# Loss reserving and Hofmann distributions

Autor(en): **Schmidt, Klaus D. / Zocher, Mathias**

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KLAUS D. SCHMIDT and MATHIAS ZOCHER, Dresden

## Loss Reserving and Hofmann Distributions

### 1 Introduction

The present paper honours and applies the work of Martin Hofmann [1955] who introduced a wide and most useful class of mixed Poisson distributions.

The mixed Poisson distributions introduced by Hofmann [1955] are called *Hofmann distributions* and they are grouped as three–parametric *Hofmann families*. The most prominent examples of Hofmann families are the family of all Poisson distributions and the family of all negativebinomial distributions, but the class of all Hofmann distributions is much larger and permits a very flexible modelling of claim numbers. An important property of all Hofmann distributions is the property that the variance is always at least as large as the expectation, and that the inequality is strict except for Poisson distributions; this property is shared by many empirical claim number distributions. Moreover, Hürlimann [1990] has shown that the maximum–likelihood estimator for the expectation of a Hofmann distribution agrees with the sample mean.

A review of the basic properties of Hofmann families and Hofmann distributions will be given in Sections 2 and 3 of this paper, and a short proof of Hürlimann's result on maximum–likelihood estimation of the expectation of a Hofmann distribution will be given in Section 4. These results will be needed in the subsequent sections on loss reserving.

Hofmann distributions have shown to be useful in the construction of models for ratemaking, in particular in motor car insurance; see Kestemont and Paris [1985] as well as Walhin and Paris [1999, 2001]. In the present paper we shall show that Hofmann distributions are equally useful in the construction of models for loss reserving; for general information on loss reserving we refer to Radtke and Schmidt [2004].

The models for loss reserving considered here are models for the prediction of the number of outstanding claims. These are claims which have already incurred but have not yet been reported to the insurance company (IBNR claims). The earliest models for the prediction of the number of outstanding claims are the Poisson model of Hachemeister and Stanard [1975], who used maximum–likelihood estimation, and the multinomial model of Witting [1987], who used credibility prediction. In a sense, the ideas of both of these papers have been combined by Schmidt and Wünsche [1998], who used maximum–likelihood estimation in a special case of the multinomial model.

As a first step in the construction of a model for the prediction of the number of outstanding claims, we consider a family of random variables $\{Z_{i,k}\}_{i,k\in\{0,1,\dots,n\}}$ where $Z_{i,k}$ is interpreted as the number of claims which have incurred in the relative *accident year* $i \in \{0,1,\dots,n\}$ and are reported with a delay of $k \in \{0,1,\dots,n\}$ years in the relative *development year* $k$. We assume that the *incremental claim numbers* $Z_{i,k}$ are *observable* for $i+k \le n$ and *non–observable* for $i+k > n$. The observable incremental claim numbers are presented in a *run–off triangle*:

| **Accident** | **Development Year** | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **Year** | $0$ | $1$ | $\dots$ | $k$ | $\dots$ | $n-i$ | $\dots$ | $n-1$ | $n$ |
| $0$ | $Z_{0,0}$ | $Z_{0,1}$ | $\dots$ | $Z_{0,k}$ | $\dots$ | $Z_{0,n-i}$ | $\dots$ | $Z_{0,n-1}$ | $Z_{0,n}$ |
| $1$ | $Z_{1,0}$ | $Z_{1,1}$ | $\dots$ | $Z_{1,k}$ | $\dots$ | $Z_{1,n-i}$ | $\dots$ | $Z_{1,n-1}$ | |
| $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ | | $\vdots$ | | | |
| $i$ | $Z_{i,0}$ | $Z_{i,1}$ | $\dots$ | $Z_{i,k}$ | $\dots$ | $Z_{i,n-i}$ | | | |
| $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ | | | | | |
| $n-k$ | $Z_{n-k,0}$ | $Z_{n-k,1}$ | $\dots$ | $Z_{n-k,k}$ | | | | | |
| $\vdots$ | $\vdots$ | $\vdots$ | | | | | | | |
| $n-1$ | $Z_{n-1,0}$ | $Z_{n-1,1}$ | | | | | | | |
| $n$ | $Z_{n,0}$ | | | | | | | | |

The quantities of interest are the *ultimate aggregate claim numbers*

$$S_{i,n} := \sum_{k=0}^{n} Z_{i,k} \,.$$

For $i \in \{1,\dots,n\}$, the ultimate aggregate claim number $S_{i,n}$ is non–observable and has to be predicted on the basis of the observable incremental claim numbers.

As a second step in the construction of a model for the prediction of the number of outstanding claims, we assume that the family of incremental claim numbers satisfies the assumption of the *multiplicative model*:

– There exist parameters $\alpha_0, \alpha_1, \ldots, \alpha_n \in (0, \infty)$ and $\vartheta_0, \vartheta_1, \ldots, \vartheta_n \in (0, 1)$ with $\sum_{k=0}^{n} \vartheta_k = 1$ such that the identity

$$E[Z_{i,k}] = \alpha_i \vartheta_k$$

holds for all $i, k \in \{0, 1, \ldots, n\}$.

In the multiplicative model, the *expected ultimate aggregate claim numbers* satisfy

$$E[S_{i,n}] = \alpha_i \,.$$

Therefore, the expected ultimate aggregate claim numbers can be estimated by estimating the parameters $\alpha_0, \alpha_1, \ldots, \alpha_n$. The different accident years are connected among each other by the common *development pattern* $\vartheta_0, \vartheta_1, \ldots, \vartheta_n$.

The assumption of the multiplicative model is, in particular, fulfilled when the family of incremental claim numbers satisfies the assumptions of the *Poisson model*:

– The family $\{Z_{i,k}\}_{i,k \in \{0,1,\ldots,n\}}$ is independent.
– There exist parameters $\alpha_0, \alpha_1, \ldots, \alpha_n \in (0, \infty)$ and $\vartheta_0, \vartheta_1, \ldots, \vartheta_n \in (0, 1)$ with $\sum_{k=0}^{n} \vartheta_k = 1$ such that, for all $i, k \in \{0, 1, \ldots, n\}$, the distribution of $Z_{i,k}$ is the Poisson distribution with expectation $\alpha_i \vartheta_k$.

Under the assumptions of the Poisson model, Hachemeister and Stanard [1975] have shown that the maximum–likelihood estimators of the parameters $\alpha_0, \alpha_1, \ldots, \alpha_n$ agree with the *chain–ladder estimators*

$$\widehat{S}^{\mathrm{CL}}_{i,n} := S_{i,n-i} \prod_{l=n-i+1}^{n} \frac{\sum_{j=0}^{n-l} S_{j,l}}{\sum_{j=0}^{n-l} S_{j,l-1}}$$

of the expected ultimate aggregate claim numbers; details on chain–ladder estimation may be found e. g. in Schmidt [2002; Abschnitt 11.3] or in Schmidt [2004a].

In Section 5 of the present paper we review the Poisson model, and in Sections 6 and 7 we introduce two general multiplicative models which generalize the Poisson model and which contain only the Poisson model as a common special case. In both models, the assumption of independent incremental claim numbers is dropped and the distributions of the incremental claim numbers are Hofmann distributions. In each of these models, we show that the maximum–likelihood estimators of certain parameters agree with the chain–ladder estimators of the expected ultimate aggregate claim numbers. We thus obtain two extensions of the result of Hachemeister and Stanard [1975] and we also capture a related result of Schmidt and Wünsche [1998] for a model in which each of the ultimate aggregate claim numbers has a negativebinomial distribution.

We also present a variety of additional results which indicate that one should be most careful with the choice of a model and a method of prediction or estimation. For example, if in the models of Sections 5, 6 and 7 all parameters are assumed to be known, then maximum–likelihood estimation is meaningless and credibility predictors may be used instead to predict the ultimate aggregate claim numbers, but the credibility predictors usually differ from the chain–ladder predictors. Also, even in the case where all parameters are unknown, the maximum–likelihood estimators of the expected ultimate aggregate claim numbers may differ from the chain–ladder estimators, as can be seen from the model considered in Section 8.

## 2 Hofmann Families Revisited

In the present section we consider a family $\{Q_t\}_{t\in\mathbf{R}_+}$ of claim number distributions $\mathcal{B}(\mathbf{R}) \to [0,1]$ and a sequence $\{\Pi_k\}_{k\in\mathbf{N}_0}$ of functions $\mathbf{R}_+ \to [0,1]$ such that the identity

$$Q_t[\{k\}] = \Pi_k(t)$$

holds for all $t \in \mathbf{R}_+$ and $k \in \mathbf{N}_0$.

The family $\{Q_t\}_{t\in\mathbf{R}_+}$ is said to be the *Hofmann family* $\mathbf{H}(a,p,c)$ with parameters $a \in \mathbf{R}_+$ and $p,c \in (0,\infty)$ if there exists a differentiable function $\nu_{a,p,c}\colon \mathbf{R}_+ \to \mathbf{R}$ such that $\nu_{a,p,c}(0) = 0$ and

$$\begin{aligned} \frac{d\nu_{a,p,c}}{dt}(t) &= \frac{p}{(1+ct)^a} \\ \Pi_0(t) &= \exp(-\nu_{a,p,c}(t)) \\ \Pi_k(t) &= \frac{(-t)^k}{k!}\,\frac{d^k\Pi_0}{dt^k}(t) \end{aligned}$$

holds for all $t \in \mathbf{R}_+$. The Hofmann family was introduced by Hofmann [1955].

Throughout this section, we assume that $\{Q_t\}_{t \in \mathbf{R}_+}$ is the Hofmann family $\mathbf{H}(a, p, c)$. Integration yields

$$\nu_{a,p,c}(t) = \begin{cases} pt & \text{if } a = 0 \\ \dfrac{p}{c} \log(1+ct) & \text{if } a = 1 \\ \dfrac{p}{c} \dfrac{(1+ct)^{1-a} - 1}{1-a} & \text{if } a \in (0,1) \cup (1, \infty) \end{cases}$$

for all $t \in \mathbf{R}_+$. The cases $a = 0$ and $a = 1$ are of special interest:

**2.1 Examples**

(1) **Poisson case.** In the case $a = 0$, one has

$$Q_t[\{k\}] = \exp(-pt) \frac{(pt)^k}{k!}$$

for all $t \in \mathbf{R}_+$ and $k \in \mathbf{N}_0$.

