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IIIb1

Fonctions d'influence pour la correction des déviations angulaires dans les ponts suspendus

Einflussfunktionen für die Berücksichtigung der Winkelabweichung bei Hängebrücken

Influence functions for the angular deviation correction in suspension bridges

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In the classical deflection theory of suspension bridges the differential equation [1]

$$(EI\eta'')'' - (H_w + H)\eta'' = Hy'' + p(x)$$

of the elastic line η of the stiffening truss is established under the assumption

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[1] This equation is usually termed Melan's equation but was earlier published and solved by A. Ritter (1877) and H. Müller-Breslau (1881).

Notations : All notation, except that which is specific to this paper, is in conformity with common usage in most American treatises on the theory of suspension bridges. Reference to the numbered equation where the magnitude is first defined or used is given in parentheses.

A_{xk}'', a_{xk}'' = angular deviation correction influence functions (10), (12)
c = incidental flexibility of one-span bridge (10)
E = modulus of elasticity of truss (1)
f = cable sag (11), $f(x)$ = angular correction load (4)
H_w = horizontal force of dead load (1), H = increment of horizontal force due to other causes (1)
I = moment of inertia of truss (1)
I_{xk} = influence functions (2), i_{xk} = influence functions for one-span bridge of unit length (13)
J_x'', j_x'' = influence functions (2)
k = abscissa of live load elements measured from the left end (2)
l = span length (11)
M = moment in stiffening truss at section x (6)
$p(k)$ = distributed live load at abscissa k (2)
t = temporarily used abscissa (10). t_1 = abscissa measured from low-point of cable (center of span) (12)

that the cable-points move along fixed verticals. Rode ([2]) seems to have first pointed out that the differential equation may be established without resorting to that assumption. Rode gave his equation for a constant stiffness EI of the truss, but it can easily be generalized to apply to variable truss stiffness and rearranged to manifest more plainly its interesting mathematical character ([3]) :

$$(EI\eta'')'' - (H_w + H)[(1 + y'^2)\eta']' = Hy'' + p(x) \qquad (1)$$

Melan's equation as well as (1) are restricted by homogeneous and linear boundary conditions namely the restraint conditions at the supports of the stiffening truss and the cable condition.

The exact solution of (1) or of Melan's equation is very complicated, since by the cable condition the live load horizontal force H is a function of all η. In fact, no solution by finite methods has yet been given. If, however, the variable H is treated as a constant, the equation (1) together with its boundary conditions may be recognized as a self-adjoint linear boundary problem with variable coefficients in the differential equation. Accordingly one enters a plausible value of H in the problem. By its solution a new value of H is calculated from the cable condition. This value is again entered in the boundary problem. By iteration of the solution with successively corrected values of H the exact solution of the fundamental equation (1) and its boundary conditions may thus be approached to any desired accuracy. In practical bridge problems the convergence of this process is rapid. More than one or two iterations are seldom required.

The general form of the solution of the *linear* boundary problem ([1]) (with an assumed constant value of H) may be obtained by an expansion of the solution according to the (orthogonal) « eigenfunctions » of the homogeneous differential equation associated to (1) (left member = 0) ([4]). It is :

$$\eta(x) = \frac{1}{H_w + H}\left[\int_s I_{xk}\, p(k)\, dk + \delta J_x\right] \qquad (2)$$

This equation indicates that influence lines may be suitably applied in the solution. The « deflection influence function » I_{xk} is a function of H (and of the section x considered and of the load position k). It is independent of the cable *yield*, because all cable yield effects are comprised in a factor δ and carried to the correction term δJ_x, J_x being a function of H (and of x). But I_{xk} cannot be made to be independent of the *form*, y, of the cable (sag-ratio, etc.). Therefore, if a set of influence functions had been computed and tabulated it could only be used for bridges with the same relative stiff-

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w = dead load per unit length of bridge (13)
x = abscissa measured from the left support to the section to be investigated (1)
y = ordinate measured downwards from a horizontal line to the cable under dead load at section x (1)
δ = cable yield constant (2)
η = vertical displacement of truss at section x due to live load, temperature, and anchorage displacement (1)
ξ = horizontal displacement of the cable point originally at x due to same causes (8)

