_{k=1}^{\infty} \frac{1}{k} d_{k-1,k} \left(\frac{2q_1\eta}{q_2} \right)^k \tag{5}$$

where $d_{k-1,k}$ is calculated according to the recursive scheme:

$$\begin{aligned} d_{j,k} &= \frac{1}{j} \sum_{s=1}^{j} (sk - j + s) c_s d_{j-s,k} \\ d_{0,k} &= 1 \end{aligned} \tag{6}$$

and

$$\begin{aligned} c_n + \sum_{k=1}^{n} c_{n-k} \frac{2q_{k+2}}{(k+2)!\, q_2} &= 0 \\ c_0 &= 1 \end{aligned} \tag{7}$$

Proof The idea of the proof is to reduce equation (3) to a power series of the safety loading η in the adjustment coefficient R and to reverse this series by use of Lagrange's theorem. This theorem is stated in the Appendix.
For convenience, we will eliminate the parameter λ in the expressions by determining the time scale so that $\lambda = 1$. Then equation (3) can easily be rewritten as follows:

$$\begin{aligned} & 1 + (1+\eta) q_1 R = \int_0^{\infty} e^{Rx}\, dF_X(x) \\ \Leftrightarrow\quad & \eta q_1 R = \int_0^{\infty} \left(e^{Rx} - 1 - Rx \right) dF_X(x) \\ \Leftrightarrow\quad & \eta = \frac{1}{q_1 R} \left[\sum_{j=2}^{\infty} \frac{R^j}{j!} \int_0^{\infty} x^j\, dF_X(x) \right] \\ \Leftrightarrow\quad & \eta = \frac{1}{q_1} \sum_{j=1}^{\infty} \frac{R^j}{(j+1)!} \int_0^{\infty} x^{j+1}\, dF_X(x). \end{aligned} \tag{8}$$

Hence we can express η as a power series in R, i.e.

$$\eta = \sum_{j=1}^{\infty} a_j R^j \tag{9}$$

where

$$\alpha_j = \frac{q_{j+1}}{q_1 (j+1)!} . \tag{10}$$

In order to apply Lagrange's theorem to this equation, we define the function $g(R)$ by means of the equation

$$\eta = \frac{R}{g(R)} . \tag{11}$$

Using (9) and the classical formulas for the inversion of power series (*Gradshteyn,* 1965, p. 14) we can write $g(R)$ so as a power series in R:

$$g(R) = \frac{1}{\sum_{j=1}^{\infty} a_j R^{j-1}} = \frac{1}{\alpha_1} \sum_{j=0}^{\infty} c_j R^j \tag{12}$$

where the coefficients c_j are given by the following recursive scheme:

$$\begin{aligned} c_0 &= 1 \\ c_n + \frac{1}{\alpha_1} \sum_{k=1}^{n} c_{n-k} \alpha_{k+1} &= 0, \end{aligned} \tag{13}$$

which after substitution of the value of α_{k+1} and α_1 reduces to formula (7). So we get for the first three coefficients

$$\begin{aligned} c_0 &= 1 \\ c_1 &= -\frac{q_3}{3q_2} \\ c_2 &= \left(\frac{q_3}{3q_2}\right)^2 - \frac{q_4}{12q_2} . \end{aligned}$$

Now we can reverse the power series (9) in R by using Lagrange's formula in order to obtain:

$$\begin{aligned} R &= \sum_{k=1}^{\infty} \frac{\eta^k}{k!} \frac{d^{k-1}}{dR^{k-1}} \left[(g(R))^k \right]_{|R=0} \\ &= \sum_{k=1}^{\infty} \frac{\eta^k}{k!} \frac{d^{k-1}}{dR^{k-1}} \left[\left(\frac{1}{\alpha_1} \sum_{j=0}^{\infty} c_j R^j \right)^k \right]_{|R=0} \end{aligned} \tag{14}$$

The k-th power of the power series which appears in this expression can again be calculated in a recursive way (*Gradshsteyn,* 1965, p. 14); we get

$$\left(\sum_{j=0}^{\infty} c_j R^j \right)^k = \sum_{j=0}^{\infty} d_{j,k} R^j \tag{15}$$

where

$$\begin{aligned} d_{0,k} &= 1 \\ d_{j,k} &= \frac{1}{j} \sum_{s=1}^{j} (sk - j + s) c_s d_{j-s,k} \, . \end{aligned} \tag{16}$$

The first coefficients in this expansion are given by

$$\begin{aligned} d_{0,k} &= 1 \\ d_{1,k} &= -\frac{kq_3}{3q_2} \\ d_{2,k} &= \frac{k(k+1)}{2} \left(\frac{q_3}{3q_2} \right)^2 - \frac{kq_4}{12q_2} \end{aligned}$$

After substitution of (15) in (14), we get

$$\begin{aligned} R &= \sum_{k=1}^{\infty} \left(\frac{1}{\alpha_1} \right)^k \frac{\eta^k}{k!} \frac{d^{k-1}}{dR_{k-1}} \left[\sum_{j=0}^{\infty} d_{j,k} R^j \right]_{|R=0} \\ &= \sum_{k=1}^{\infty} \left(\frac{1}{\alpha_1} \right)^k \frac{\eta^k}{k!} \left[\sum_{j=k-1}^{\infty} d_{j,k} \, j(j-1) \cdots (j-k+2) R^{j-(k-1)} \right]_{|R=0} \\ &= \sum_{k=1}^{\infty} \left(\frac{1}{\alpha_1} \right)^k \frac{\eta^k}{k!} (k-1)! \, d_{k-1,k} \end{aligned} \tag{17}$$

As $\alpha_1 = \frac{q_2}{2q_1}$ we finally have

$$R = \sum_{k=1}^{\infty} \left(\frac{2q_1\eta}{q_2}\right)^k \frac{1}{k} d_{k-1,k} \,. \tag{18}$$

Writing this formula in an explicit form until the third degree we have

$$R = \left(\frac{2\eta q_1}{q_2}\right) - \frac{1}{3}\left(\frac{q_3}{q_2}\right)\left(\frac{2\eta q_1}{q_2}\right)^2 + \left(\frac{2q_3^2}{9q_2^2} - \frac{q_4}{12q_2}\right)\left(\frac{2\eta q_1}{q_2}\right)^3 + \ldots \tag{19}$$

which indeed corresponds to Taylor's formula (4).

Appendix

Lagrange's theorem for the reversion of series states that

"if $f(z)$ is regular in a neighbourhood of z_0 and if $f(z_0) = w_0$, $f'(z_0) \neq 0$, then the equation

$$f(z) = w$$

has a unique solution, regular in a neighbourhood of w_0, of the form

$$z = z_0 + \sum_{n=1}^{\infty} \frac{(w - w_0)^n}{n!} \left[\frac{d^{n-1}}{dz^{n-1}} \{g(z)\}^n\right]\Big|_{z=z_0}$$

where $f(z) - w_0 = \frac{z - z_0}{g(z)}$."

(See *Copson*, 1970, p. 125).

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Summary

Recurrence relations are given for the coefficients in the development of the adjustment coefficient as a power series in the safety loading.

Zusammenfassung

Es werden rekursive Beziehungen hergeleitet für die Koeffizienten in der Entwicklung des Anpassungskoeffizienten als Potenzreihe des Sicherheitszuschlages.

Résumeé

L'auteur présente certaines relations de récurrence pour les coefficients figurant dans le développement du "coefficient d'ajustement"en série de puissance du chargement de sécurité.