(2) **Negativebinomial case.** In the case $a = 1$, one has

$$Q_t[\{k\}] = \binom{p/c + k - 1}{k} \left(\frac{1}{1+ct}\right)^{p/c} \left(\frac{ct}{1+ct}\right)^k$$

for all $t \in \mathbf{R}_+$ and $k \in \mathbf{N}_0$.

The Poisson case and the negativebinomial case are in a sense singular cases within the collection of all Hofmann families: In the Poisson case the parameter $c$ does not matter, and in the negativebinomial case the function $\nu_{a,p,c}$ depends on the logarithm instead of a power of $1 + ct$.

The key to the analysis of the Hofmann family $\mathbf{H}(a, p, c)$ is the following result:

**2.2 Proposition** *There exists a probability measure* $Q_{a,p,c} : \mathcal{B}(\mathbf{R}) \to [0, 1]$ *with* $Q_{a,p,c}[\mathbf{R}_+] = 1$ *and such that*

$$Q_t[\{k\}] = \int_{\mathbf{R}} \exp(-\lambda t) \frac{(\lambda t)^k}{k!} \, dQ_{a,p,c}(\lambda)$$

*holds for all* $t \in \mathbf{R}_+$ *and* $k \in \mathbf{N}_0$.

Since the function $\Pi_0$ is completely monotone, Proposition 2.2 follows from the Bernstein–Widder Theorem; see e.g. Berg, Christensen and Ressel [1984].

The mixed Poisson representation of the claim number distributions of the Hofmann family $\mathbf{H}(a,p,c)$ given by Proposition 2.2 yields the following fundamental result on their probability generating functions:

**2.3 Proposition** *The probability generating function of $Q_t$ satisfies*

$$m_{Q_t}(z) \;=\; \exp(-\nu_{a,p,c}(t-tz))\,.$$

Proposition 2.3 yields all further properties of the distributions of a Hofmann family which will be presented in the subsequent section.

## 3 Hofmann Distributions

In the present section we consider claim number distributions

$$\pi_{a,p,c,t} := \{\pi_{a,p,c,t}(k)\}_{k\in\mathbf{N}_0}$$

with parameters $a,t \in \mathbf{R}_+$ and $p,c \in (0,\infty)$ such that $\{\pi_{a,p,c,t}\}_{t\in\mathbf{R}_+}$ is the Hofmann family $\mathbf{H}(a,p,c)$. We put

$$\nu_{a,p,c,t} := \nu_{a,p,c}(t)\,.$$

Since $\pi_{a,p,c,0}(0) = 1$, we assume throughout this section that $t \in (0,\infty)$.

As a first and immediate consequence of Proposition 2.3 we obtain a formula for the expectation of $\pi_{a,p,c,t}$:

**3.1 Corollary** *The expectation of $\pi_{a,p,c,t}$ satisfies*

$$E[\pi_{a,p,c,t}] \;=\; pt$$

*and the variance of $\pi_{a,p,c,t}$ satisfies*

$$\mathrm{var}[\pi_{a,p,c,t}] \;=\; pt + acpt^2\,.$$

To proceed further, we define an auxiliary sequence $h_{a,p,c,t} = \{h_{a,p,c,t}(k)\}_{k\in\mathbf{N}_0}$ by letting $h_{a,p,c,t}(0) := 0$ and

$$h_{a,p,c,t}(k) := \begin{cases} \delta_{1,k} & \text{if } a = 0 \\[2ex] \dfrac{1}{k\,\log(1+ct)}\left(\dfrac{ct}{1+ct}\right)^k & \text{if } a = 1 \\[2ex] \dfrac{\dbinom{a-1+k-1}{k}\left(\dfrac{1}{1+ct}\right)^{a-1}\left(\dfrac{ct}{1+ct}\right)^k}{1-\left(\dfrac{1}{1+ct}\right)^{a-1}} & \text{if } a \in (0,1)\cup(1,\infty) \end{cases}$$

for all $k \in \mathbf{N}$ (where $\delta_{1,k}$ is the Kronecker symbol). Then $h_{a,p,c,t}$ is a claim number distribution.

Kestemont and Paris [1985] have pointed out that every distribution of the Hofmann family $\mathbf{H}(a,p,c)$ is a compound Poisson distribution. The following result makes this statement more precise.

**3.2 Corollary** *The identity*

$$\pi_{a,p,c,t}(k) = \sum_{j=0}^{\infty} h_{a,p,c,t}^{*j}(k)\,\exp(-\nu_{a,p,c,t})\,\frac{(\nu_{a,p,c,t})^j}{j!}$$

*holds for all* $k \in \mathbf{N}_0$.

Corollary 3.2 can be obtained by comparing the probability generating functions of both sides of the identity; see Hess, Liewald and Schmidt [2002].

The following formula for the recursive computation of the Hofmann distribution $\pi_{a,p,c,t}$ is immediate from Corollary 3.2 and Panjer's recursion; see e. g. Schmidt [1996; Theorem 5.4.2] or Schmidt [2002; Folgerung 7.3.2]:

**3.3 Corollary** *The identity*

$$\pi_{a,p,c,t}(k) = \sum_{j=1}^{k} \nu_{a,p,c,t}\,\frac{j}{k}\,\pi_{a,p,c,t}(k-j)\,h_{a,p,c,t}(j)$$

*holds for all* $k \in \mathbf{N}$.

The identities of Corollaries 3.2 and 3.3 suggest to define another auxiliary sequence $g_{a,p,c,t} = \{g_{a,p,c,t}(k)\}_{k\in\mathbf{N}_0}$ by letting

$$g_{a,p,c,t}(k) := \nu_{a,p,c,t}\, h_{a,p,c,t}(k)$$

for all $k \in \mathbf{N}_0$. Combining Corollaries 3.2 and 3.3 we thus obtain the following result which is due to Hürlimann [1990] and which will be the starting point of the discussion in Section 4:

**3.4 Corollary** *The identities*

$$\pi_{a,p,c,t}(k) = \exp(-\nu_{a,p,c,t}) \sum_{j=0}^{\infty} \frac{g_{a,p,c,t}^{*j}(k)}{j!}$$

*and*

$$\pi_{a,p,c,t}(k) = \frac{1}{k} \sum_{j=1}^{k} j\, g_{a,p,c,t}(j)\, \pi_{a,p,c,t}(k-j)$$

*hold for all* $k \in \mathbf{N}$.

In Section 4 we will also need the explicit form of the sequence $g_{a,p,c,t}$ which is easily obtained from the definition of $h_{a,p,c,t}$:

**3.5 Corollary** *The sequence* $g_{a,p,c,t}$ *satisfies* $g_{a,p,c,t}(0) = 0$ *and*

$$g_{a,p,c,t}(k) = \frac{pt}{k} \binom{a+k-1-1}{k-1} \left(\frac{1}{1+ct}\right)^{a} \left(\frac{ct}{1+ct}\right)^{k-1}$$

*for all* $k \in \mathbf{N}$.

The final result of this section shows that the collection $\{\pi_{a,p,c,t}\}_{a\in\mathbf{R}_+,p,c,t\in(0,\infty)}$ of all Hofmann distribution with parameter $t \in (0,\infty)$ is identical with the collection $\{\pi_{a,p,c,1}\}_{a\in\mathbf{R}_+,p,c\in(0,\infty)}$ of all Hofmann distribution with parameter $t = 1$:

**3.6 Lemma** *For each* $t \in (0,\infty)$, *the Hofmann distributions* $\pi_{a,p,c,t}$ *and* $\pi_{a,pt,ct,1}$ *are identical.*

**Proof** We have $\nu_{a,p,c,t} = \nu_{a,pt,ct,1}$ and Corollary 3.5 yields $g_{a,p,c,t} = g_{a,pt,ct,1}$. Now the assertion follows from Corollary 3.4. □

## 4 Maximum–Likelihood Estimation of the Expectation of a Hofmann Distribution

The Hofmann distribution $\pi_{a,p,c,0}$ is concentrated in 0 and is thus without interest. Throughout this section we assume that $t \in (0, \infty)$. In this case, Lemma 3.6 yields

$$\pi_{a,p,c,t} = \pi_{a,pt,ct,1} \,.$$

To estimate the expectation of a Hofmann distribution, it is therefore sufficient to consider the collection of all Hofmann distributions $\pi_{a,p,c,1}$ with parameters $a \in \mathbf{R}_+$ and $p, c \in (0, \infty)$. To simplify the notation, we put

$$\pi_{a,p,c} := \pi_{a,p,c,1}$$

$$\nu_{a,p,c} := \nu_{a,p,c,1}$$

$$g_{a,p,c} := g_{a,p,c,1}$$

In the present section, we consider maximum–likelihood estimation of the expectation $p$ of the Hofmann distribution $\pi_{a,p,c}$ on the basis of a sample $X_1, \ldots, X_m$ from the Hofmann distribution $\pi_{a,p,c}$ with parameters $a, p, c$. We therefore assume that the parameter $p$ is unknown but we do not make such an assumption on the parameters $a$ and $c$.

By Corollary 3.4, the likelihood function is given by

$$L_{a,p,c}(x_1, \ldots, x_m) := \prod_{i=1}^{m} \left( \exp(-\nu_{a,p,c}) \sum_{j=0}^{\infty} \frac{g_{a,p,c}^{*j}(x_i)}{j!} \right)$$

and it follows that the log–likelihood function satisfies

$$\log\Bigl(L_{a,p,c}(x_1, \ldots, x_m)\Bigr) = -\, m\, \nu_{a,p,c} + \sum_{i=1}^{m} \log\left( \sum_{j=0}^{\infty} \frac{g_{a,p,c}^{*j}(x_i)}{j!} \right) .$$

The partial derivatives of the log–likelihood function with respect to the parameters vanish if and only if the partial derivatives of the log–likelihood function with respect to a bijective transformation of the parameters vanish.

