Solid domes, cylindrical reservoirs and similar constructions

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Solid Domes, Cylindrical Reservoirs and Similar Constructions.

Massive Kuppeln, zylindrische Behälter und ähnliche Konstruktionen.

Coupoles massives, réservoirs cylindriques et constructions semblables.

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The precise calculation of the bending stresses in a massive dome is a very difficult matter. These difficulties are brought out in a thesis[1] presented to the Royal Technical College Stockholm, and it may well be asked whether the practising engineer ever has the time and opportunity of working out the dimensions of a dome in terms of exact theories. Even to draw up the fundamental equations is a fairly complicated business, and their exact integration leads to series which are often difficult to handle, and which slowly converge. Even though their convergence is satisfactory for many wall-thicknesses (gauges), any alteration in gauge may result in this good convergence being lost. Even where the engineer has the mathetical equipment necessary for dealing with the problem, the amount of work necessary for working out a definite case of loading is much too great; and it may not be possible at all to arrive at practical methods in the way indicated by *Meissner, Bolle, Dubois, Honegger, Ekström* and others. In the case of spherical domes, for instance, integration, even in the simplest cases. gives hypergeometric series which do not constitute the proper equipment of the engineer owing to their slow convergence.

In view of these facts, it is particularly necessary that the further development of the dome theory should be based upon solutions that fully meet practical requirements, even though this involves introducing certain approximations. As *Geckeler*[2] has shown, it is possible, even with comparatively simple mathematical expedients, to arrive at a solution which differs only inappreciably from the true one, and which can be easily and conveniently employed, in cases where the wall-thickness and radius are constant. The good agreement between *Geckeler*'s theory and the exact theory may justify our discussing the former in greater detail, provided we are clear as to what approximations are introduced. A still further step in the direction of the true result is achieved by using *Blumenthal*'s and *Steuermann*'s method of asymptotic integration, which is applicable to

[1] *John Erik Ekström:* Studien über dünne Schalen von rotationssymetrischer Form und Belastung mit konstanter und veränderlicher Wandstärke. Stockholm 1932.

[2] See, inter alia, Handbuch für Eisenbetonbau, Vol, 6, Berlin. 1928.

variable wall thicknesses as well. We actually get farther with this method than we do with the methods that are based on solutions in the form of infinite series, in which connection the wall thickness was always assumed to vary in terms of a definite function if the solution had to be worked out.

Closer examination of *Geckeler*'s final equations reveals that these are of the same type as the equations for an elastically supported beam. Nor is it difficult to appreciate the physical analogy. The meridian of the dome may be regarded as a girder supported by the parallel circles or rings. As these may be compressed or expanded, they correspond, statically, to elastic supports.

When the dome is regarded in this way, its statics may be elucidated with sufficient accuracy. It is not then necessary to revert to *Meissner*'s differential equations for drawing up the equations of equilibrium, but all the necessary equations may be set out directly, simply with the aid of the theory of the elastically supported beam. For the practising engineer, this means that he need not attempt to understand the fairly complicated classic theory of the dome, but can work out the necessary equations for himself.

Geckeler's published works show that he himself has not fully appreciated the high importance of the approximations he suggested; that is, he has not understood that the dome, considered broadly, acts like a steady series of girders on elastic supports. The method of treatment which I suggest can of course be extended by regarding the meridian, not as a girder, but as an arch supported elastically by the ring elements of the dome.

By considering the dome in this way, it is possible to get a more accurate idea of the statics of the structure, and the equations obtained as the result of doing so are the same as *Meissner*'s.

It is obviously necessary to introduce this latter method of conception especially in the case of very flat domes; that is to say, where the arch effect is very manifest in the elements of the meridian, if the desired accuracy is to be achieved. The more inclined the tangent of the dome at its support, the more accurate will be the method where the meridian is regarded as a girder on an elastic support; and in the special case where the tangent of the cupola is everywhere vertical, i. e., when the dome merges into a cylinder, this particular method of considering the dome is perfectly exact.

In order to show more closely how simply the dome problem can be dealt with in this way, I have worked out a few problems and compared the results with those obtained in accordance with the strict theory. The agreement is extremely satisfactory throughout.

As our first example, we shall select a spherical concrete dome of uniform thickness, wall-thickness $\delta = 16$ cm, radius $r = 1000$ cm, angle of opening 40^0. We shall suppose the dome to be loaded with a constant fluid pressure of $p = 1.0$ kg/cm^2, and to be firmly restrained around the edge (see Fig. 1).

