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Pension Funding and Optimal Control

1 Introduction

Although control theory has been applied to actuarial problems, there are few papers illustrating the applicability of deterministic or stochastic optimal control theory. Among the exceptions is the paper written by *O'Brien* [1987]: he demonstrates how pension costs can be determined by means of stochastic optimal control in order to minimize deviations from a target fund and to minimize the fluctuations of the contributions. The stochastic components involved are the stochastic capitalization rate for the fund – although the discount rate is deterministic – and the stochastic component in the benefits. This model is however only manageable if the benefits are represented by a linear function of time.

The model used in this paper is very similar but differs on two points. The first alteration concerns the objective function: most objective functions that occur in studies on pension funding minimize the total discounted cost or the fluctuations of the contributions. This is only fair if just one generation is involved: it is not hard to conceive a situation where total discounted costs can be minimized by paying a lump sum at the start of the pension plan. As pension plans, and especially national social security plans, involve several generations, we aim at a fair contribution plan by minimizing the deviations from a fixed contribution level that is expressed as a percentage of total salary. To fix this contribution level we adopt the second criterion that was used by *O'Brien* [1987]. He proposed to apply the results of a paper by *Trowbridge* [1963] in which it was shown that the different funding methods can be approximated by determining the funding level. This funding level is defined, for all times t, by the ratio of the fund over the present value of the future benefits of all members – active and retired – of the plan at time t. It was shown e.g. that terminal funding has a funding level between 30 % and 35 % and initial funding of 100 %.

As linear growth of the pension benefits is not realistic, the second difference with the model of *O'Brien* concerns the function representing the future benefits. We will use general functions, although this prevents us from introducing a stochastic component. As a matter of fact it is very well possible to add a stochastic component in the benefit function but it is virtually impossible to derive the exact relations between the stochastic components in total salary, total benefits and the present value of future benefits. In any case the resulting stochastic model would be very hard to solve.

As to the stochastic component in the fund growth rate, there is no great difficulty introducing it in our model (see *Vandebroek* [1989]) but we preferred to deal with the deterministic version in this paper because it appears to us that this model is more useful. The most important difference between the solutions of the deterministic and the stochastic model is that for the first model the contribution function can be determined uniquely for all future times whereas for the second model the optimal contribution for every moment is given as a function of the fund because the evolution of the latter is unknown due to stochastic interest rates.

We will derive the optimal contribution function for this deterministic model by means of optimal control theory in the next section. Afterwards we apply these results to a Belgian social security model and analyse the influence of some parameters on the results.

2 Notation and results

We adopt the notations of *Bowers* et al. [1986] to describe a defined benefit pension plan:

- $W(t)$ = the total salary rate at time t
- $B(t)$ = the benefit outgo at time t
- $A(t)$ = the present value at time t of future benefits for active and retired members at t
- $F(t)$ = fund at time t
- $C(t)$ = annual contribution rate at t

The fund ratio will be denoted by η and the contribution level by α. As was explained in the introduction we try to minimize deviations between $\eta A(t)$ and $F(t)$ and between $\alpha W(t)$ and $C(t)$.

The differential equation describing the development of the fund is

$$F'(t) = \delta F(t) + C(t) - B(t) \tag{1}$$

where δ stands for the force of interest.

If we denote by φ the discount rate, by $\beta(\geq 0)$ the weighing factor to reflect the relative importance of the two criteria and by T the considered time horizon, we end up with the following problem:

$$\min_{C,\alpha} \int_0^T e^{-\varphi t}\{[C(t) - \alpha W(t)]^2 + \beta[\eta A(t) - F(t)]^2\}dt. \tag{2}$$

This optimization problem involves both the function $C(t)$ and the level α. Of course it is also possible to fix the contribution level as a constant percentage of salary, to obtain the problem

$$\min_{\alpha} \int_0^T e^{-\varphi t}[\eta A(t) - F(t)]^2 dt. \tag{3}$$

If we take into account the initial fund F_0, these two problems can be solved (see *Vandebroek* [1989]) but the solutions are very unrealistic because the fund becomes strongly negative at the end of the considered period. To overcome this problem we also prescribe the fund level at $t = T$, denoted by F_T. In the last section we will demonstrate the influence of this amount on the solution. It is clear that introducing the height of $F(T)$ determines the constant level α in (3) uniquely. This value is given by the following theorem