Following Hürlimann [1990], we consider the map $\varrho : \mathbf{R}_+ \times (0,\infty) \times (0,\infty) \to \mathbf{R}_+ \times (0,\infty) \times (0,1)$ given by

$$\varrho \begin{pmatrix} a \\ p \\ c \end{pmatrix} := \begin{pmatrix} a \\ p \left(\dfrac{1}{1+c}\right)^a \left(\dfrac{c}{1+c}\right)^{-1} \\ \dfrac{c}{1+c} \end{pmatrix} .$$

Then $\varrho$ is a diffeomorphism with

$$\varrho^{-1} \begin{pmatrix} a \\ b \\ q \end{pmatrix} = \begin{pmatrix} a \\ b\,q \left(\dfrac{1}{1-q}\right)^a \\ q\,\dfrac{1}{1-q} \end{pmatrix} .$$

Therefore, the partial derivatives of $\log\bigl(L_{a,p,c}(x_1,\ldots,x_n)\bigr)$ with respect to $a,p,c$ vanish if and only if the partial derivatives of

$$\begin{aligned} &\log\Bigl(L_{\varrho^{-1}(a,b,q)}(x_1,\ldots,x_m)\Bigr) \\ &= -\,m\,\nu_{\varrho^{-1}(a,b,q)} + \sum_{i=1}^{m} \log\left(\sum_{j=0}^{\infty} \frac{g^{*j}_{\varrho^{-1}(a,b,q)}(x_i)}{j!}\right) \end{aligned}$$

with respect to $a,b,q$ vanish.

We shall now determine the partial derivative of $\log\bigl(L_{\varrho^{-1}(a,b,q)}(x_1,\ldots,x_m)\bigr)$ with respect to the parameter $q$. To this end, we establish two lemmas:

**4.1 Lemma**

$$\frac{\partial \nu_{\varrho^{-1}(a,b,q)}}{\partial q} = b\left(\frac{1}{1-q}\right)^a .$$

**Proof** Using the transformation $\varrho$, we obtain

$$\nu_{\varrho^{-1}(a,b,q)} = \begin{cases} b\,q & \text{if } a = 0 \\ b\,(-\log(1-q)) & \text{if } a = 1 \\ b\,\dfrac{1-(1-q)^{1-a}}{1-a} & \text{if } a \in (0,1)\cup(1,\infty) \end{cases}$$

and hence

$$\frac{\partial \nu_{\varrho^{-1}(a,b,q)}}{\partial q} = b\left(\frac{1}{1-q}\right)^a$$

as was to be shown. □

In accordance with the definition of scalar multiples or sums of sequences, we define the partial derivative of the sequence $g_{\varrho^{-1}(a,b,q)}$ with respect to $q$ to be the sequence

$$\frac{\partial g_{\varrho^{-1}(a,b,q)}}{\partial q} := \left\{\frac{\partial g_{\varrho^{-1}(a,b,q)}(k)}{\partial q}\right\}_{k\in\mathbf{N}_0}.$$

Then we have the following lemma:

**4.2 Lemma** *The identity*

$$\frac{\partial\left(\log\sum_{j=0}^{\infty}\frac{1}{j!}\,g^{*j}_{\varrho^{-1}(a,b,q)}(k)\right)}{\partial q} = \frac{k}{q}$$

*holds for all* $k \in \mathbf{N}_0$.

**Proof** The definition of the partial derivative of the sequence $g_{\varrho^{-1}(a,b,q)}$ yields

$$\frac{\partial g^{*0}_{\varrho^{-1}(a,b,q)}(k)}{\partial q} = 0$$

for all $k \in \mathbf{N}_0$ and

$$\frac{\partial g^{*j}_{\varrho^{-1}(a,b,q)}(k)}{\partial q} = j\left(\frac{\partial g_{\varrho^{-1}(a,b,q)}}{\partial q} * g^{*(j-1)}_{\varrho^{-1}(a,b,q)}\right)(k)$$

for all $j \in \mathbf{N}$ and for all $k \in \mathbf{N}_0$. Furthermore, Corollary 3.5 yields $g_{\varrho^{-1}(a,b,q)}(0) = 0$ and

$$g_{\varrho^{-1}(a,b,q)}(k) = \frac{b}{k}\binom{a+k-1-1}{k-1}q^k$$

for all $k \in \mathbf{N}$, hence

$$\frac{\partial g_{\varrho^{-1}(a,b,q)}}{\partial q}(k) = \frac{k}{q}\,g_{\varrho^{-1}(a,b,q)}(k)$$

for all $k \in \mathbf{N}_0$, and now the second identity of Corollary 3.4 yields

$$\left(\frac{\partial g_{\varrho^{-1}(a,b,q)}}{\partial q} * \pi_{\varrho^{-1}(a,b,q)}\right)(k) = \frac{k}{q}\,\pi_{\varrho^{-1}(a,b,q)}(k)$$

for all $k \in \mathbf{N}_0$. Using the identities established so far and the first identity of Corollary 3.4, we obtain

$$\begin{aligned}
\frac{\partial\left(\log \sum_{j=0}^{\infty} \frac{1}{j!}\, g^{*j}_{\varrho^{-1}(a,b,q)}(k)\right)}{\partial q}
&= \frac{\sum_{j=0}^{\infty} \frac{1}{j!} \frac{\partial g^{*j}_{\varrho^{-1}(a,b,q)}(k)}{\partial q}}{\sum_{j=0}^{\infty} \frac{1}{j!}\, g^{*j}_{\varrho^{-1}(a,b,q)}(k)} \\
&= \frac{\sum_{j=1}^{\infty} \frac{1}{(j-1)!}\left(\frac{\partial g_{\varrho^{-1}(a,b,q)}}{\partial q} * g^{*(j-1)}_{\varrho^{-1}(a,b,q)}\right)(k)}{\sum_{j=0}^{\infty} \frac{1}{j!}\, g^{*j}_{\varrho^{-1}(a,b,q)}(k)} \\
&= \frac{\left(\frac{\partial g_{\varrho^{-1}(a,b,q)}}{\partial q} * \sum_{j=0}^{\infty} \frac{g^{*j}_{\varrho^{-1}(a,b,q)}}{j!}\right)(k)}{\left(\sum_{j=0}^{\infty} \frac{g^{*j}_{\varrho^{-1}(a,b,q)}}{j!}\right)(k)} \\
&= \frac{\left(\frac{\partial g_{\varrho^{-1}(a,b,q)}}{\partial q} * \pi_{\varrho^{-1}(a,b,q)}\right)(k)}{\pi_{\varrho^{-1}(a,b,q)}(k)} \\
&= \frac{\frac{k}{q}\,\pi_{\varrho^{-1}(a,b,q)}(k)}{\pi_{\varrho^{-1}(a,b,q)}(k)} \\
&= \frac{k}{q}
\end{aligned}$$

for all $k \in \mathbf{N}_0$. $\square$

We are now ready to determine the partial derivative of the log-Likelihood function $\log\big(L_{\varrho^{-1}(a,b,q)}(x_1, \ldots, x_m)\big)$ with respect to the parameter $q$:

**4.3 Lemma** *The partial derivative of* $\log\bigl(L_{\varrho^{-1}(a,b,q)}(x_1,\ldots,x_m)\bigr)$ *with respect to the parameter* $q$ *satisfies the identity*

$$\frac{\partial\Bigl(\log L_{a,p,c}(x_1,\ldots,x_m)\Bigr)}{\partial q} = -m\,b\left(\frac{1}{1-q}\right)^{a} + \frac{1}{q}\sum_{i=1}^{m} x_i\,.$$

**Proof** The assertion follows from Lemmas 4.1 and 4.2. □

We thus obtain the following result:

**4.4 Theorem** *If* $\widehat{p}$ *is a maximum likelihood estimator of* $p$, *then*

$$\widehat{p} = \frac{1}{m}\sum_{i=1}^{m} X_i\,.$$

**Proof** Since

$$p = b\,q\left(\frac{1}{1-q}\right)^{a}$$

the assertion follows from Lemma 4.3. □

Theorem 4.4 is due to Hürlimann [1990], but its proof presented here avoids his general results on maximum–likelihood estimation.

## 5 The Poisson Model

We now turn to the problem of loss reserving for claim numbers and consider a family $\{Z_{i,k}\}_{i,k\in\{0,1,\ldots,n\}}$ of random variables taking their values in $\mathbf{N}_0$. The random variables $Z_{i,k}$ are called *incremental claim numbers* and the sum

$$S_{i,n} := \sum_{k=0}^{n} Z_{i,k}$$

is called the *ultimate aggregate claim number* of *accident year* $i \in \{0,1,\ldots,n\}$. We assume that the incremental claim numbers are *observable* for $i+k \leq n$ and that they are *non–observable* for $i+k > n$.

In the *Poisson model* introduced by Hachemeister and Stanard [1975], it is assumed that the joint distribution of the family $\{Z_{i,k}\}_{i,k\in\{0,1,\ldots,n\}}$ satisfies

$$P\left[\bigcap_{i=0}^{n}\bigcap_{k=0}^{n}\{Z_{i,k}=z_{i,k}\}\right]=\prod_{i=0}^{n}\prod_{k=0}^{n}\exp(-\alpha_i\vartheta_k)\,\frac{(\alpha_i\vartheta_k)^{z_{i,k}}}{z_{i,k}!}$$

where $\alpha_0,\alpha_1,\ldots,\alpha_n\in(0,\infty)$ and $\vartheta_0,\vartheta_1,\ldots,\vartheta_n\in(0,1)$ are parameters satisfying $\sum_{k=0}^{n}\vartheta_k=1$. In this model,

- the family $\{Z_{i,k}\}_{i,k\in\{0,1,\ldots,n\}}$ of incremental claim numbers is independent,
- for all $i,k\in\{0,1,\ldots,n\}$, the incremental claim number $Z_{i,k}$ has the Poisson distribution with expectation $\alpha_i\vartheta_k$, and
- the expected ultimate aggregate claim number satisfies

$$E[S_{i,n}]=\alpha_i\,.$$

In particular, the expectation of any incremental claim number satisfies

$$E[Z_{i,k}]=\alpha_i\vartheta_k\,.$$

Therefore, the Poisson model is a multiplicative model.