If the stresses in this dome be calculated in accordance with the membrane theory, we get a compressive stress at the meridian of $T_1 = \frac{p\,r}{2}$ and an annular compressive stress of $T_2 = \frac{p\,r}{2}$. These meridian and ring stresses are constant throughout the dome, and the solution in terms of the membrane theory is thus very simple. Due to these compressive stresses T_1 and T_2, the dome is compressed, so that its radius is reduced by $\frac{T \cdot r}{E\delta}$, i. e., by $\frac{pr^2}{2E}$. This reduction in radius is not very great, amounting to only 0.15 cm under the assumptions given, and for $E = 210.000$ kg/cm². As the dome is secured about its edge, it is not capable of freely altering its shape; the parts nearest to the edge will

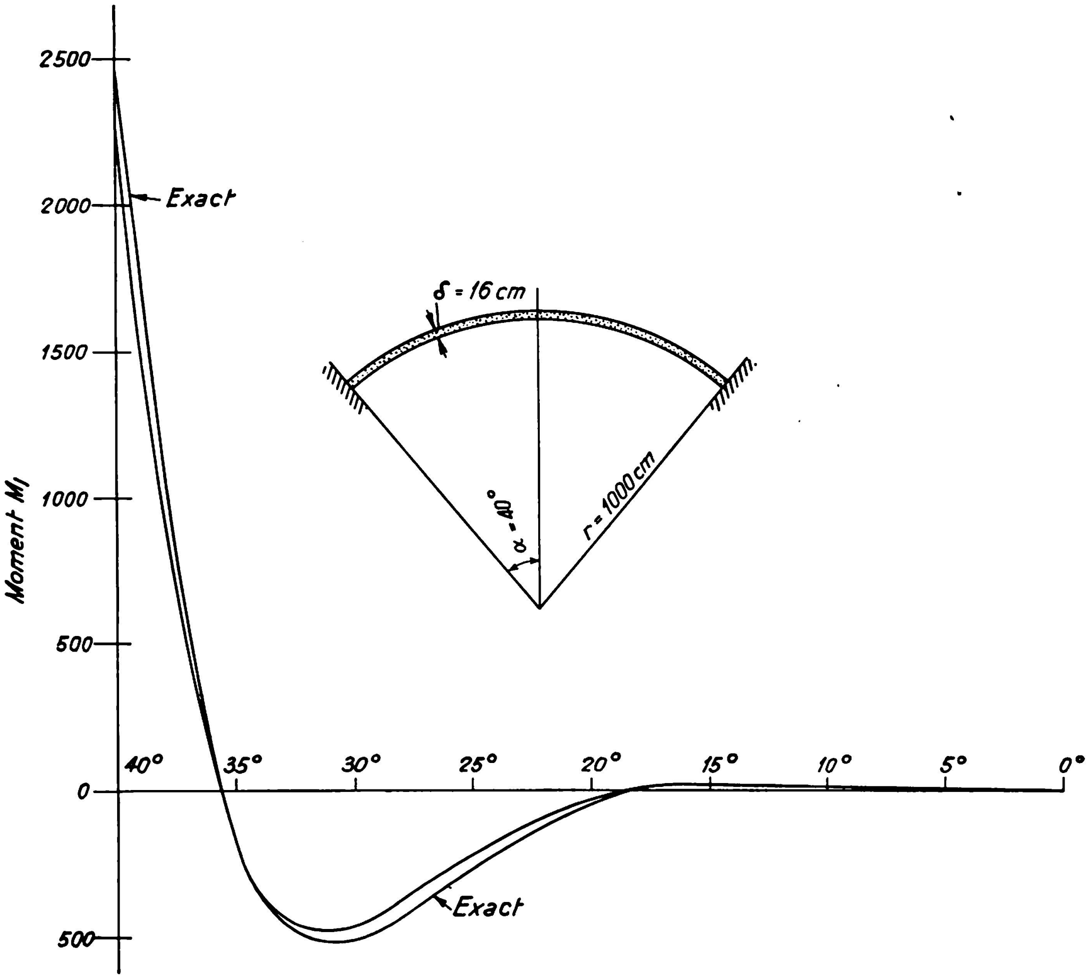

Fig. 1.

Comparison between the values of the Meridian-moments, (1) calculated according to equation and (2) according to the exact method by means of hypergeometrical series. The deviations are of no practical avail.

retain their original radius; but the farther we get from the edge, the more freely will the structure be able to move, and the more freely deformation can take place. Although the compression of the radius is fairly small in this case, certain disturbances are set up near the edges which may lead to bending moments of such magnitude that they cannot be ignored.