Theorem 1. *If the fund level at $t = T$ is prescribed, the constant contribution level α is given by*

$$\alpha = \frac{F_T - F_0 e^{\delta T} + e^{\delta T} \int_0^T e^{-\delta u} B(u)\, du}{e^{\delta T} \int_0^T e^{-\delta u} W(u)\, du} \tag{4}$$

where $F(0) = F_0$ and *$F(T) = F_T$ stand for the initial and final fund level.*

Proof This result can easily be derived from the following expression which is the solution to (1), with the boundary condition $F(0) = F_0$:

$$F(t) = F_0 e^{\delta t} \int_0^t e^{-\delta u} \left[\alpha W(u) - B(u)\right] du. \tag{5}$$

This function must equal F_T if $t = T$, determining α. $\square$

This solution is easy to compute, but as will be shown in the next section, this gives unrealistic results because the increase in the contribution level is too abrupt. Therefore the solution to the other problem, although much harder to solve and to compute, is more realistic:

Theorem 2. *The solution to*

$$\min_C \int_0^T e^{-\varphi t}\{[C(t) - \alpha W(t)^2] + \beta[\eta A(t) - F(t)]^2\}\, dt \tag{6}$$

with

$$F'(t) = \delta F(t) + C(t) - B(t), \quad F(0) = F_0 \quad \text{and} \quad F(T) = F_T \tag{7}$$

is given by

$$C(t) = \alpha W(t) - \frac{\lambda_0 e^{-(\delta-\varphi)t}}{2} - \beta e^{-(\delta-\varphi)t} \int_0^t e^{(\delta-\varphi)u}[\eta A(u) - F(u)]\, du \tag{8}$$

where $F(t)$ is the solution to the differential equation

$$F''(t) = \varphi F'(t) + (\beta + \delta(\delta - \varphi))F(t) + \alpha W'(t) - B'(t) + (\delta - \varphi)(\alpha W(t) - B(t)) - \beta\eta A(t) \tag{9}$$

satisfying the boundary conditions

$$F(0) = F_0 \quad \text{and} \quad F(T) = F_T \tag{10}$$

and where the constant λ_0 is determined by

$$\lambda_0 = -2(\varphi - 2\delta)/(e^{(\delta-\varphi)T - e^{\delta T}}) \times \left\{F_0 e^{\delta T} - F_T + \int_0^T e^{\delta(t-u)}[\alpha W(u) - B(u)]\, du + \beta \int_0^T e^{\delta(t-u)} e^{-(\delta-\varphi)u} \int_0^u e^{(\delta-\varphi)v}[\eta A(v) - F(v)]\, dv\, du\right\} \tag{11}$$

Proof This problem can be solved by optimal control theory (see *Kamien/Schwartz* [1981]). There exists at least one solution as the integrand in (6) and the righthandside of the differential equation (7) are convex in (F, C). The Hamiltonian is

$$\mathscr{H}(t) = e^{-\varphi t}\{[C(t) - \alpha W(t)]^2 + \beta[\eta A(t) - F(t)]^2\} + \lambda(t)[\delta F(t) + C(t) - B(t)]. \tag{12}$$

The optimal $C(t)$ can be found from $\mathscr{H}_C \equiv \partial\mathscr{H}/\partial\mathscr{C} = 0$:

$$\lambda(t) = -2e^{-\varphi t}[C(t) - \alpha W(t)]. \tag{13}$$

The derivative $\lambda'(t)$ must satisfy $-\mathscr{H}_F = \lambda'(t)$ which yields the following differential equation:

$$\lambda'(t) = 2\beta e^{-\varphi t}[\eta A(t) - F(t)] - \delta\lambda(t). \tag{14}$$