In the case where the parameters of the Poisson model are unknown, the parameters may be estimated by the maximum–likelihood method. The following result is due to Hachemeister and Stanard [1975]:

**5.1 Proposition** *If $\widehat{\alpha}_0,\widehat{\alpha}_1,\ldots,\widehat{\alpha}_n$ are maximum–likelihood estimators of $\alpha_0,\alpha_1,\ldots,\alpha_n$, then $\widehat{\alpha}_0,\widehat{\alpha}_1,\ldots,\widehat{\alpha}_n$ are the chain–ladder estimators of $\alpha_0,\alpha_1,\ldots,\alpha_n$.*

The link between chain–ladder estimation and maximum–likelihood estimation in an arbitrary multiplicative model (and not only in the Poisson model) is provided by *marginal–sum estimation*:

The assumptions of the multiplicative model imply

$$\sum_{k=0}^{n-i}E[Z_{i,k}]=\sum_{k=0}^{n-i}\alpha_i\vartheta_k$$

for all $i\in\{0,1,\ldots,n\}$ and

$$\sum_{i=0}^{n-k}E[Z_{i,k}]=\sum_{i=0}^{n-k}\alpha_i\vartheta_k$$

for all $k \in \{0, 1, \ldots, n\}$. Because of these identities, it is quite natural to estimate the parameters $\alpha_0, \alpha_1, \ldots, \alpha_n$ and $\vartheta_0, \vartheta_1, \ldots, \vartheta_n$ by the *marginal–sum estimators* $\widehat{\alpha}_0, \widehat{\alpha}_1, \ldots, \widehat{\alpha}_n$ and $\widehat{\vartheta}_0, \widehat{\vartheta}_1, \ldots, \widehat{\vartheta}_n$ which are defined to be solutions of the *marginal–sum equations*

$$\sum_{k=0}^{n-i} Z_{i,k} = \sum_{k=0}^{n-i} \widehat{\alpha}_i \widehat{\vartheta}_k$$

with $i \in \{0, 1, \ldots, n\}$ and

$$\sum_{i=0}^{n-k} Z_{i,k} = \sum_{i=0}^{n-k} \widehat{\alpha}_i \widehat{\vartheta}_k$$

with $k \in \{0, 1, \ldots, n\}$ under the condition $\sum_{k=0}^{n} \widehat{\vartheta}_k = 1$. It is well–known and has been pointed out by Schmidt and Wünsche [1998] that marginal–sum estimators exist and are unique, and that the marginal–sum estimators of $\alpha_0, \alpha_1, \ldots, \alpha_n$ agree with the chain–ladder estimators of the expected ultimate aggregate claim numbers.

In order to prove that the maximum–likelihood estimators of the expected ultimate aggregate claim numbers agree with the chain–ladder estimators, it is therefore sufficient to show that the maximum–likelihood estimators of the parameters of the multiplicative model agree with the marginal–sum estimators.

In the case where the parameters of the Poisson model are known (which may be the case when they are provided by external information which is not contained in the run–off triangle), the ultimate aggregate claim numbers may be predicted by the credibility method:

**5.2 Theorem** *For every accident year $i \in \{0, 1, \ldots, n\}$, the credibility predictor $S_{i,n}^*$ of the ultimate aggregate claim number $S_{i,n}$ satisfies*

$$S_{i,n}^* = \sum_{k=i}^{n-i} Z_{i,k} + \left( \sum_{k=n-i+1}^{n} \vartheta_k \right) \alpha_i \, .$$

By Theorem 5.2, which is a special case of Theorems 6.6 and 7.6 given below, the credibility predictors of the ultimate aggregate claim numbers are predictors of the *Bornhuetter–Ferguson* type; see Schnaus [2004].

With regard to the first extension of the Poisson model which will be studied in Section 6, we note that the assumption of the Poisson model is equivalent to the assumption that the joint distribution of the family $\{Z_{i,k}\}_{i,k\in\{0,1,\ldots,n\}}$ satisfies

$$P\left[\bigcap_{i=0}^{n}\bigcap_{k=0}^{n}\{Z_{i,k}=z_{i,k}\}\right]=\prod_{i=0}^{n}\prod_{k=0}^{n}\exp(-\lambda\beta_i\vartheta_k)\,\frac{(\lambda\beta_i\vartheta_k)^{z_{i,k}}}{z_{i,k}!}$$

where $\lambda\in(0,\infty)$ as well as $\beta_0,\beta_1,\ldots,\beta_n\in(0,1)$ and $\vartheta_0,\vartheta_1,\ldots,\vartheta_n\in(0,1)$ are parameters satisfying $\sum_{i=0}^{n}\beta_i=\sum_{k=0}^{n}\vartheta_k=1$. Then we have $E[Z_{i,k}]=\lambda\beta_i\vartheta_k$ for all $i,k\in\{0,1,\ldots,n\}$, and hence

$$E\left[\sum_{i=0}^{n}\sum_{k=0}^{n}Z_{i,k}\right]=\lambda\,.$$

This means that $\lambda$ is the *expected total number of claims*. The parameters $\beta_i$ and $\vartheta_k$ may then be interpreted as the parts of the expected total number of claims which belong to accident year $i$ or to development year $k$, respectively.

## 6 The First Extension of the Poisson Model

In the present section, we consider a first extension of the Poisson model and we assume that the joint distribution of the family $\{Z_{i,k}\}_{i,k\in\{0,1,\ldots,n\}}$ satisfies

$$\begin{aligned}P\left[\bigcap_{i=0}^{n}\bigcap_{k=0}^{n}\{Z_{i,k}=z_{i,k}\}\right]\\ =\int_{\mathbf{R}}\prod_{i=0}^{n}\prod_{k=0}^{n}\exp(-\lambda\beta_i\vartheta_k)\,\frac{(\lambda\beta_i\vartheta_k)^{z_{i,k}}}{z_{i,k}!}\,dQ_{a,p,c}(\lambda)\end{aligned}$$

where $\beta_0,\beta_1,\ldots,\beta_n\in(0,1)$ and $\vartheta_0,\vartheta_1,\ldots,\vartheta_n\in(0,1)$ are parameters satisfying $\sum_{i=0}^{n}\beta_i=\sum_{k=0}^{n}\vartheta_k=1$ and $Q_{a,p,c}$ is the mixing distribution of the Hofmann family $\mathbf{H}(a,p,c)$. In the case $a=0$ this model coincides with the Poisson model.

Let us first study the present model in more detail.

**6.1 Lemma** *The expectation of any incremental claim number satisfies*

$$E[Z_{i,k}]\;=\;(\beta_i\vartheta_k)\,p$$

*and the covariance of any two incremental claim numbers satisfies*

$$\mathrm{cov}[Z_{i,k}, Z_{j,l}] = (\beta_i \vartheta_k)(\beta_j \vartheta_l)\, acp + (\beta_i \vartheta_k)\, p\, \delta_{i,j}\, \delta_{k,l}\,.$$

**Proof** The formulas for the expectation and the variance of $Z_{i,k}$ are evident from $P_{Z_{i,k}} = \pi(a, p, c, \beta_i \vartheta_k)$ and Corollary 3.1. Furthermore, since

$$\begin{aligned}
&(\beta_i \vartheta_k)^2\, acp + (\beta_i \vartheta_k)\, p \\
&\quad = \mathrm{var}[Z_{i,k}] \\
&\quad = E[Z_{i,k}(Z_{i,k}-1)] + E[Z_{i,k}] - \Big(E[Z_{i,k}]\Big)^2 \\
&\quad = (\beta_i \vartheta_k)^2 \int_{\mathbf{R}} \lambda^2\, dQ_{a,p,c}(\lambda) + (\beta_i \vartheta_k)\, p - \Big((\beta_i \vartheta_k)\, p\Big)^2
\end{aligned}$$

we obtain $\int_{\mathbf{R}} \lambda^2\, dQ_{a,p,c}(\lambda) = acp + p^2$ and hence

$$\begin{aligned}
\mathrm{cov}[Z_{i,k}, Z_{j,l}] &= E[Z_{i,k} Z_{j,l}] - E[Z_{i,k}]\, E[Z_{j,l}] \\
&= (\beta_i \vartheta_k)(\beta_j \vartheta_l) \int_{\mathbf{R}} \lambda^2\, Q_{a,p,c}(\lambda) - (\beta_i \vartheta_k)\, p\, (\beta_j \vartheta_l)\, p \\
&= (\beta_i \vartheta_k)(\beta_j \vartheta_l)\, acp
\end{aligned}$$

for all $i, j, k, l \in \{0, 1, \ldots, n\}$ such that $i \neq j$ or $k \neq l$. □

Lemma 6.1 implies that, except for the case $a = 0$, any two distinct incremental claim numbers are strictly positively correlated and are hence dependent. The lemma also yields a characterization of the Poisson model as a special case of the present model:

**6.2 Theorem** *The following are equivalent:*

(a) *There exist two distinct incremental claim numbers which are uncorrelated.*

(b) *The family of all incremental claim numbers is uncorrelated.*

(c) *The family of all incremental claim numbers is independent.*

(d) *The family of all incremental claim numbers satisfies the assumption of the Poisson model.*

**Proof** Assume first that (a) holds. Then there exist two incremental claim numbers $Z_{i,k}$ and $Z_{j,l}$ such that $i \neq j$ or $k \neq l$ with $\mathrm{cov}[Z_{i,k}, Z_{j,l}] = 0$.Because of Lemma 6.1, this yields $a = 0$. Therefore, (a) implies (d). The remaining implications are obvious. □

Because of Lemma 6.1, the expectation of any incremental claim number satisfies

$$E[Z_{i,k}] = (p\,\beta_i)\,\vartheta_k\,.$$

Therefore, the present model is a multiplicative model, and summation yields

$$E\left[\sum_{i=0}^{n}\sum_{k=0}^{n} Z_{i,k}\right] = p\,.$$

Therefore, the parameter $p$ of the present model takes the role of the parameter $\lambda$ in the Poisson model and the interpretation of the parameters $\beta_0, \beta_1, \ldots, \beta_n$ and $\vartheta_0, \vartheta_1, \ldots, \vartheta_n$ is the same as in the Poisson model.

In the case where the parameters of the present model are unknown, the parameters may be estimated by the maximum–likelihood method:

**6.3 Lemma** *The identity*

$$P\left[\bigcap_{i=0}^{n}\bigcap_{k=0}^{n-i}\{Z_{i,k} = z_{i,k}\}\right] = \frac{\left(\sum_{i=0}^{n}\sum_{k=0}^{n-i} z_{i,k}\right)!}{\prod_{i=0}^{n}\prod_{k=0}^{n-i} z_{i,k}!}\prod_{i=0}^{n}\prod_{k=0}^{n-i}\left(\frac{\beta_i\vartheta_k}{\sum_{j=0}^{n}\sum_{l=0}^{n-j}\beta_j\vartheta_l}\right)^{z_{i,k}}$$

$$\cdot\,\pi_{a,\,p\sum_{i=0}^{n}\sum_{k=0}^{n-i}\beta_i\vartheta_k,\,c\sum_{i=0}^{n}\sum_{k=0}^{n-i}\beta_i\vartheta_k}\left(\sum_{i=0}^{n}\sum_{k=0}^{n-i} z_{i,k}\right)$$

*holds for every family* $\{z_{i,k}\}_{i,k\in\{0,1,\ldots,n\},i+k\leq n} \subseteq \mathbf{N}_0$, *and the identity*

$$P\left[\left\{\sum_{i=0}^{n}\sum_{k=0}^{n-i} Z_{i,k} = z\right\}\right] = \pi_{a,\,p\sum_{i=0}^{n}\sum_{k=0}^{n-i}\beta_i\vartheta_k,\,c\sum_{i=0}^{n}\sum_{k=0}^{n-i}\beta_i\vartheta_k}(z)$$

*holds for all* $z \in \mathbf{N}_0$.