We shall now investigate how large moments are set up in an elastically supported girder assuming that it is deflected in accordance with the values $\frac{p\,r^2}{2E}$

calculated above. The moment and the deflection are connected by the formula

$$EJ \cdot \frac{d^2 y}{dx^2} = - M_1 \tag{1}$$

and the effect of the elastic supporting of the ring elements is expressed by the equation:

$$\frac{d^2 M_1}{dx^2} = \frac{E\delta}{r^2} \cdot y \tag{2}$$

Eliminating M_1 from these two equations, we get:

$$\frac{d^2}{dx^2}\left[EJ\,\frac{d^2 y}{dx^2}\right] + \frac{E\delta}{r^2} \cdot y = o \tag{3a}$$

or, assuming the bending rigidity EI to be constant and equal to $\frac{Em^2}{m^2 - 1} \cdot \frac{\delta^3}{12}$, we have:

$$\frac{d^4 y}{dx^4} + 4k^4\, y = o$$

where
$$k^4 = \frac{3\,(m^2 - 1)}{m^2} \cdot \frac{1}{r^2\,\delta^2} \tag{3b}$$

The general integral of equation (3b) can be written in the following form:

$$y = e^{-kx}\,(A \cos kx + B \sin kx) + e^{-kx}\,(C \cos kx + D \sin kx) \tag{4a}$$

which means that the deflection may be regarded as the sum of two sine vibrations, one having a damped and the other an increasing amplitude. Generally speaking, the coefficients C and D may be taken as = 0, provided the girder is not too short and that the origin is located at the point from which the disturbance proceeds. For closed domes, therefore, the integral can be written with sufficient accuracy in the following form:

$$y = e^{-kx}\,(A \cos kx + B \sin kx) \tag{4b}$$

Here x is the arc length of the meridian measured from the edge of the dome. In this case the arbitrary constants A and B can easily be determined from the boundary condition, so that

$$y = -\frac{pr^2}{2E\delta} \text{ and } y' = 0 \text{ for } x = 0.$$

This gives $A = B = -\frac{pr^2}{2E\delta}$, and the deflection at the meridian is therefore

$$y = -\frac{pr^2}{2E\delta} \cdot e^{-kx}\,(\cos kx + \sin kx).$$

By inserting in equation (1) we get the following expression for the meridian moment:

$$M_1 = \frac{\sqrt{3}}{12}\, pr\delta e^{-kx}\,(-\cos kx + \sin kx) \tag{5}$$

In this expression, the effect of the transverse compression of the material is ignored, i. e., Poisson's factor m is taken as equal to infinity.

Table I.

Values of functions $e^{-kx}\cos kx$, $e^{-kx}\sin kx$, $e^{-kx}(\cos kx - \sin kx)$ and $e^{-kx}(\cos kx + \sin kx)$.

kx	$e^{-kx}\cos kx$	$e^{-kx}\sin kx$	$e^{-kx}(\cos kx - \sin kx)$	$e^{-kx}(\cos kx + \sin kx)$
0	1.0000	0.0000	1.0000	1,0000
$\frac{\pi}{8}$	0.6239	0.2584	0.3655	0.8823
$\frac{\pi}{4}$	0.3225	0.3225	0.0000	0.6450
$\frac{3\pi}{8}$	0.1179	0.2845	— 0.1665	0.4024
$\frac{\pi}{2}$	0.0000	0.2079	— 0.2079	0.2079
$\frac{5\pi}{8}$	— 0.0536	0.1297	— 0.1833	0.0761
$\frac{3\pi}{4}$	— 0.0671	0.0671	— 0.1342	0.0000
$\frac{7\pi}{8}$	— 0.0592	0.0245	— 0.0837	— 0.0347
π	— 0.0432	0.0000	— 0.0432	— 0.0432
$\frac{9\pi}{8}$	— 0.0269	— 0.0112	— 0.0157	— 0.0381
$\frac{5\pi}{4}$	— 0.0139	— 0.0139	0.0000	— 0.0279
$\frac{11\pi}{8}$	— 0.0051	— 0.0123	0.0072	— 0.0174
$\frac{3\pi}{2}$	0.0000	— 0.0090	0.0090	— 0.0090
$\frac{13\pi}{8}$	0.0023	— 0.0056	0.0079	— 0.0033
$\frac{7\pi}{4}$	0.0029	— 0.0029	0.0058	0.0000
$\frac{15\pi}{8}$	0.0026	— 0.0011	0.0037	0.0015
2π	0.0019	0.0000	0.0019	0.0019
$\frac{17}{8}\pi$	0.0011	0.0005	0.0006	0.0016
$\frac{9}{4}\pi$	0.0006	0.0006	0.0000	0.0012
$\frac{19}{8}\pi$	0.0002	0.0005	— 0.0003	0.0007
$\frac{5}{2}\pi$	0.0000	0.0004	— 0.0004	0.0004
$\frac{21}{8}\pi$	— 0.0001	0.0003	— 0.0004	0.0002
$\frac{11}{4}\pi$	— 0.0001	0.0001	— 0.0002	0.0000
$\frac{23}{8}\pi$	— 0.0001	0.0001	— 0.0002	0.0000
3π	— 0.0001	0.0000	— 0.0001	— 0.0001