The solution to this equation is

$$\lambda(t) = \lambda_0 e^{-\delta t} + 2\beta e^{-\delta t}\int_0^t e^{(\delta-\varphi)u}[\eta A(u) - F(u)]\,du. \tag{15}$$

This solution, together with (13) yields $C(t)$ as given in (8). Inserting this solution for $C(t)$ in (7) one obtains the differential equation

$$F'(t) = \delta F(t) + \alpha W(t) - B(t) - \frac{\lambda_0 e^{(\delta-\varphi)t}}{2} + \beta e^{-(\delta-\varphi)t}\int_0^t e^{(\delta-\varphi)u}[\eta A(u) - F(u)]\,du. \tag{16}$$

Deriving this function with respect to t one obtains

$$F''(t) = \delta F'(t) + \alpha W'(t) - B'(t) + \lambda_0\frac{(\delta-\varphi)e^{-(\delta-\varphi)t}}{2} - \beta(\eta A(t) - F(t)) + \beta(\delta-\varphi)e^{-(\delta-\varphi)t}\int_0^t e^{(\delta-\varphi)u}[\eta A(u) - F(u)]\,du \tag{17}$$

where the integral in the righthandside can be eliminated using (16) yielding (9) as all terms containing λ_0 disappear.
Rests us to fix the constant λ_0. From the equations (7) and (8) it follows that

$$F(t) = F_0 e^{\delta t} - F_T - \lambda_0 \frac{e^{(\delta-\varphi)T} - e^{\delta T}}{2(\varphi - 2\delta)} + \int_0^T e^{\delta(t-u)}[\alpha W(u) - B(u)]\,du + \beta \int_0^T e^{\delta(t-u)} e^{-(\delta-\varphi)u} \int_0^u e^{(\delta-\varphi)v}[\eta A(v) - F(v)\,dv\,du. \tag{18}$$

As everything in this expression is known, the constant λ_0 can be solved. Remark that this equation holds for every t, so one can chose which value of t to use. □

This solution can be computed analytically for some pension plans such as the exponential model because all functions involved, W, A and B, are of the exponential type in this case. For these plans the optimal value for α can also be determined anlytically but for more realistic cases the function $C(t)$ can not be computed analytically, so the search for the optimal value for α will be done numerically.

3 Applied to a Belgian social security model

All predictions about the pension contributions in the Belgian social security system are alarming, see e.g. *K.V.B.A.* [1982], *Gollier* [1987] and *Pepermans* et al. [1987]. The charges for account of the future generations are very high. As new initiatives in this matter do not tend to improve the situation from the benefit side, we will derive a contribution plan where the spread of the costs is more appropriate.
To represent the future salaries and benefits we use the model that was built by *Pepermans* et al. [1987]. In this model demographic and economic factors are taken into account. The future population is predicted assuming a decreasing birthing-rate and an increasing expected lifetime. For the evolution of the total pension costs and the active population the influence of the expected economic expansion is determined. We will use their standard model for the

period 1990 – 2049. As their study produces discrete values for the future expected salary and benefits, we performed two linear regressions, resulting in (in 1982 – Belgian francs, in thousand millions):