([2]) *Proceedings Am. Soc. of Civ. Eng.*, 1928, p. 393.
([3]) S. O. Asplund, *On the Deflection Theory of Suspension Bridges* (*Ingenjörsvetenskaps-akademins Handlingar* 184, Stockholm, 1945, p. 23).
([4]) See ([3]), p. 31-37.

ness variation *and* cable sag ratios in the different spans. Moreover, although the general form (2) of the solution is known, influence functions have not yet been published for any specific bridge, however simple. A mathematically interesting « exact » (proviso constant H) solution of special cases of (1) has been undertaken but it is not possible to see how it could be advantageously applied.

For practical reasons the author considers that the mathematically concise form (1) should be given up in favor of

$$(EI\eta'')'' - (H_w + H)\eta'' = Hy'' + p(x) + f(x) \tag{3}$$

$$f(x) = (H_w + H)(y'^2\eta')' \tag{4}$$

in suspension bridge problems. The solution of this form still obtains the same form (2) with the exceptions that I_{xk} is computed from the homogeneous part (left hand side) of (3) only and therefore *no* function of the cable form y, and that instead of $p(x)$ the « load » $p(x) + f(x)$ should be applied under the integral sign of (2). In actual bridge problems the « correction load » $f(x)$ is much smaller than $p(x)$. If $f(x)$ is first neglected, η can be solved by (2) and η' and $(y'^2\eta')' = f(x)/(H_w + H)$ computed. When this function is entered together with $p(x)$ in (2) an improved solution is obtained which is acceptable for practical purposes even if it can be made more precise by iteration.

This procedure makes possible the treatment of suspension bridges by much larger classes, inasmuch as the influence functions I_{xk} will be entirely independent of the cables (size, yield *and* form): *All* bridges with the same relative stiffness variation of the truss may be treated by the same set of tables I_{xk}.

If $f(x)$ is omitted the solution will be the same as that of Melan's equation. The application of $f(x)$ is equivalent to a consideration of changes ξ' in the horizontal projections of the cable elements and can suitably be termed the cable angular deviation correction load.

If the solution according to Melan's equation is first established, the addition of an angular deviation correction obviously yields the solution according to (1).

By differentiation of (2) the change of grade of the truss becomes

$$\eta'(x) = \frac{1}{H_w + H}\left\{ \int_s I'_{xk}\,[p(k) + f(k)]\,dk + \delta J_x' \right\} \tag{5}$$

and for instance the moment in the truss

$$M = \int_s I''_{xk}\,[p(k) + f(k)]\,dk + \delta J_x'' \tag{6}$$

where

$$I'_{xk} = \frac{\partial I_{xk}}{\partial x}, \qquad J_x' = \frac{dJ_x}{dx}, \qquad I''_{xk} = -\frac{EI(x)}{H_w + H}\frac{\partial I'_{xk}}{\partial x},$$

$$\text{and} \quad J_x' = -\frac{EI(x)}{H_w + H}\frac{dJ_x}{dx}.$$

Instead of by the laborious computation of $f(x)$ just explained and application to (2), (5), (6), etc., the angular deviation corrections may be evaluated separately and directly from the load $p(x)$ by the use of influence functions. Entering (4) in (6) yields

Row	t/l		0	0.05	0.1	0.15	0.2
1	$2\,t/l - 1 = 2\,t_1/l$		-1.0	-0.9	-0.8	-0.7	-0.6
2	$-(t_1/l)^2\,c^2$		-25	-20.3	-16	-12.3	-9
3	i'_{tk}	$10^{-3}\times$	327	285	146	-6	-88
4	i''_{k}	$10^{-3}\times$	0	17.2	40.0	22.2	11.5
5	$i'_{tk}\cdot 2\,t_1/l$	$10^{-3}\times$	-327	-257	-117	4	53
6	$-i''_{tk}\cdot(t_1/l)^2\,c^2$	$10^{-3}\times$	0	-347	-640	-272	-103
7	Sum of rows 6,7	$10^{-3}\times$	-327	-604	-757	-268	-50
8	i''_{xt}	$10^{-3}\times$	0	4.8	11.5	22.4	40.9
9	Integrand of a''_{xk}	$10^{-3}\times$	0	-2.89	-8.68	-6.01	-2.00

$$M = \int_s I''_{xk}\,p(k)\,dk + \int_s I''_{xk}\,(H_w + H)\,[(y'^2\,\eta')']_{x=k}\,dk + \delta J_x'' \,. \tag{7}$$