From the values of the functions $e^{-kx}\cos kx$ and $e^{-kx}\sin kx$ given in Table 1, it is an easy matter to plot equation (5) graphically. Fig. 1 shows how the meridian moment M_1 varies with the distance from the edge of the dome. The exact values obtained by *Bolle*'s method with hypergeometrical series are given by way of comparison[3]. It will be seen that the agreement between the exact results and the approximate values is surprisingly good, so that there is no occasion to make the dome problem a complicated mathematical business. For domes with a bigger angle of opening than 40°, the agreement between the exact and the approximate values is better still. Only in the case of domes whose angle of inclination to the supports is very small does the effect of the approximations achieve practical significance. Incidentally, such domes are impracticable due to the serious disturbances at the edges set up when the dome is connected to its supports.

For the calculation of the stresses in the dome, we have to consider not only the meridian moment M_1 but also the ring moments M_2 and the additions to the meridian compressive stress and ring compressive stress set up through the boundary conditions not corresponding to the assumptions of the membrane theory. These quantities, M_2, ΔT_1 and ΔT_2 can be calculated directly from the equations below. The agreement between the figures obtained by this approximation method and the exact values is also very satisfactory, as may be seen from the comparative figures given in Table 2.

It is simplest to derive the mathematical expressions for the additional stresses ΔT_1 and ΔT_2 by assuming that the meridian is a girder with an elastic support. The increase in the compressive stress at the meridian, ΔT_1, may thus be regarded as the shearing stress in the girder multiplied by cot α, where α is the angle of inclination of the meridian to the horizontal plane. We therefore get:

$$\Delta T_1 = \cot\alpha \cdot EJ \cdot \frac{d^3 y}{dx^3} \tag{6}$$

The increase in the ring compressive stress ΔT_2 is a measure of the elastically supporting effect of the base, and, hence, ΔT_2 is directly proportional to the deflection y of the meridian, so that

$$\Delta T_2 = \frac{E\delta}{r} \cdot y \tag{7}$$

The ring moment is most simply obtained by determining the alteration in the curvature of the rings[4], and, neglecting the effect of the transverse compression, we get:

$$M_2 = \cot\alpha \cdot \frac{EJ}{r} \cdot \frac{dy}{dx} \tag{8}$$

Inserting the equation for the deflection of the meridian, viz.,

$$y = -\frac{pr^2}{2E\delta} e^{-kx} (\cos kx + \sin kx)$$

[3] See *Ekström*, loc. cit., p. 124.

[4] See, inter alia, *Föppl*: Drang und Zwang, Vol. 2. Berlin 1928.

in equations (6), (7) and (8), we obtain the following expressions for ΔT_1, ΔT_2 and M_2:

$$\Delta T_1 = -\cot\alpha \frac{pr^2\delta^2}{6} k^3 e^{-kx} \cos kx \tag{6a}$$

$$\Delta T_2 = -\frac{pr}{2} e^{-kx} (\cos kx + \sin kx) \tag{7a}$$

$$M_2 = \cot\alpha \cdot \frac{pr\,\delta^2}{12} k\, e^{-kx} \sin kx \tag{8a}$$

Table 2 contains the values of the meridian and ring stresses and ring moments worked out in this way, in comparison with the exact figures.

Table 2.
Comparison between the Proximate and Exact Values of the Meridian and Ring Stresses and Ring moments.

Angle of Inclination of the Meridian	$T_1+\Delta T_1$ Proximate kg/cm	$T_1+\Delta T_1$ Exakt kg/cm	$T_2+\Delta T_2$ Proximate	$T_2+\Delta T_2$ Exakt	M_2 Proximate kg cm/cm	M_2 Exakt
40°	443	439	0	0	0	0
35°	474	481	215	193	99	113
30°	503	504	437	427	62	73
25°	506	508	517	520	12	17
20°	503	504	518	523	— 8	— 10
15°	501	501	511	510	— 9	— 14
10°	499	499	501	501	— 5	— 9
5°	499	498	499	498	0	— 3

The problem worked out above relates to the simplest conceivable edge conditions. In order to show the applicability of the method for complicated edge conditions as well, I have worked out a dome connected to a circular cylinder all round, as Fig. 2. To simplify the problem to a certain extent, the water pressure on the dome was assumed to be constant. This problem has been dealt with by Ekström under the same assumptions. The calculated values for the meridian moment M_1 and the ring stress T_2 are given in Table 3, with the exact values for comparison.