$$W_{1982}(t) = -53384 + 28.248569 \cdot t \qquad R^2 = 99.68\ \% \tag{19}$$

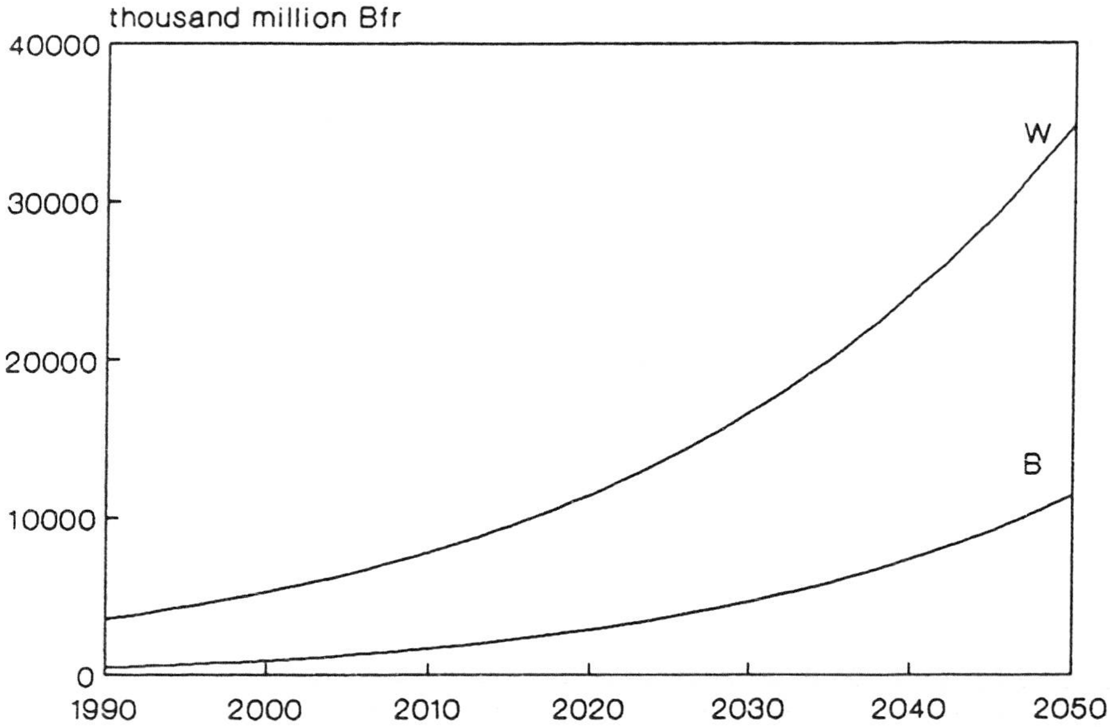

Figure 1. Evolution of total salary W and of total benefits B

$$B_{1982}(t) = -36183 + 18.364318 \cdot t \qquad R^2 = 98.19\ \% \tag{20}$$

with the corresponding – very satisfying – R^2-values.
In these functions the productivity changes and the merit components are reflected. To cope with inflation we assume that 3 % is a realistic annual inflation rate. This provides us with a total salary rate $W(t)$ and total benefit rate $B(t)$

$$W(t) = e^{0.03(t-1982)}\, W_{1982}(t) \tag{21}$$

$$B(t) = e^{0.03(t-1982)}\, B_{1982}(t) \tag{22}$$

Both functions are depicted in figure 1 which reveals the sharply increasing contributions in our pay-as-you-go system: $B(t)/W(t)$ is 13.49 % in 1990, 25.36 % in 2020 and 32.79 % in 2050.
We will therefore apply the two theorems of the previous section to this model. Because it is not realistic to built up large funds, we fixed the fund ratio η as being zero which means that we do not want to deviate very much from the current pay-as-you-go model. For the fund at $t = T$ we will also use $F_T = 0$. For the capitalization rate of the fund and for the discount rate we take 6 %.
We compared 3 models. Model 1 stands for the solution of theorem 2: the contributions fluctuating around the optimal percentage α of total salary and funds fluctuating around zero. Model 2 gives the solution of theorem 1 where $C \equiv \alpha W$, whereas model 3 represents the solution to the criterion of *O'Brien* [1987]: model 1 with $\alpha = 0$. With this criterion the fluctuations in the contributions are minimized, without taking into account the corresponding evolution in the salaries.
Of course, the results depend on the value of β which reflects the relative importance we attach to the prescribed fund ratio. This is also clear from the figures 2 and 3 which show the contributions and funds for $\beta = 1$ % and the figures 4 and 5 which show the same functions for $\beta = 5$ %.
The optimal value for α in the first model is 25.30 % if $\beta = 1$ % and 25.87 % if $\beta = 5$ %. The contribution level is however smaller than this percentage until ±2020 whereas it is higher afterwards.
With the second model we obtain an optimal value for the contribution level of 22.58 %. This stands for the percentage that should be paid from now on to charge all generations equally until 2050. This clearly is a substantial increase in the current contribution level which is 13.49 %.
The contribution function resulting from the third model starts at a higher level and ends up lower as could be expected.
It is clear from these figures that for $\beta = 1$ % and $\beta = 5$ % the second criterion still dominates the objective function as the contribution function does not deviate very much from the benefit function. If one wants to deviate more from the pay-as-you-go system one simply has to lower the value for β.