The middle term on the right hand side is $M_{\xi'}$, the angular deviation correction for moment,

$$M_{\xi'} = \int_s I''_{xk}\,(H_w + H)(2\,y'y''\eta' + y'^2\,\eta'')_{x=k}\,dk \,. \tag{8}$$

Substituting (5) in (8) yields

$$M_{\xi'} = \int_s I''_{xk}\,\Big[(2\,y'y'')_{x=k}\int_s I_{kt}\,p(t) + f(t)\Big]\,dt$$
$$- (y'^2)_{x=k}\int_s \frac{H_w + H}{EI(k)}\,I''_{kt}\,[p(t) + f(t)]\,dt\Big\}\,dk + \delta J_x'\,[y'^2\,\eta']_s \,.$$

The relatively small influence of the correction load $f(t)$ and the small δ-term could be accounted for by this formula. If they are dropped and k and t interchanged the comparatively simple formula

$$M_{\xi'} = \int_s A''_{xk}\,p(k)\,dk \tag{9}$$

remains with

$$A''_{xt} = \int_s I''_{xt}\left[(2\,y'y'')_{x=t}\,I'_{tk} - (y'^2)_{x=t}\,\frac{H_w + H}{EI(t)}\,I''_{tk}\right]dt \tag{10}$$

as an influence function for the angular deviation correction for moment.

Such influence functions shall now be numerically evaluated for the case of a symmetrical one-span bridge with constant stiffness EI. If the span-length is l, the cable sag f, and the uniform dead load w one has

$$y'' = -\frac{w}{H_w} = -\frac{8\,f}{l}\,, \qquad y' = -\frac{8\,f}{l^2}\,x_1\,, \qquad 2\,y'y'' = \frac{128\,f^2}{l^4}\,x_1\,,$$
$$y'^2 = \frac{64\,f^2}{l^4}\,x_1^2$$

0.3	0.4	0.5	0.6	0.8	1.0		t/l	Row
— 0.4	— 0.2	0	0.2	0.6	1.0		$2\,t/l - 1 = 2\,t_1/l$	1
— 4	— 1	0	— 1	— 9	— 25		$-(t_1/l)^2\,c^2$	2
— 142	— 129	— 92				$10^{-3}\times$	i'_{tk}	3
1.0	— 2.9	— 4.3				$10^{-3}\times$	i''_{tk}	4
57	26					$10^{-3}\times$	$i'_{tk}\cdot 2\,t_1/l$	5
— 4	3					$10^{-3}\times$	$-\,i''_{tk}\cdot(t_1/l)^2\,c^2$	6
53	29	0	— 4	68	113	$10^{-3}\times$	Sum of rows 6,7	7
7.1	— 6.0	— 10.8	— 11.7	— 8.1	0	$10^{-3}\times$	i''_{xt}	8
0.38	— 0.17	0	0.65	— 0.55	0	$10^{-3}\times$	Integrand of a''_{xk}	9

x_1 (and t_1) being measured from the low point of the cable at the center of the span. The influence functions $i_{xk}' = I_{xk}'$ and $i_{xk}'' = I_{xk}''/l$ are previously tabulated ([5]) for different values of

$$c = l\sqrt{\frac{H_w + H}{EI}}$$

Equation (10) becomes

$$A''_{xk} = \int_0^l l i''_{xt}\left(\frac{64\,f^2}{l^4}\,2\,t_1 i'_{tk'} - \frac{64\,f^2}{l^4}\,t_1^2\,\frac{c^2}{l^2}\,l i''_{tk}\right) dt$$

$$A''_{xk} = l\,\frac{64\,f^2}{l^2}\,a''_{xk} \tag{11}$$

$$a''_{xk} = \int i''_{xt}\left[\frac{2\,t_1}{l}\,i'_{tk} - \left(\frac{t_1}{l}\right)^2 c^2\,i''_{tk}\right] d\,\frac{t}{l}\;. \tag{12}$$