The index 1 will subsequently be used for all constants of the dome, and the index 2 for all constants of the cylinder.

This design of dome is worked out as follows. When the inner dome and the cylinder are freed from each other and can deform unhindered under the effect of the load, the membrane theory shows a decrease in the radius of the dome of

$$\frac{pr_1^2}{2E\delta_1} = \frac{p \cdot 10^4}{E} \cdot 3{,}12 \text{ cm}$$

and an increase in the radius of the cylinder of

$$\frac{pr_2^2}{E\delta_2} = \frac{p \cdot 10^4}{E} \cdot 1{,}72 \text{ cm.}$$

The wall of the cylinder thereby forms a small angle to the perpendicular $= \frac{10}{E} \cdot 1.72$ (see Fig. 2).

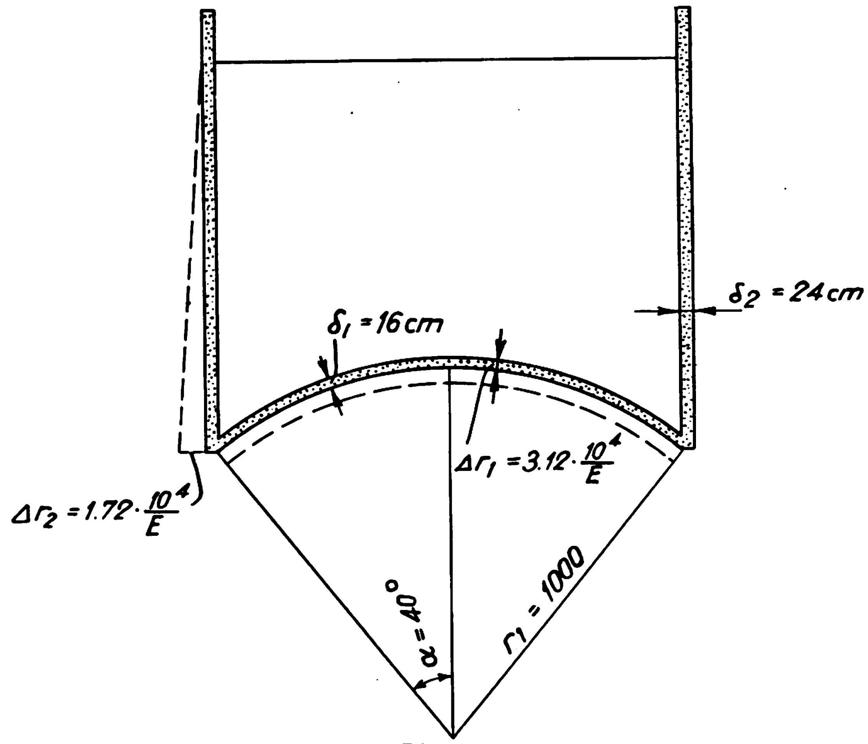

Fig. 2.

As this state of deformation is incompatible with the actual conditions of support, certain additional forces and additional moments must be introduced to satisfy the conditions of steadiness. These conditions of steadiness are as follows: The cylinder and the dome should have the same outward deflection and alteration of angle at the point of junction, and the point of junction should also be in equilibrium as regards the moments and applied forces. This involves four edge conditions, which may be expressed by means of four equations, from which all unknown deformations, moments, etc. may be determined.

To facilitate drawing up the equations, we now give the general expressions for the deflection and their derivation. We have:

$$\begin{aligned} y &= e^{-kx} [A \cos kx + B \sin kx] \\ y' &= k\, e^{-kx} [(B - A) \cos kx - (A + B) \sin kx] \\ y'' &= 2k^2 e^{-kx} [- B \cos kx + A \sin kx] \\ y''' &= 2k^3 e^{-kx} [(A + B) \cos kx + (B - A) \sin kx] \end{aligned} \tag{9}$$

The first condition, viz., that the deflections of the cylinder and the dome must be the same at the edge, may be expressed by the following equation:

$$- A_1 \sin 40^0 + A_2 = \frac{p \cdot 10^4}{E} (3{,}12 \sin 40^0 + 1{,}72).$$

So that the angular modifications may be the same in extent, we must get

$$k_1 (B_1 - A_1) = k_2 (B_2 - A_2) - \frac{10}{E} \cdot 1{,}72$$

and for the equilibrium of moment we get

$$k_1^2 EJ_1 B_1 = k_2^2 EJ_2 B_2.$$

The remaining condition should express the fact that the horizontal reaction due to the loading of the inner dome should be taken up by the shearing stress in the cylinder and by the shearing stress and the meridian stress in the dome; i. e.,

$$-2k_1^3 EJ_1 (A_1 + B_1) \cdot \frac{1}{\sin 40^0} - 2 k_2^3 EJ_2 (A_2 + B_2) = p \cdot 500 \cdot \cos 40^0.$$