4 Sensitivity analysis

To investigate to what extend the solutions depend on the considered period and the prescribed fund at time T, we compared the contributions for several values of T and F_T.
In figure 6 the contribution functions $C(t)$ during the first 60 years are depicted for $T = 2050$, $T = 2070$ and $T = 2100$ with $F_T = 0$, $\beta = 1\%$ and $\eta = 0$ as they result from theorem 2. To predict $W(t)$ and $B(t)$ beyond 2050 we used the formulas in (21) and (22).
In figure 7 de contributions $C(t)$ are shown for $F_T = 0$, $F_T = 25000$, $F_T = 50\,000$ and $F_T = 75\,000$ with $T = 2050$.
From these figures it is clear that the contributions during the first part of the considered period do not really depend on these factors which makes the results more reliable for the first say 30 years.

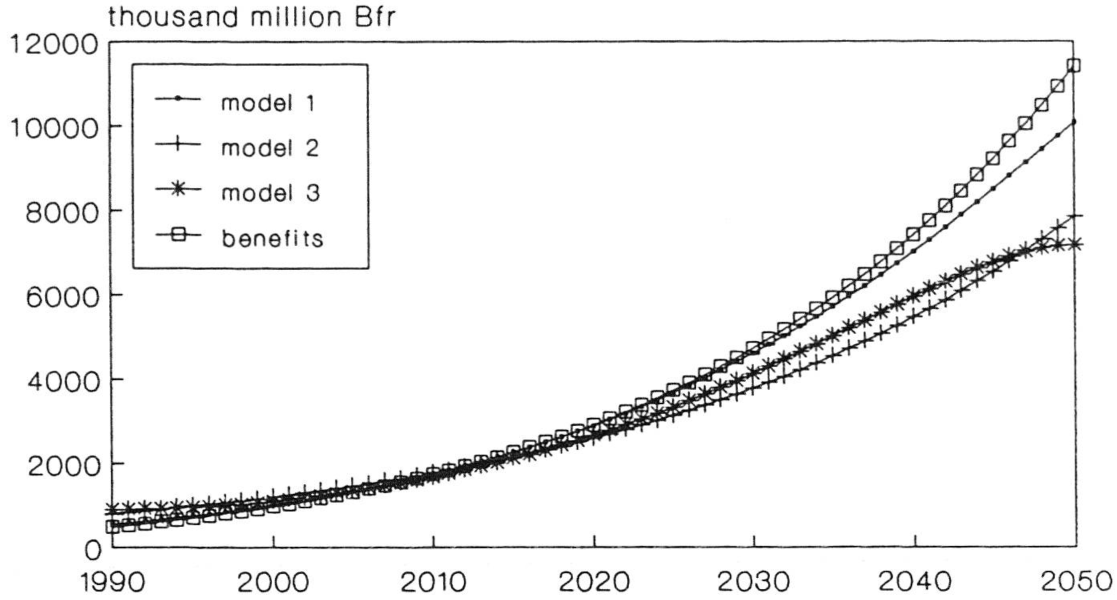

Figure 2. Contributions ($\beta = 1\ \%$, $F_T = 0$)