**Proof** Summation yields

$$
\begin{aligned}
&P\left[\bigcap_{i=0}^{n}\bigcap_{k=0}^{n-i}\{Z_{i,k}=z_{i,k}\}\right]\\
&= \int_{\mathbf{R}} \prod_{i=0}^{n}\prod_{k=0}^{n-i} \exp(-\lambda\beta_i\vartheta_k)\,\frac{(\lambda\beta_i\vartheta_k)^{z_{i,k}}}{z_{i,k}!}\,dQ_{a,p,c}(\lambda)\\
&= \frac{\left(\sum_{i=0}^{n}\sum_{k=0}^{n-i} z_{i,k}\right)!}{\prod_{i=0}^{n}\prod_{k=0}^{n-i} z_{i,k}!}\prod_{i=0}^{n}\prod_{k=0}^{n-i}\left(\frac{\beta_i\vartheta_k}{\sum_{j=0}^{n}\sum_{l=0}^{n-j}\beta_j\vartheta_l}\right)^{z_{i,k}}\\
&\qquad \cdot \int_{\mathbf{R}} \exp\left(-\lambda\sum_{i=0}^{n}\sum_{k=0}^{n-i}\beta_i\vartheta_k\right)\frac{\left(\lambda\sum_{i=0}^{n}\sum_{k=0}^{n-i}\beta_i\vartheta_k\right)^{\sum_{i=0}^{n}\sum_{k=0}^{n-i} z_{i,k}}}{\left(\sum_{i=0}^{n}\sum_{k=0}^{n-i} z_{i,k}\right)!}\,dQ_{a,p,c}(\lambda)\,.
\end{aligned}
$$

This proves the first identity, and the second identity follows by summation. □

Because of the first identity of Lemma 6.3, the likelihood function, which depends on the parameters $a, p, c$ and on the parameters $\beta_0, \beta_1, \ldots, \beta_n$ and $\vartheta_0, \vartheta_1, \ldots, \vartheta_n$, satisfies

$$
\begin{aligned}
L = &\frac{\left(\sum_{i=0}^{n}\sum_{k=0}^{n-i} z_{i,k}\right)!}{\prod_{i=0}^{n}\prod_{k=0}^{n-i} z_{i,k}!}\prod_{i=0}^{n}\prod_{k=0}^{n-i}\left(\frac{\beta_i\vartheta_k}{\sum_{j=0}^{n}\sum_{l=0}^{n-j}\beta_j\vartheta_l}\right)^{z_{i,k}}\\
&\cdot\pi_{a,\,p\sum_{j=0}^{n}\sum_{l=0}^{n-j}\beta_j\vartheta_l,\,c\sum_{j=0}^{n}\sum_{l=0}^{n-j}\beta_j\vartheta_l}\left(\sum_{j=0}^{n}\sum_{l=0}^{n-j} z_{j,l}\right)
\end{aligned}
$$

and it follows that the log–likelihood function satisfies

$$
\begin{aligned}
\log(L) = g &+ \sum_{i=0}^{n}\sum_{k=0}^{n-i} z_{i,k}\left(\log(\beta_i\vartheta_k) - \log\left(\sum_{j=0}^{n}\sum_{l=0}^{n-j}\beta_j\vartheta_l\right)\right)\\
&+ \log\left(\pi_{a,\,p\sum_{j=0}^{n}\sum_{l=0}^{n-j}\beta_j\vartheta_l,\,c\sum_{j=0}^{n}\sum_{l=0}^{n-j}\beta_j\vartheta_l}\left(\sum_{j=0}^{n}\sum_{l=0}^{n-j} z_{j,l}\right)\right)
\end{aligned}
$$

where $g$ is a constant not depending on the parameters.

**6.4 Lemma** *If* $\widehat{p}, \widehat{\beta}_0, \widehat{\beta}_1, \dots, \widehat{\beta}_n, \widehat{\vartheta}_0, \widehat{\vartheta}_1, \dots, \widehat{\vartheta}_n$ *are maximum–likelihood estimators of* $p, \beta_0, \beta_1, \dots, \beta_n, \vartheta_0, \vartheta_1, \dots, \vartheta_n$, *then* $\widehat{p}\,\widehat{\beta}_0, \widehat{p}\,\widehat{\beta}_1, \dots, \widehat{p}\,\widehat{\beta}_n$ *and* $\widehat{\vartheta}_0, \widehat{\vartheta}_1, \dots, \widehat{\vartheta}_n$ *are the marginal–sum estimators of* $p\beta_0, p\beta_1, \dots, p\beta_n$ *and* $\vartheta_0, \vartheta_1, \dots, \vartheta_n$.

**Proof** Define

$$\widetilde{p} := p \sum_{j=0}^{n} \sum_{l=0}^{n-j} \beta_j \vartheta_l$$

$$\widetilde{c} := c \sum_{j=0}^{n} \sum_{l=0}^{n-j} \beta_j \vartheta_l \,.$$

Then we have

$$\frac{\partial(\log L)}{\partial p} = \frac{\partial\left(\log \pi_{a,\widetilde{p},\widetilde{c}}\left(\sum\limits_{j=0}^{n}\sum\limits_{l=0}^{n-j} z_{j,l}\right)\right)}{\partial \widetilde{p}} \cdot \sum_{j=0}^{n} \sum_{l=0}^{n-j} \beta_j \vartheta_l$$

$$\frac{\partial(\log L)}{\partial c} = \frac{\partial\left(\log \pi_{a,\widetilde{p},\widetilde{c}}\left(\sum\limits_{j=0}^{n}\sum\limits_{l=0}^{n-j} z_{j,l}\right)\right)}{\partial \widetilde{c}} \cdot \sum_{j=0}^{n} \sum_{l=0}^{n-j} \beta_j \vartheta_l \,.$$

Since $\sum_{j=0}^{n} \sum_{l=0}^{n-j} \beta_j \vartheta_l \neq 0$, the maximum–likelihood conditions yield

$$\frac{\partial\left(\log \pi_{a,\widetilde{p},\widetilde{c}}\left(\sum\limits_{j=0}^{n}\sum\limits_{l=0}^{n-j} z_{j,l}\right)\right)}{\partial \widetilde{p}} = 0$$

$$\frac{\partial\left(\log \pi_{a,\widetilde{p},\widetilde{c}}\left(\sum\limits_{j=0}^{n}\sum\limits_{l=0}^{n-j} z_{j,l}\right)\right)}{\partial \widetilde{c}} = 0 \,.$$

Using the second identity of Lemma 6.3 and applying Theorem 4.4 to the Hofmann distribution $\pi_{a,\widetilde{p},\widetilde{c}}$, we now obtain $\widetilde{p} = \sum_{j=0}^{n} \sum_{l=0}^{n-j} z_{j,l}$ and hence

$$p \sum_{j=0}^{n} \sum_{l=0}^{n-j} \beta_j \vartheta_l = \sum_{j=0}^{n} \sum_{l=0}^{n-j} z_{j,l} \,.$$

We also have

$$\frac{\partial(\log L)}{\partial\beta_i} = \frac{1}{\beta_i}\sum_{k=0}^{n-i} z_{i,k} - \frac{\sum\limits_{k=0}^{n-i}\vartheta_k}{\sum\limits_{j=0}^{n}\sum\limits_{l=0}^{n-j}\beta_j\vartheta_l}\sum_{j=0}^{n}\sum_{l=0}^{n-j} z_{j,l}$$
$$+ \frac{\partial\left(\log \pi_{a,\widetilde{p},\widetilde{c}}\left(\sum\limits_{j=0}^{n}\sum\limits_{l=0}^{n-j} z_{j,l}\right)\right)}{\partial\widetilde{p}}\cdot\frac{\partial\widetilde{p}}{\partial\beta_i}$$
$$+ \frac{\partial\left(\log \pi_{a,\widetilde{p},\widetilde{c}}\left(\sum\limits_{j=0}^{n}\sum\limits_{l=0}^{n-j} z_{j,l}\right)\right)}{\partial\widetilde{c}}\cdot\frac{\partial\widetilde{c}}{\partial\beta_i}$$

for all $i \in \{0, 1, \dots, n\}$ and

$$\frac{\partial(\log L)}{\partial\vartheta_k} = \frac{1}{\vartheta_k}\sum_{i=0}^{n-k} z_{i,k} - \frac{\sum\limits_{i=0}^{n-k}\beta_i}{\sum\limits_{j=0}^{n}\sum\limits_{l=0}^{n-j}\beta_j\vartheta_l}\sum_{j=0}^{n}\sum_{l=0}^{n-j} z_{j,l}$$
$$+ \frac{\partial\left(\log \pi_{a,\widetilde{p},\widetilde{c}}\left(\sum\limits_{j=0}^{n}\sum\limits_{l=0}^{n-j} z_{j,l}\right)\right)}{\partial\widetilde{p}}\cdot\frac{\partial\widetilde{p}}{\partial\vartheta_k}$$
$$+ \frac{\partial\left(\log \pi_{a,\widetilde{p},\widetilde{c}}\left(\sum\limits_{j=0}^{n}\sum\limits_{l=0}^{n-j} z_{j,l}\right)\right)}{\partial\widetilde{c}}\cdot\frac{\partial\widetilde{c}}{\partial\vartheta_k}$$

for all $k \in \{0, 1, \dots, n\}$. Inserting the identities obtained before into the previous ones, we obtain

$$\sum_{k=0}^{n-i} z_{i,k} = \sum_{k=0}^{n-i}(p\beta_i)\,\vartheta_k$$

for all $i \in \{0, 1, \dots, n\}$ and

$$\sum_{i=0}^{n-k} z_{i,k} = \sum_{i=0}^{n-k}(p\beta_i)\,\vartheta_k$$

for all $k \in \{0, 1, \dots, n\}$. These are the marginal–sum equations. □

The previous lemma yields the following result:

**6.5 Theorem** *If $\widehat{p}$ and $\widehat{\beta}_0, \widehat{\beta}_1, \ldots, \widehat{\beta}_n$ are maximum–likelihood estimators of $p$ and $\beta_0, \beta_1, \ldots, \beta_n$, then $\widehat{p}\,\widehat{\beta}_0, \widehat{p}\,\widehat{\beta}_1, \ldots, \widehat{p}\,\widehat{\beta}_n$ are the chain–ladder estimators of $p\beta_0, p\beta_1, \ldots, p\beta_n$.*

In the case where the parameters of the present model are known, the ultimate aggregate claim numbers may be predicted by the credibility method:

**6.6 Theorem** *For every accident year $i \in \{0, 1, \ldots, n\}$, the credibility predictor $S^*_{i,n}$ of the ultimate aggregate claim number $S_{i,n}$ satisfies*

$$S^*_{i,n} = \sum_{k=0}^{n-i} Z_{i,k} + \left( \sum_{k=n-i+1}^{n} \vartheta_k \right) \beta_i \frac{p + ac \sum\limits_{j=0}^{n} \sum\limits_{l=0}^{n-j} Z_{j,l}}{1 + ac \sum\limits_{j=0}^{n} \sum\limits_{l=0}^{n-j} \beta_j \vartheta_l} .$$

**Proof** For all $i, k \in \{0, 1, \ldots, n\}$, we define the normalized incremental claim numbers $X_{i,k}$ by letting

$$X_{i,k} := \frac{Z_{i,k}}{\beta_i \vartheta_k} .$$

By Lemma 6.1, we obtain

$$\begin{aligned} E[X_{i,k}] &= p \\ \mathrm{cov}[X_{i,k}, X_{j,l}] &= acp + \frac{p}{\beta_i \vartheta_k} \delta_{i,j}\, \delta_{k,l} . \end{aligned}$$

Because of these identities, the family $\{X_{i,k}\}_{i,k \in \{0,1,\ldots,n\}}$ satisfies the assumptions of the standard credibility model considered by Schmidt [2004b] and it follows that the credibility predictor $X^*_{i,k}$ of $X_{i,k}$ with $i, k \in \{0, 1, \ldots, n\}$ such that $i + k > n$ satisfies

$$\begin{aligned} X^*_{i,k} &= \frac{1}{1 + acp \sum\limits_{j=0}^{n} \sum\limits_{l=0}^{n-j} (\beta_j \vartheta_l)/p}\, p + \sum_{h=0}^{n} \sum_{m=0}^{n-h} \frac{acp\,(\beta_h \vartheta_m)/p}{1 + acp \sum\limits_{j=0}^{n} \sum\limits_{l=0}^{n-j} (\beta_j \vartheta_l)/p} X_{h,m} \\ &= \frac{1}{1 + ac \sum\limits_{j=0}^{n} \sum\limits_{l=0}^{n-j} \beta_j \vartheta_l}\, p + \sum_{h=0}^{n} \sum_{m=0}^{n-h} \frac{ac}{1 + ac \sum\limits_{j=0}^{n} \sum\limits_{l=0}^{n-j} \beta_j \vartheta_l} Z_{h,m} . \end{aligned}$$

Therefore, the credibility predictor $Z^*_{i,k}$ of $Z_{i,k}$ with $i,k \in \{0,1,\ldots,n\}$ such that $i+k>n$ satisfies

$$Z^*_{i,k} = \beta_i \vartheta_k \frac{p + ac \sum\limits_{j=0}^{n} \sum\limits_{l=0}^{n-j} Z_{j,l}}{1 + ac \sum\limits_{j=0}^{n} \sum\limits_{l=0}^{n-j} \beta_j \vartheta_l}$$

and it follows that the credibility predictor $S^*_{i,n}$ of $S_{i,n}$ with $i \in \{0,1,\ldots,n\}$ satisfies

$$S^*_{i,n} = \sum_{k=0}^{n-i} Z_{i,k} + \left( \sum_{k=n-i+1}^{n} \vartheta_k \right) \beta_i \frac{p + ac \sum\limits_{j=0}^{n} \sum\limits_{l=0}^{n-j} Z_{j,l}}{1 + ac \sum\limits_{j=0}^{n} \sum\limits_{l=0}^{n-j} \beta_j \vartheta_l} .$$

This is the assertion. □

By Theorem 6.6, the credibility predictors of the ultimate aggregate claim numbers are predictors of the Bornhuetter–Ferguson type. Due to the covariance structure of the incremental claim numbers, the credibility predictor of the ultimate aggregate claim number of a given accident year also depends on the data from all other accident years.

We conclude this section with another look at the model under consideration.

A family of random variables $\{N_t\}_{t\in\mathbf{R}_+}$ is said to be a *Hofmann process* if the family $\{P_{N_t}\}_{t\in\mathbf{R}_+}$ is a Hofmann family $\mathbf{H}(a,p,c)$ and if the identity

$$P\left[\bigcap_{h=1}^{m} \{N_{t_h} - N_{t_{h-1}} = k_h\}\right] = \int_{\mathbf{R}} \prod_{h=1}^{m} \exp(-\lambda(t_h - t_{h-1})) \frac{(\lambda(t_h - t_{h-1}))^{k_h}}{k_h!} \, dQ_{a,p,c}(\lambda)$$

holds for all $m \in \mathbf{N}$, for all $t_0, t_1, \ldots, t_m \in \mathbf{R}_+$ satisfying $0 = t_0 < t_1 < \ldots < t_m$, and for all $k_1, \ldots, k_m \in \mathbf{N}_0$.

Let $\{N_t\}_{t\in\mathbf{R}_+}$ be a Hofmann process.

– First, define $\tau_{-1} := 0$ as well as $\tau_i := \sum_{j=0}^{i} \beta_j$ and

$$U_{i,n} := N_{\tau_i} - N_{\tau_{i-1}}$$

for all $i \in \{0, 1, \ldots, n\}$. Then the joint distribution of the family $\{U_{i,n}\}_{i\in\{0,1,\ldots,n\}}$ satisfies

$$P\left[\bigcap_{i=0}^{n}\{U_{i,n} = u_{i,n}\}\right] = \int_{\mathbf{R}} \prod_{i=0}^{n} \exp(-\lambda\beta_i)\,\frac{(\lambda\beta_i)^{u_{i,n}}}{u_{i,n}!}\,dQ_{a,p,c}(\lambda)\,.$$

If $U_{i,n}$ is interpreted as the ultimate aggregate claim number of accident year $i$, then the ultimate aggregate claim numbers of all accident years are increments of the Hofmann process $\{N_t\}_{t\in\mathbf{R}_+}$ restricted to the unit interval $[0, 1]$.

– Second, define $\gamma_{-1} := 0$ as well as $\gamma_k := \sum_{l=0}^{k} \vartheta_l$ and

$$V_{i,k} := N_{\tau_{i-1}+\gamma_k(\tau_i-\tau_{i-1})} - N_{\tau_{i-1}+\gamma_{k-1}(\tau_i-\tau_{i-1})}$$

for all $k \in \{0, 1, \ldots, n\}$. Then the joint distribution of the family $\{V_{i,k}\}_{k\in\{0,1,\ldots,n\}}$ satisfies

$$P\left[\bigcap_{k=0}^{n}\{V_{i,k} = v_{i,k}\}\right] = \int_{\mathbf{R}} \prod_{k=0}^{n} \exp(-\lambda\beta_i\vartheta_k)\,\frac{(\lambda\beta_i\vartheta_k)^{v_{i,k}}}{v_{i,k}!}\,dQ_{a,p,c}(\lambda)$$

for all $i \in \{0, 1, \ldots, n\}$. If $V_{i,k}$ is interpreted as the incremental claim number of accident year $i$ and development year $k$, then the incremental claim numbers of accident year $i$ are increments of the Hofmann process $\{N_t\}_{t\in\mathbf{R}_+}$ restricted to the interval $[\tau_{i-1}, \tau_i]$.

The definitions yield

$$U_{i,n} = \sum_{k=0}^{n} V_{i,k}$$

and the joint distribution of the family $\{V_{i,k}\}_{i,k\in\{0,1,\ldots,n\}}$ satisfies

$$P\left[\bigcap_{i=0}^{n}\bigcap_{k=0}^{n}\{V_{i,k} = v_{i,k}\}\right] = \int_{\mathbf{R}} \prod_{i=0}^{n}\prod_{k=0}^{n} \exp(-\lambda\beta_i\vartheta_k)\,\frac{(\lambda\beta_i\vartheta_k)^{v_{i,k}}}{v_{i,k}!}\,dQ_{a,p,c}(\lambda)\,.$$

This shows that the model considered in the present section can be obtained from a Hofmann process on the unit interval.

## 7 The Second Extension of the Poisson Model

In the present section, we consider a second extension of the Poisson model and we assume that the joint distribution of the family $\{Z_{i,k}\}_{i,k\in\{0,1,\dots,n\}}$ satisfies

$$\begin{aligned} & P\left[\bigcap_{i=0}^{n}\bigcap_{k=0}^{n}\{Z_{i,k}=z_{i,k}\}\right] \\ & \quad = \prod_{i=0}^{n}\int_{\mathbf{R}}\prod_{k=0}^{n}\exp(-\lambda\vartheta_k)\,\frac{(\lambda\vartheta_k)^{z_{i,k}}}{z_{i,k}!}\,dQ_{a_i,p_i,c_i}(\lambda) \end{aligned}$$

where $\vartheta_0,\vartheta_1,\dots,\vartheta_n \in (0,1)$ are parameters satisfying $\sum_{k=0}^{n}\vartheta_k = 1$ and $Q_{a_i,p_i,c_i}$ is the mixing distribution of the Hofmann family $\mathbf{H}(a_i,p_i,c_i)$. In the case where $a_i = 0$ holds for all $i \in \{0,1,\dots,n\}$ this model coincides with the Poisson model.

Throughout this section we omit all proofs which are essentially identical to those given in Section 6.

Let us first study the present model in more detail.