The complete moment formula derived from (7), (9) and (11), $j_x'' = J_x''/l$, thus becomes

$$\frac{M}{wl^2} = \int i''_{xk}\,\frac{p(k)}{w}\,d\,\frac{k}{l} + \frac{64\,f^2}{l^2}\int a''_{xk}\,\frac{p(k)}{w}\,d\,\frac{k}{l} + \frac{\delta}{wl}\,j_x'' \tag{13}$$

The computation of just one value of a_{xk}'' will be explained here, namely for $c = 10$ at $x/l = 0.2$ and $k/l = 0.1$.

Intermediate values of the integrand may be determined from diagrams of rows 7 and 8. Finally the integrands of a_{xk}'' are carefully plotted and integrated by quadrature, yielding the integral $a_{xk}'' = -\,1.03 \times 10^{-3}$. The result of similar computations for $x/l = 0.2$, $k/l = 0.05$, 0.1, 0.15, 0.2, 0.3, 0.4, 0.5, 0.6 and 0.8, and $c = 10$ and 20 is diagrammated in fig. 1 and 2.

These a_{xk}'' influence diagrams can be used for a fairly accurate eva-

([5]) See ([3]), p. 85.

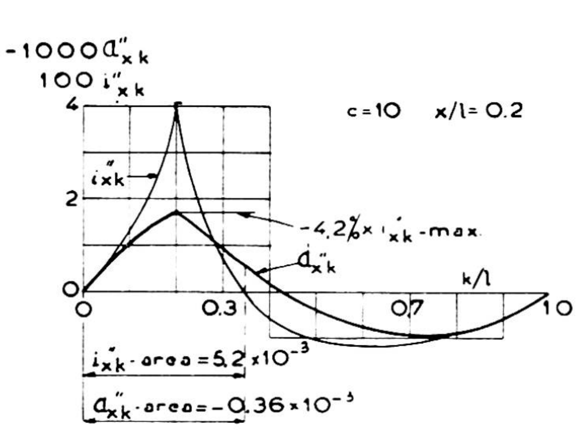

Fig. 1.

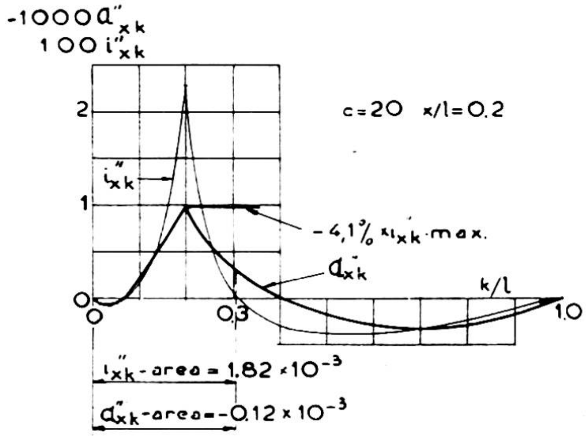

Fig. 2.

luation of the angular deviation correction for moment at any given load, for instance composed of concentrations and loaded segments.

For comparison the influence functions i_{ik}'' to be used simultaneously in (13) are shown in fig. 1 and 2. It is seen that the shape of the a_{xk}''-diagram is roughly the same as that of the i_{xk}''-diagram.

The maximum ordinate of a_{xk}'' is — 4.2 % and — 4.1 % of that of i_{xk}'' for $c = 10$ and 20, respectively, and $x = 0.2\,l$. A calculation for $x = 0.15\,l$ gives the percentages — 5.3 and — 6.2 for $c = 10$ and 15, respectively.