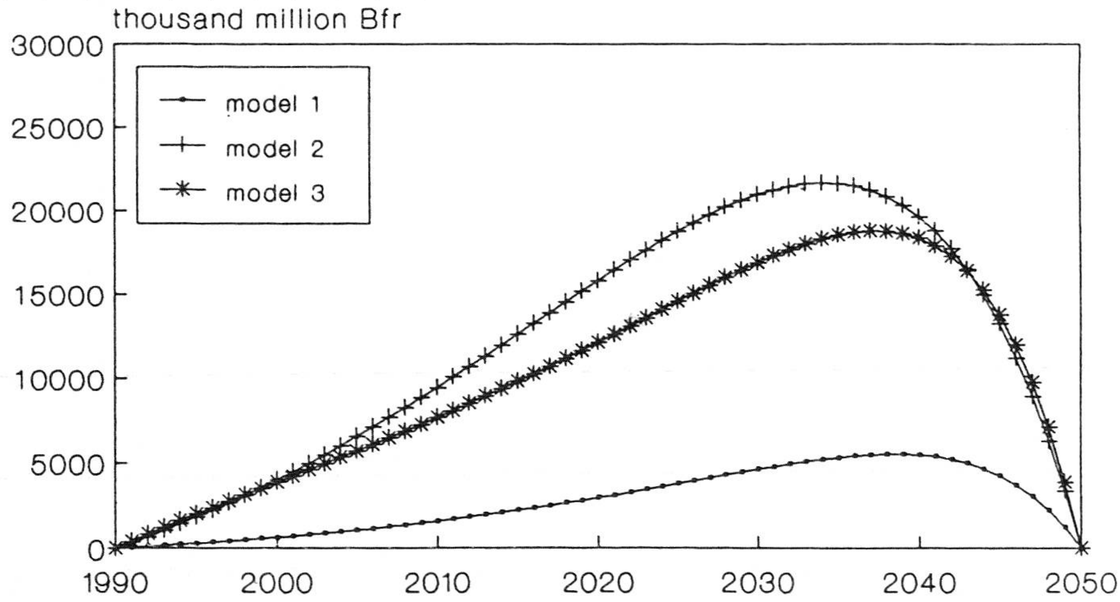

Figure 3. Funds ($\beta = 1\ \%$, $F_T = 0$)

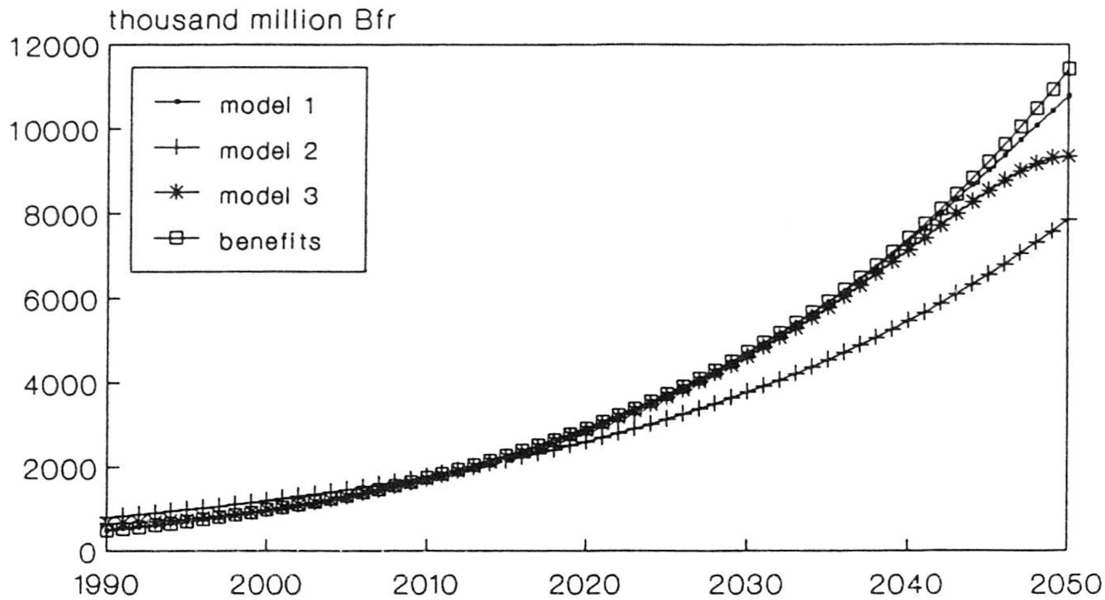

Figure 4. Contributions ($\beta = 5\,\%$, $F_T = 0$)