**7.1 Lemma** *The expectations of any incremental claim number satisfies*

$$E[Z_{i,k}] = (\beta_i\vartheta_k)\,p_i$$

*and the covariance of any two incremental claim numbers of the same accident year satisfies*

$$\mathrm{cov}[Z_{i,k},Z_{i,l}] = (\beta_i\vartheta_k)(\beta_i\vartheta_l)\,a_i c_i p_i + (\beta_i\vartheta_k)\,p_i\,\delta_{k,l}\,.$$

*Moreover, the accident years are independent.*

Lemma 7.1 implies that, except for the case $a_i = 0$, any two distinct incremental claim numbers of accident year $i \in \{0,1,\dots,n\}$ are strictly positively correlated and are hence dependent. The lemma also yields a characterization of the Poisson model as a special case of the present model:

**7.2 Theorem** *The following are equivalent:*

(a) *For every accident year there exist two distinct incremental claim numbers which are uncorrelated.*

(b) *For every accident year the family of incremental claim numbers is uncorrelated.*

(c) *For every accident year the family of incremental claim numbers is independent.*

(d) *The family of all incremental claim numbers satisfies the assumption of the Poisson model.*

Because of Lemma 7.1, the expectation of any incremental claim number satisfies

$$E[Z_{i,k}] = p_i\, \vartheta_k\,.$$

Therefore, the present model is a multiplicative model, and summation yields

$$E\left[\sum_{k=0}^{n} Z_{i,k}\right] = p_i$$

for all $i \in \{0, 1, \ldots, n\}$, which means that $p_i$ is the expected ultimate aggregate claim number of accident year $i$. The interpretation of the parameters $\vartheta_0, \vartheta_1, \ldots, \vartheta_n$ is the same as in the Poisson model.

The assumption of the model considered here is equivalent with the assumption that the joint distribution of the family $\{Z_{i,k}\}_{i,k\in\{0,1,\ldots,n\}}$ satisfies

$$P\left[\bigcap_{i=0}^{n}\bigcap_{k=0}^{n}\{Z_{i,k} = z_{i,k}\}\right] = \prod_{i=0}^{n}\left(\frac{\left(\sum\limits_{k=0}^{n} z_{i,k}\right)!}{\prod\limits_{k=0}^{n} z_{i,k}!}\prod_{k=0}^{n}\vartheta_k^{z_{i,k}}\cdot\int_{\mathbf{R}}\exp(-\lambda)\,\frac{\lambda^{\sum_{k=0}^{n} z_{i,k}}}{\sum\limits_{k=0}^{n} z_{i,k}!}\,dQ_{a_i,p_i,c_i}(\lambda)\right).$$

This means that

- the accident years are independent,
- for every accident year, the incremental claim numbers have a conditional multinomial distribution with respect to the ultimate aggregate claim number and with a development pattern which is identical for all accident years, and

- for every accident year, the ultimate aggregate claim number has a Hofmann distribution.

As mentioned before, in the case where $a_i = 0$ holds for all $i \in \{0, 1, \ldots, n\}$ we are back to the Poisson model. Moreover, in the case where $a_i = 1$ holds for all $i \in \{0, 1, \ldots, n\}$, all ultimate aggregate claim numbers have a negativebinomial distribution and this is the case considered by Schmidt and Wünsche [1998] as a first modification of the Poisson model; this case is of interest since negativebinomial distributions and many empirical claim number distributions share the property that the variance exceeds the expectation. The model considered here is much more general than the model of Schmidt and Wünsche [1998] since it allows for arbitrary values of the parameters $a_i$ and even for different values of $a_i$ for different accident years.

In the case where the parameters of the present model are unknown, the parameters may be estimated by the maximum–likelihood method:

**7.3 Lemma** *The identity*

$$P\left[\bigcap_{k=0}^{n-i}\{Z_{i,k} = z_{i,k}\}\right] = \frac{\left(\sum_{k=0}^{n-i} z_{i,k}\right)!}{\prod_{k=0}^{n-i} z_{i,k}!} \prod_{k=0}^{n-i} \left(\frac{\vartheta_k}{\sum_{l=0}^{n-i} \vartheta_l}\right)^{z_{i,k}} \cdot \pi_{a_i, p_i \sum_{k=0}^{n-i} \vartheta_k, c_i \sum_{k=0}^{n-i} \vartheta_k}\left(\sum_{k=0}^{n-i} z_{i,k}\right)$$

*holds for all* $i \in \{0, 1, \ldots, n\}$ *and for every family* $\{z_{i,k}\}_{k\in\{0,1,\ldots,n-i\}} \subseteq \mathbf{N}_0$, *and the identity*

$$P\left[\left\{\sum_{k=0}^{n-i} Z_{i,k} = z\right\}\right] = \pi_{a_i, p_i \sum_{k=0}^{n-i} \vartheta_k, c_i \sum_{k=0}^{n-i} \vartheta_k}(z)$$

*holds for all* $i \in \{0, 1, \ldots, n\}$ *and for all* $z \in \mathbf{N}_0$.

Since the accident years are independent, it follows from the first identity of Lemma 7.3 that the joint distribution of all observable incremental claim numbers satisfies

$$P\left[\bigcap_{i=0}^{n}\bigcap_{k=0}^{n-i}\{Z_{i,k}=z_{i,k}\}\right]$$

$$=\prod_{i=0}^{n}\left(\frac{\left(\sum_{k=0}^{n-i} z_{i,k}\right)!}{\prod_{k=0}^{n-i} z_{i,k}!}\prod_{k=0}^{n-i}\left(\frac{\vartheta_k}{\sum_{l=0}^{n-i}\vartheta_l}\right)^{z_{i,k}}\cdot\pi_{a_i,\,p_i\sum_{k=0}^{n-i}\vartheta_k,\,c_i\sum_{k=0}^{n-i}\vartheta_k}\left(\sum_{k=0}^{n-i} z_{i,k}\right)\right).$$

Therefore, the likelihood function, which depends on the parameters $a_0, a_1, \ldots, a_n$, $p_0, p_1, \ldots, p_n$, $c_0, c_1, \ldots, c_n$ and $\vartheta_0, \vartheta_1, \ldots, \vartheta_n$, satisfies

$$L=\prod_{i=0}^{n}\left(\frac{\left(\sum_{k=0}^{n-i} z_{i,k}\right)!}{\prod_{k=0}^{n-i} z_{i,k}!}\prod_{k=0}^{n-i}\left(\frac{\vartheta_k}{\sum_{l=0}^{n-i}\vartheta_l}\right)^{z_{i,k}}\cdot\pi_{a_i,\,p_i\sum_{l=0}^{n-i}\vartheta_l,\,c_i\sum_{l=0}^{n-i}\vartheta_l}\left(\sum_{l=0}^{n-i} z_{i,l}\right)\right)$$

and it follows that the log–likelihood function satisfies

$$\begin{aligned}\log(L) = g + \sum_{i=0}^{n}\Biggl(\sum_{k=0}^{n-i} z_{i,k}\log(\vartheta_k) - \left(\sum_{l=0}^{n-i} z_{i,l}\right)\log\left(\sum_{l=0}^{n-i}\vartheta_l\right)\\ + \log\left(\pi_{a_i,\,p_i\sum_{l=0}^{n-i}\vartheta_l,\,c_i\sum_{l=0}^{n-i}\vartheta_l}\left(\sum_{l=0}^{n-i} z_{i,l}\right)\right)\Biggr)\end{aligned}$$

where $g$ is constant not depending on the parameters.

**7.4 Lemma** *If $\widehat{p}_0, \widehat{p}_1, \ldots, \widehat{p}_n$ and $\widehat{\vartheta}_0, \widehat{\vartheta}_1, \ldots, \widehat{\vartheta}_n$ are maximum–likelihood estimators of $p_0, p_1, \ldots, p_n$ and $\vartheta_0, \vartheta_1, \ldots, \vartheta_n$, then $\widehat{p}_0, \widehat{p}_1, \ldots, \widehat{p}_n$ and $\widehat{\vartheta}_0, \widehat{\vartheta}_1, \ldots, \widehat{\vartheta}_n$ are the marginal–sum estimators of $p_0, p_1, \ldots, p_n$ and $\vartheta_0, \vartheta_1, \ldots, \vartheta_n$.*

**Proof** For $i \in \{0, 1, \ldots, n\}$ define

$$\widetilde{p}_i := p_i \sum_{l=0}^{n-i}\vartheta_l$$

$$\widetilde{c}_i := c_i \sum_{l=0}^{n-i}\vartheta_l\,.$$

Then we have

$$\frac{\partial(\log L)}{\partial p_i} = \frac{\partial\left(\log \pi_{a_i,\widetilde{p}_i,\widetilde{c}_i}\left(\sum\limits_{l=0}^{n-i} z_{i,l}\right)\right)}{\partial \widetilde{p}_i} \cdot \sum_{l=0}^{n-i} \vartheta_l$$

$$\frac{\partial(\log L)}{\partial c_i} = \frac{\partial\left(\log \pi_{a_i,\widetilde{p}_i,\widetilde{c}_i}\left(\sum\limits_{l=0}^{n-i} z_{i,l}\right)\right)}{\partial \widetilde{c}_i} \cdot \sum_{l=0}^{n-i} \vartheta_l$$

for all $i \in \{0, 1, \ldots, n\}$. Since $\sum_{l=0}^{n-i} \vartheta_l \neq 0$, the maximum–likelihood conditions yield

$$\frac{\partial\left(\log \pi_{a_i,\widetilde{p}_i,\widetilde{c}_i}\left(\sum\limits_{l=0}^{n-i} z_{i,l}\right)\right)}{\partial \widetilde{p}_i} = 0$$

$$\frac{\partial\left(\log \pi_{a_i,\widetilde{p}_i,\widetilde{c}_i}\left(\sum\limits_{l=0}^{n-i} z_{i,l}\right)\right)}{\partial \widetilde{c}_i} = 0$$

for all $i \in \{0, 1, \ldots, n\}$. Using the second identity of Lemma 7.3 and applying Theorem 4.4 to the Hofmann distribution $\pi_{a_i,\widetilde{p}_i,\widetilde{c}_i}$, we now obtain $\widetilde{p}_i = \sum_{l=0}^{n-i} z_{i,l}$ and hence

$$p_i \sum_{l=0}^{n-i} \vartheta_l = \sum_{l=0}^{n-i} z_{i,l}$$

for all $i \in \{0, 1, \ldots, n\}$. This yields

$$\sum_{k=0}^{n-i} z_{i,k} = \sum_{k=0}^{n-i} p_i \vartheta_k$$

for all $i \in \{0, 1, \ldots, n\}$, which is one of the marginal–sum equations.