The zero point of a_{xk}'' is to the right of that of i_{xk}''. The branches leading to the maximum point are more straight for a_{xk}''. When the positive area of i_{xk}'' is loaded the engaged area of a_{xk}'' is —6.9 % and —6.6 % of that of i_{xk}'' for $c = 10$ and 20, respectively, and $x = 0.2\,l$ (fig. 1 and 2). For ordinary cases of loading the maximum positive moment in a one-span suspension bridge of even stiffness occurs near the point $x = 0.2\,l$, while the larger part of the i_{xk}''-area is loaded with concentrated or uniform loads. The given percentages indicate that the angular deviation correction between $c = 10$ and 20 amounts to between — 6 % $\times$ 64 f^2/l^2 and — 7 % $\times f^2/l^2$ of the maximum positive moment at $x = 0.2\,l$. These values can be roughly used if influence lines a_{xk}'' are lacking. In a one-span bridge of flexibility between $c = 10$ and 20 and with a sag-ratio $f/l = 0.12$ the angular deviation correction is thus about — 6 % of the maximum moment.

The completed solution, according to this paper, of the angular deviation correction can be directly applied to bridges of any sag-ratio. The direct mathematical solution of (1) mentioned above, would be even more difficult if the influence of a variable f/l had to be taken into account. When influence functions are determined by tests on model ([6]), (1) and its boundary conditions is so to speak solved directly by a model machine. Then the resulting influence functions I_{xk} of (2) will also include the angular deviation correction, the magnitude of which will not be disclosed. Therefore the set of influence lines obtained will not be exactly applicable to any other bridges than such with the *same* sag-ratios of the cable as in the model bridge used.

In a one-span bridge the maximum moment is seen to *reduce* the angular deviation correction. An omission of this correction therefore is

([6]) See ([3]). p. 110

on the safe side. It should not be overlooked that the absolute value of the minimum moment at the tower of a continuous bridge is *increased* by the angular deviation correction.

Résumé

L'équation différentielle classique et quasi linéaire des ponts suspendus ancrés (équation de Melan) peut se compléter par une expression qui tient compte des variations d'angles des éléments des câbles. Il se présente alors le problème de la valeur aux limites, dont la solution purement mathématique n'est, cependant, pas avantageuse. Si, au lieu de cela, l'on ajoute ladite expression à la partie non homogène de l'équation comme fonction d'interférence, cette interférence peut se calculer rapidement en se servant à plusieurs reprises des fonctions d'influence. L'auteur indique le calcul numérique d'une valeur de la fonction d'influence. Deux courbes complètes de l'influence, utilisables pour des ponts avec des flèches différentes, sont indiquées sur des schémas. Pour un pont ayant une ouverture et une flexibilité c comprise entre 10 et 20, la dimension de rectification de la variation d'angle pour les plus grands moments absolus, avec $x = 0{,}2\ l$ peut atteindre entre — 6 et — 7 % de 64 f^2/l^2 fois la valeur du moment, qui se calcule en se servant de la théorie classique des moments fléchissants.

Zusammenfassung

Die klassische quasilineare Differentialgleichung der verankerten Hängebrücke (Gleichung von Melan) kann durch einen die Winkelabweichungen der Kabelelemente erfassenden Ausdruck ergänzt werden. Dabei entsteht ein selbstadjungiertes Randwert-Problem, dessen direkte mathematische Lösung sich aber als unvorteilhaft erweist. Wenn statt dessen der genannte Ausdruck dem inhomogenen Teil der Gleichung als Störungsfunktion angefügt wird, kann die Störung schnell durch wiederholte Anwendung von Einflussfunktionen berechnet werden. Die numerische Berechnung eines Wertes der Einflussfunktion wird gezeigt. Zwei bestimmte vollständige Einflusslinien, anwendbar auf Brücken mit verschieden grossem Durchhang sind in Diagrammen dargestellt. Bei einer Brücke mit einer Oeffnung und einer Biegsamkeit c zwischen 10 und 20 kann die Korrekturgrösse der Winkelabweichung für die grössten positiven Momente bei $x = 0{,}2\ l$ zwischen — 6 und — 7 % von 64 f^2/l^2 mal den Wert des Moments erreichen, das mit der klassischen Durchbiegungstheorie berechnet wird.

Summary

A term accounting for the angular deviations of the cable elements may be included in the classical quasi-linear differential equation of the truss (Melan's equation). A self-adjoining boundary problem is then formed but a direct mathematical solution appears disadvantageous. If the term in question is instead carried to the non-homogeneous part of the equation as a disturbance function, the disturbance may be expediently calculated by iterated use of influence functions. The numerical evaluation of one influence function value is demonstrated. Two significant complete influence lines, applicable to bridges of various sag ratio are given in diagrams. In a one-span bridge with flexibility c between 10 and 20 the angular deviation correction for maximum positive moments at $x = 0.2\,l$ may amount to between — 6 and — 7 % of 64 f^2/l^2 times the moment as calculated by the classical deflection theory.