By elimination from these four conditional equations, we get, for $p = 1$ kg/cm² the following values of the constants:

$$A_1 = -15{,}35 \cdot \frac{10^4}{E} \qquad B_1 = -7{,}16 \cdot \frac{10^4}{E}$$

$$A_2 = -6{,}13 \cdot \frac{10^4}{E} \qquad B_2 = 2{,}05 \cdot \frac{10^4}{E}.$$

This completely solves the problem. The moments, etc. can now be worked out without difficulty for any point of the cylinder and the dome. Table 3 contains a comparison of the calculated and true values for meridian moment and ring stress in the dome. The agreement is satisfactory at all points.

Table 3.

Meridian Moments and Ring Stresses of the Dome as Fig. 2.

Angle of Inclination of the Meridian	M_1 Proximate kgcm/cm	M_1 Exakt kgcm/cm	$T_2 + \Delta T_2$ Proximate kg/cm	$T_2 + \Delta T_2$ Exakt kg/cm
40°	— 5280	— 5560	— 1950	— 1930
35°	1450	2250	— 800	— 540
30°	1980	2200	401	613
25°	597	764	618	639
20°	— 6	9	572	593
15°	— 99	— 141	520	526
10°	— 54	— 80	498	498
5°	— 8	— 15	495	493

These two examples indicate that the method explained here for dealing with the problem gives results which are practically applicable and easy to find. As already mentioned, the proximate solution comes closer to the true values, the steeper the dome and the thinner the shell. This latter factor in particular is of great importance, as *Steuermann*[5] and others have pointed out. Unlike Equation (3 b), the exact equation for the outward deflection of the meridian contains not only expressions of the fourth and zero order, but also expressions with derivitives of the first, second and third degree, which,

[5] *E. Steuermann:* Some Considerations on the Calculation of Elastic Shells. International Conference for Technical Mechanics, Stockholm, 1930.

however, are all multiplied by polynomes of cot α. The significance of these expressions decreases with increasing values of α, and for $\alpha = 90^0$, i. e., for the cylinder, they drop out altogether, which means that equation (3 b) applies exactly. A reduction in the wall thickness of the dome has a similar effect on the complete differential equation. It is easy to see why this should be the case; it is simply due to the fact that, for small wall thickness, the compression at the meridian and the influence of the change in curvature are less pronounced in their effect. Put differently, this means that the work of the normal stresses due to compression of the meridian, together with the work of the meridian moment and the ring stresses may be ignored in thin-walled domes.

In the problems dealt with previously, the wall thickness was assumed to be constant throughout. Where the wall thickness δ is variable, we cannot start from equation (3b), must apply equation (3a). As the simple theory of the elastically supported girder gave sufficiently accurate results in the above casses, i. e., for constant wall thickness, there was reason for assuming that this would also be the case for variable wall thicknesses.

The theory of the elastically supported girder with variable moment of inertia and variable support has previously been studied by various researchers,[6] mainly with the aid of series. Unfortunately the results obtained are more or less useless for practical purposes. Due to the close affinity of equations (3 a) and (3 b), however, it is only natural that the solutions of both equations should have substantially the same mathematical basis. It may therefore be supposed that the solution of equation (3 a), for instance, may be written in the following form:

$$y = u e^{\pm z} (A \cos z + B \sin z) \tag{12}$$

where u and z are certain functions of x. By adopting *Blumenthal*'s "asymptotic process of integration", the functions of u and z can be ascertained, so that equation (12) represents an integral of equation (3 a) with very good approximation.

By introducing, as above, the bending rigidity of the girder $E\,J = \frac{E\,\delta^3}{12}$, we get the following expressions for the functions of u and z:

$$u = \frac{1}{\sqrt[4]{\delta^3}} \tag{13}$$

and

$$z = \sqrt[4]{3} \int \frac{dx}{\sqrt{r\delta}} \tag{14}$$

This result is obtained in the following way. Carring out the derivation of equation (3 a), and simplifying, we obtain the equation:

$$y^{IV} + p_1 y''' + p_2 y'' + p_3 y' + p_4 y = 0 \tag{15}$$

where $\quad p_1 = 6 \frac{\delta'}{\delta}$

[6] See, for instance, *Hayashi:* Theorie des Trägers auf elastischer Unterlage, Berlin, 1921.

$$p_2 = 3\left(\frac{\delta'^2}{\delta^2} + \frac{\delta''}{\delta}\right)$$
$$p_3 = 0$$
$$p_4 = \frac{12}{r^2\,\delta^2}$$