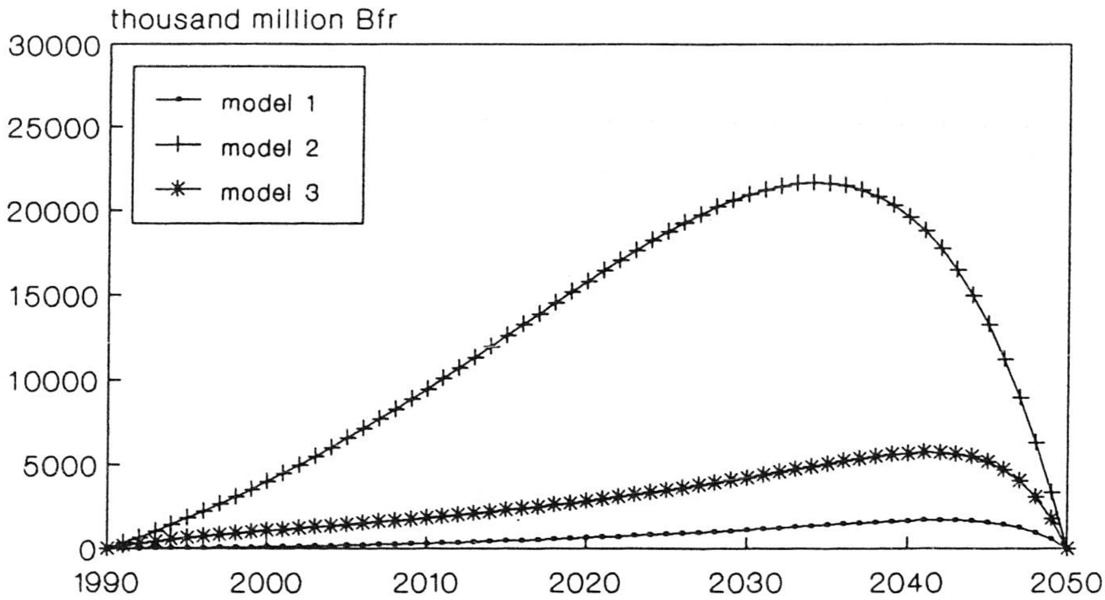

Figure 5. Funds ($\beta = 5\,\%$, $F_T = 0$)