We also have

$$\frac{\partial(\log L)}{\partial\vartheta_k} = \sum_{i=0}^{n-k}\left(\frac{z_{i,k}}{\vartheta_k} - \frac{\sum\limits_{l=0}^{n-i} z_{i,l}}{\sum\limits_{l=0}^{n-i}\vartheta_l}\right.$$
$$+ \frac{\partial\left(\log \pi_{a_i,\widetilde{p}_i,\widetilde{c}_i}\left(\sum\limits_{l=0}^{n-i} z_{i,l}\right)\right)}{\partial\widetilde{p}_i}\cdot\frac{d\widetilde{p}_i}{d\vartheta_k}$$
$$\left.+ \frac{\partial\left(\log \pi_{a_i,\widetilde{p}_i,\widetilde{c}_i}\left(\sum\limits_{l=0}^{n-i} z_{i,l}\right)\right)}{\partial\widetilde{c}_i}\cdot\frac{d\widetilde{c}_i}{d\vartheta_k}\right)$$

for all $k \in \{0, 1, \dots, n\}$. Inserting the identities obtained so far into the previous one, we obtain

$$\sum_{i=0}^{n-k}\left(\frac{z_{i,k}}{\vartheta_k} - p_i\right) = 0$$

and hence

$$\sum_{i=0}^{n-k} z_{i,k} = \sum_{i=0}^{n-k} p_i\vartheta_k$$

for all $k \in \{0, 1, \dots, n\}$, which is the other marginal–sum equation. □

The previous lemma yields the following result:

**7.5 Theorem** *If* $\widehat{p}_0, \widehat{p}_1, \dots, \widehat{p}_n$ *are maximum–likelihood estimators of* $p_0, p_1, \dots, p_n$, *then* $\widehat{p}_0, \widehat{p}_1, \dots, \widehat{p}_n$ *are the chain–ladder estimators of* $p_0, p_1, \dots, p_n$.

In the case where the parameters of the present model are known, the ultimate aggregate claim numbers may be predicted by the credibility method:

**7.6 Theorem** *For every accident year $i \in \{0, 1, \ldots, n\}$, the credibility predictor $S^*_{i,n}$ of the ultimate aggregate claim number $S_{i,n}$ satisfies*

$$S^*_{i,n} = \sum_{k=0}^{n-i} Z_{i,k} + \left( \sum_{k=n-i+1}^{n} \vartheta_k \right) \frac{p_i + a_i c_i \sum_{k=0}^{n-i} Z_{i,k}}{1 + a_i c_i \sum_{k=0}^{n-i} \vartheta_k} .$$

By Theorem 7.6, the credibility predictors of the ultimate aggregate claim numbers are predictors of the Bornhuetter–Ferguson type. Due to the independence of the accident years, the credibility predictor of a given ultimate aggregate claim number does not depend on data from other accident years.

Let us finally note that the discussion at the end of Section 6 can be adopted to the model considered here. It turns out that the model considered in the present section can be obtained from $n+1$ Hofmann processes on the unit interval such that each Hofmann process corresponds to an accident year and the Hofmann processes are independent.

## 8 The Poisson Model With Identical Parameters

With regard to the model considered in Section 7, it is natural to investigate also the special case in which the parameters of the Hofmann distributions are assumed to be identical for all accident years. In this case, the joint distribution of the incremental claim numbers of a given accident year is the same for all accident years and one would say that the accident years are not only independent but also identically distributed.

To illustrate the effect of this additional assumption, we consider here the Poisson model and assume that the joint distribution of the family $\{Z_{i,k}\}_{i,k \in \{0,1,\ldots,n\}}$ satisfies

$$P\left[ \bigcap_{i=0}^{n} \bigcap_{k=0}^{n} \{Z_{i,k} = z_{i,k}\} \right] = \prod_{i=0}^{n} \prod_{k=0}^{n} \exp(-\alpha\vartheta_k) \frac{(\alpha\vartheta_k)^{z_{i,k}}}{z_{i,k}!}$$

where $\alpha \in (0, \infty)$ and $\vartheta_0, \vartheta_1, \ldots, \vartheta_n \in (0, 1)$ are parameters satisfying $\sum_{k=0}^{n} \vartheta_k = 1$. In this model, we have $E[Z_{i,k}] = \alpha\vartheta_k$ for all $i, k \in \{0, 1, \ldots, n\}$

and hence

$$E\left[\sum_{k=0}^{n} Z_{i,k}\right] = \alpha$$

for all $i \in \{0, 1, \ldots, n\}$.

In the case where the parameters of the present model are unknown, the parameters may be estimated by the maximum–likelihood method:

**8.2 Lemma** *The identity*

$$P\left[\bigcap_{i=0}^{n} \bigcap_{k=0}^{n-i} \{Z_{i,k} = z_{i,k}\}\right] = \prod_{i=0}^{n} \prod_{k=0}^{n-i} \exp(-\alpha\vartheta_k) \frac{(\alpha\vartheta_k)^{z_{i,k}}}{z_{i,k}!}$$

*holds for every family* $\{z_{i,k}\}_{i,k\in\{0,1,\ldots,n\},i+k\leq n} \subseteq \mathbf{N}_0$.

Because of Lemma 8.1, the likelihood function, which depends on the parameters $\alpha$ and $\vartheta_0, \vartheta_1, \ldots, \vartheta_n$, satisfies

$$L = \prod_{i=0}^{n} \prod_{k=0}^{n-i} \exp(-\alpha\vartheta_k) \frac{(\alpha\vartheta_k)^{z_{i,k}}}{z_{i,k}!}$$

and it follows that the log–likelihood function satisfies

$$\log(L) = g + \sum_{i=0}^{n} \sum_{k=0}^{n-i} \Big(-\alpha\vartheta_k + z_{i,k} \log(\alpha\vartheta_k)\Big)$$

where $g$ is a constant not depending on the parameters.

**8.2 Lemma** *If* $\widehat{\alpha}$ *and* $\widehat{\vartheta}_0, \widehat{\vartheta}_1, \ldots, \widehat{\vartheta}_n$ *are maximum–likelihood estimators of* $\alpha$ *and* $\vartheta_0, \vartheta_1, \ldots, \vartheta_n$, *then*

$$\widehat{\alpha} = \sum_{l=0}^{n} \frac{1}{n-l+1} \sum_{i=0}^{n-l} Z_{i,l}$$

*and*

$$\widehat{\vartheta}_k = \frac{\dfrac{1}{n-k+1} \displaystyle\sum_{i=0}^{n-k} Z_{i,k}}{\displaystyle\sum_{l=0}^{n} \frac{1}{n-l+1} \sum_{i=0}^{n-l} Z_{i,l}}$$

*holds for all* $k \in \{0, 1, \ldots, n\}$.

**Proof** The maximum–likelihood condition with respect to $\vartheta_k$ yields

$$\alpha\vartheta_k = \frac{1}{n-k+1}\sum_{i=0}^{n-k} z_{i,k}$$

for all $k \in \{0, 1, \ldots, n\}$. Since $\sum_{k=0}^{n} \vartheta_k = 1$, we obtain

$$\alpha = \sum_{k=0}^{n} \frac{1}{n-k+1}\sum_{i=0}^{n-k} z_{i,k}$$

and hence

$$\vartheta_k = \frac{\dfrac{1}{n-k+1}\displaystyle\sum_{i=0}^{n-k} Z_{i,k}}{\displaystyle\sum_{l=0}^{n}\frac{1}{n-l+1}\sum_{i=0}^{n-l} Z_{i,l}}$$

for all $k \in \{0, 1, \ldots, n\}$. Furthermore, the maximum–likelihood condition with respect to $\alpha$ is fulfilled as well. □

It can be seen from Lemma 8.2 that the maximum–likelihood estimator of $\alpha$ is *not* identical with any of the chain–ladder estimators of the expected ultimate aggregate claim numbers. This is not really surprising since the chain–ladder estimators of the expected ultimate aggregate claim numbers of different accident years are distinct.

Nevertheless, the discussion of the present section shows that one has to be careful with the choice of the model: If one is convinced that the Poisson model with identical expected ultimate aggregate claim numbers is an appropriate model, then chain–ladder estimation and maximum–likelihood estimation yield different results and one has to make a choice between these methods of estimation.

With regard to the discussion at the end of Sections 6 and 7, we remark that the model considered in the present section can be obtained from $n+1$ independent copies of a single Hofmann process on the unit interval.

## 9 Remark

The results on credibility prediction presented in Sections 5, 6 and 7 can easily be extended to the case where the mixing distributions of Hofmann families are replaced by arbitrary mixing distributions concentrated on $(0, \infty)$.

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Klaus D. Schmidt
Mathias Zocher
Lehrstuhl für Versicherungsmathematik
Technische Universität Dresden
D–01062 Dresden

e–mail: schmidt@math.tu-dresden.de

## Zusammenfassung

Die vorliegende Arbeit untersucht die Übereinstimmung der Chain–Ladder und Maximum–Likelihood Schätzer für die erwarteten Endschadenzahlen in der Schadenreservierung. Wir stellen zwei allgemeine Modelle vor, die beide das Poisson–Modell von Hachemeister und Stanard [1975] verallgemeinern und auf der von Hofmann [1955] eingeführten Klasse gemischter Poisson–Verteilungen beruhen. Wir zeigen, dass in jedem dieser Modelle die Maximum–Likelihood Schätzer der erwarteten Endschadenzahlen mit den Chain–Ladder Schätzern übereinstimmen.

## Summary

The present paper is concerned with the coincidence of chain–ladder and maximum–likelihood estimators for the expected ultimate aggregate claim numbers in loss reserving. We propose two general models which extend the Poisson model considered by Hachemeister and Stanard [1975] and which are based on the mixed Poisson distributions introduced by Hofmann [1955]. For each of these models, we show that the maximum–likelihood estimators of the expected ultimate aggregate claim numbers agree with the chain–ladder estimators.

## Résumé

Dans cet article on étudie, dans la théorie des provisions, la coïncidence des estimateurs de chain–ladder et des estimateurs de maximum de vraisemblance pour les espérance des nombres ultimes de sinistres. Nous proposons deux modèles généraux qui contiennent le modèle de Poisson étudié par Hachemeister et Stanard [1975] et qui reposent sur la classe de distributions de Poisson mixtes introduite par Hofmann [1955]. Il est démontré que dans chacun de ces modèles les estimateurs de maximum de vraisemblance des espérances des nombres ultimes de sinistres coïncident avec les estimateurs de chain–ladder.