Multiplying the equations

$$v = f(z)$$
$$v' = f'\,z'$$
$$v'' = f'\,z'' + f''\,z'^2$$
$$v''' = f'\,z''' + 3\,f'\,z'\,z'' + f'''\,z'^3$$
$$v^{IV} = f'\,z^{IV} + f''(4\,z'\,z''' + 3\,z''^2) + 6\,f'''\,z'^2\,z'' + f^{IV}\,z'^4,$$

$\left(\text{where } f' \text{ is equivalent to } \frac{df}{dz} \text{ and } z' \text{ to } \frac{dz}{dx}\right)$ in turn by the factors Q_4, Q_3, Q_2, Q_1, and 1, and adding them, then, when the member on the left is written as equal to zero, we obtain (1) equation:

$$v^{IV} + v'''\,Q_1 + v''\,Q_2 + v'\,Q_3 + v\,Q_4 = o \tag{16}$$

and (2), when each of the factors f', f'' and f''' are made zero:

$$\begin{aligned} &z^{IV} + z'''\,Q_1 + z''\,Q_2 + z'\,Q_3 = o \\ &(4\,z'\,z''' + 3\,z''^2) + 3\,z'\,z''\cdot Q_1 + z'^2\,Q_2 = o \\ &6\,z'^2\,z'' + z'^3\,Q_1 = o \end{aligned} \tag{17}$$

Q_1, Q_2 and Q_3 can be solved from these equations, whereas the function f (z) is determined by the remaining condition

$$f^{IV}\,z'^4 + Q_4\cdot f = o \tag{18}$$

If the factor Q_4 is taken as being equal to $4\,z'^4$, equation (18) is then transformed into

$$\frac{d^4 f}{dz^4} + 4\,f = o$$

that is
$$f(z) = e^{\pm z}\,(A\cos z + B\sin z) \tag{19}$$

z being determined by the condition

$$\frac{dz}{dx} = \sqrt[4]{\frac{Q_4}{4}} \tag{20}$$

Inserting $y = uv$ in equation (15) and dividing by u, we get:

$$v^{IV} + v'''\left(\frac{4\,u'}{u} + p_1\right) + v''\left(\frac{6\,u''}{u} + \frac{3\,u'}{u}\,p_1 + p_2\right)$$

$$+ v'\left(\frac{4\,u'''}{u} + \frac{3\,u''}{u}\,p_1 + \frac{2\,u'}{u}\,p_2 + p_3\right) + v\,p_4 = o \tag{21}$$

The unknown functions Q_4 and u can be determined by equalising the coefficients for v and v‴ in equations (16) and (21). This gives us $Q_4 = p_4$ and, consequently, as equation (20):

$$z = \int \sqrt[4]{\frac{p_4}{4}}\, dx$$

$$\text{or, with} \quad p_4 = \frac{12}{r^2\,\delta^2}; \quad z = \sqrt[4]{3} \int \frac{dx}{\sqrt{r\delta}} \tag{14}$$

From the condition $\frac{4u'}{u} + p_1 = Q_1$ and adopting the last of the equations (17), we get:

$$\frac{4u'}{u} = -p_1 - \frac{3}{2}(\log p_4)'$$

$$\text{or} \quad u = \frac{1}{\sqrt[4]{\delta^3}} \tag{13}$$

Summarising the result of the above calculations, the solution of equation (3 a) can be written in the following form by neglecting the expressions containing the factor e^{+z}:

$$y = \frac{1}{\sqrt[4]{\delta^3}} e^{-z} (A \cos z + B \sin z) \tag{12a}$$

in which z is determined by the condition $z = \sqrt[4]{3} \int \frac{dx}{\sqrt{r\delta}}$.

At first glance, equation (12 a) may perhaps appear involved and not very suitable for practical purposes, due to the complicated structure of the function z and of the additional factor $\frac{1}{\sqrt[4]{\delta^3}}$. The case becomes simpler in actual practice, however. The function z need never be indicated other than numerically, so that it can easily be calculated from equation (14), say, by the trapeze rule. In calculating the derivatives of equation (12 a), fairly complex expressions are obtained where no approximations are introduced. But when it is remembered that the derivations z″, z‴, u″ and u‴ are small for the dimensions involved in actual practice, and can therefore be neglected, the derivations of y are obtained in the following form:

$$\begin{aligned}
y &= u\, e^{-z} (A \cos z + B \sin z) \\
y' &= u\, z'\, e^{-z} [(B - \mu A) \cos z - (A + \mu B) \sin z] \\
y'' &= 2\, u\, z'^2\, e^{-z} [-(\mu B + \gamma A) \cos z + (\mu A - \gamma B) \sin z] \\
y''' &= 2\, u\, z'^3\, e^{-z} [(A + \mu_1 B) \cos z + (B - \mu_1 A) \sin z]
\end{aligned} \tag{9a}$$

$$\text{where} \quad \begin{aligned} v &= \frac{u'}{u\, z'} \\ \mu &= 1 - v \\ \mu_1 &= 1 - 3v. \end{aligned}$$

In cases where the wall thickness is constant, $v = 0$ and $\mu = \mu_1 = 1$, the above equations becoming exactly the same as the equations (9).