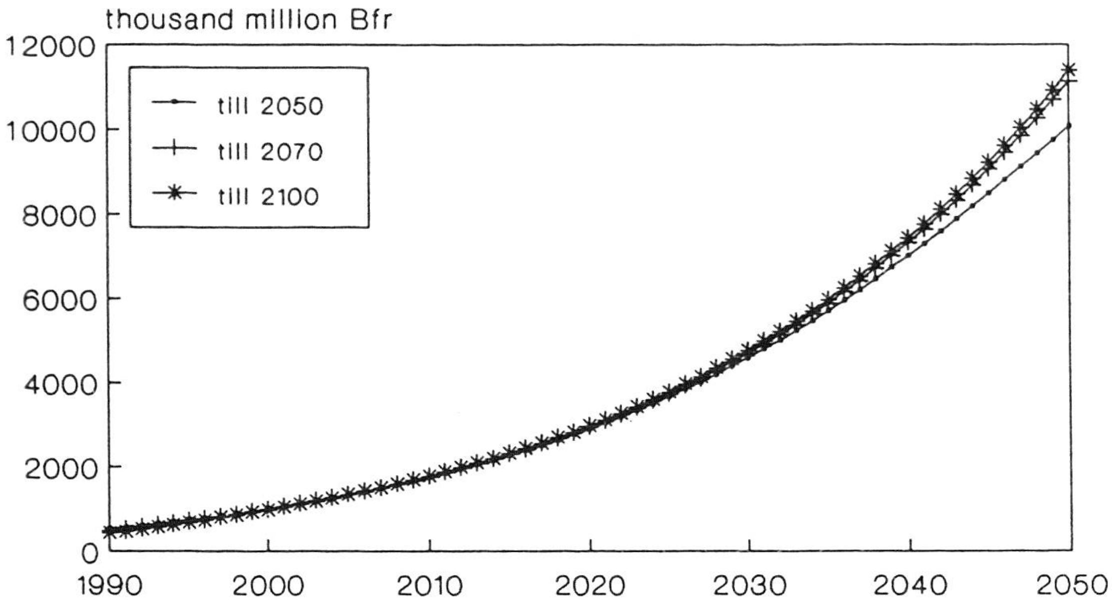

Figure 6. Evolution of the contributions for different values of T ($\beta = 1\,\%$, $F_T = 0$)

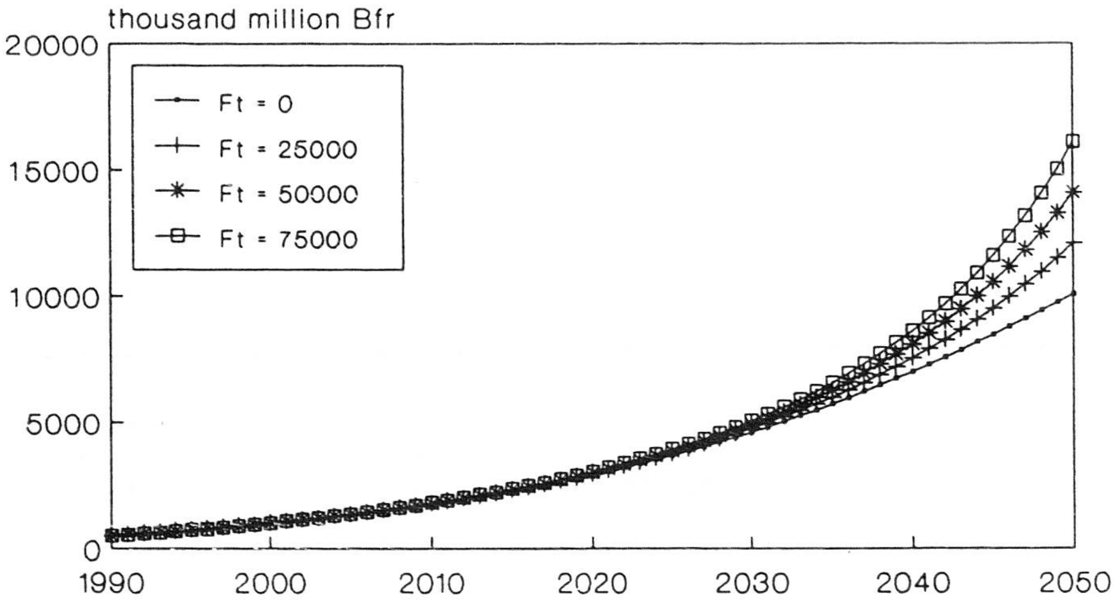

Figure 7. Evolution of the contributions for different values of F_T ($\beta = 1\,\%$, $T = 2050$)

5 Conclusion

The methods that were proposed in this paper to derive optimal contribution plans yield interesting tools to analyse the evolution of the costs and the corresponding fund. Although the results depend on the reliability of the underlying model that was used to predict the evolution in the salary and benefits, these methods can provide more insight in this complicated matter.

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Summary

It is shown how the costs for pension funding can be determined by means of optimal control theory in order to charge all generations equally, taking into account a required level of funding.

Zusammenfassung

Es wird gezeigt, wie mittels optimaler Kontrolltheorie die Beiträge an ein Sozialversicherungssystem so bestimmt werden können, dass alle Generationen gleichermassen belastet werden und gleichzeitig ein bestimmtes Beitragsniveau berücksichtigt wird.

Résumé

L'auteur montre comment les coûts de constitution d'un fonds de pension peuvent être déterminés par la théorie du contrôle optimal en vue de mettre à contribution de façon égale toutes les générations et en conservant au fonds un niveau donné.