The equations (9 a) are therefore built up in the same way as the derivations for a girder of constant bending rigidity given in the equations (9). A dome of variable wall thickness can consequently be worked out in the same way and with very little more trouble than one of constant wall thickness. The examples given above (see Figs. 1 and 2) are thus typical of the case where δ is variable, and the equilibrium equations should be drawn up in the same way, but with the modifications necessitated by the difference between equations (9) and (9 a).

We have not yet considered the dome problem in cases where the girders at the meridian taper upwards and their width is nil at the apex of the dome, but have rather assumed them to be of constant width. This is only true when the dome is cylindrical; but for domes in general a certain process of approximation is inherent in this assumption. When allowing for the taper, we can, for spherical domes, express the moment of inertia of the meridian girder at a definite angular distance α from the apex by the following equation:

$$J = \frac{\delta^3}{12} \cdot \frac{\sin \alpha}{\sin \alpha_0} \tag{21}$$

With this expression for the moment of inertia, we obtain, for functions u and z:

$$u = \frac{1}{\sqrt[4]{\delta^3}} \cdot \frac{1}{\sqrt[8]{\sin \alpha}}$$

and

$$z = \sqrt[4]{3} \int \frac{1}{\sqrt{r\delta}} \cdot \sqrt[4]{\frac{\sin \alpha_0}{\sin \alpha}}\, dx.$$

The above derivations, which apply mainly to the dome problem, may of course also be applied to cylindrical tanks and similar structures, which should be regarded as special cases of the dome. The usual methods[7] for calculating such containers, based on the developments of mathematical series, may advantageously be replaced by the method given above. A special and interesting case of this particular problem is met with in the calculation of solid arched dams. The usual method of dealing with these problems was to start from equation (3 b) and introduce a mean value for the wall thickness[8].

In dealing with equation (3 a) by the above method, it is possible, without difficulty, to allow for the anisotropy of the structure which occurs in various directions and at different points. The anisotropy may be purely a phenomenon of the material, or a purely constructive anisotropy. When different quantities of reinforcing steel are inserted in different directions, for example, the apparent modulus of elasticity of the material will vary in different directions, and this we may term anisotropy of the material; while a certain constructional or design anisotropy may be introduced into a cylindrical tank or a dome by fitting reinforcing girders in the direction of the generatrix or the meridian

[7] See, *Lorenz*: Technische Elastizitätslehre, Berlin 1913. *H. Reißner:* Beton und Eisen 7, 150, 1908. *T. Pöschl* and *K. Terzaghi:* Berechnung von Behältern, Berlin 1913.

[8] *N. Royen:* Tvärödammen vid Norrfors kraftverk (Der Damm von Tvärö am Kraftwerk Norrfors), Zeitschr. Betong, vol. 2, 1926,

(ribbed domes). In such circumstances, equation (3 a) cannot be written in the form in which it is contained in equation (15), but the coefficients p_1 to p_4 assume the following aspect:

$$
\begin{aligned}
p_1 &= \frac{2\,(E_1\,J)'}{E_1\,J} \\
p_2 &= \frac{(E_1\,J)''}{E_1\,J} \\
p_3 &= o \\
p_4 &= \frac{E_2\,\delta}{r^2\,E_1\,J}
\end{aligned}
\tag{22}
$$

and the functions z and u accordingly appear in the following form:

$$
z = \int \sqrt[4]{\frac{E_2\,\delta}{4\,r^2\,E_1\,J}}\,dx
\tag{23}
$$

and
$$
u = \sqrt[8]{\frac{r^6}{E_1\,J \cdot E_2{}^3\,\delta^3}}
$$

Since no mathematical expression is necessary either for u or z, the introduction of equations (22) and (23) does not make the calculations more difficult.

Summary.

By dividing the shell into two systems of intersecting beams and by applying the well-known theories of the elastically supported beam, we get a clear idea of the statical behaviour of the construction and obtain results that are sufficiently correct. As the exact theories lead to solutions in form of infinite series, which under certain conditions converge only slowly, this way of dealing with the problem offers great simplification.