IIIb2

Contribution à la statique des ponts suspendus à poutre de rigidité

Beitrag zur Statik der Hängebrücken mit Versteifungsträger

Contribution to the statics of suspension bridges with stiffening girders

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§ 1. Deux principes fondamentaux interviennent dans le calcul des ponts suspendus : d'une part le théorème de Godart qui fournit l'expression de la déformation, du moment fléchissant et de l'effort tranchant dans une travée suspendue en fonction de la poussée; d'autre part l'équation qui sert à la détermination de la poussée (composante horizontale de la traction des câbles).

Rappelons d'abord brièvement les résultats déduits du théorème de Godart.

En l'absence de toute surcharge et à la température de réglage, le câble porteur d'une travée, suspendue de portée l, supposée sur appuis simples et dont la charge permanente est d'intensité constante p', décrit une parabole dont l'équation rapportée à sa corde (l'origine des abscisses étant à l'extrémité de gauche de la travée) est (fig. 1)

$$y = \frac{4f}{l^2} x (l - x) . \tag{1}$$

Dans ces conditions la poussée du câble est $Q' = p' \dfrac{l^2}{8f}$.

Supposons maintenant que la température vienne à varier ou que des surcharges (constituées par des charges isolées, ou des charges d'intensité constante réparties sur tout ou partie de la travée) ; le câble se déforme, son ordonnée devenant $y + v$; la poussée devient $Q' + Q$. On sait alors que si μ désigne le moment fléchissant que produiraient les surcharges dans la poutre de rigidité si elle n'était pas suspendue, le moment fléchissant M dans la poutre a pour expression

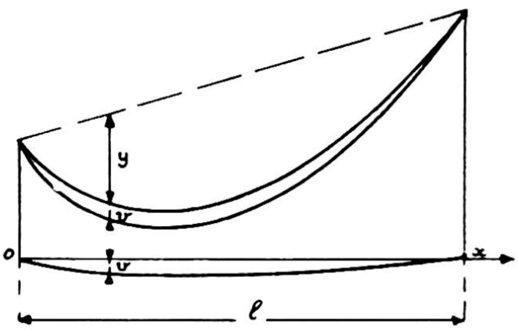

Fig. 1.

$$M = \mu - (Q' + Q)\, v - Qy \tag{2}$$

et que la déformation v satisfait à l'équation différentielle

$$\frac{d^2 v}{dx^2} - K^2 v = - \frac{K^2}{Q' + Q} [\mu - Qy], \tag{3}$$

$$\text{avec } K^2 = \frac{Q' + Q}{EI},$$

I désignant le moment d'inertie de la poutre de rigidité, que nous supposerons constant et E son module d'élasticité.

Dans les conditions de surcharges définies précédemment, on peut partager la poutre en un certain nombre d'intervalles à l'intérieur de chacun desquels la surcharge répartie garde une intensité uniforme, sans qu'il y ait de surcharges isolées. Les limites des intervalles sont donc, outre les extrémités de la poutre, les points d'application des surcharges concentrées et les extrémités des zones de surcharges d'intensité constante. A l'intérieur de chacun des intervalles ainsi définis, le second membre de l'équation différentielle (3) est un polynôme du second degré, et son intégrale générale peut s'écrire :

$$v = \frac{Q}{Q' + Q} \left[\frac{A}{K^2} e^{Kx} + \frac{B}{K^2} e^{-Kx} + \frac{\mu}{Q} - y + \frac{1}{K^2} \left(\frac{p'}{Q'} - \frac{p}{Q} \right) \right] \tag{4}$$

p désigne l'intensité de la surcharge dans l'intervalle considéré (qui peut être nulle) et A et B deux constantes. Il y a autant de couples de constantes que d'intervalles, et on les détermine en écrivant que v est nul pour $x = 0$ et $x = l$, et que v et $\frac{dv}{dx}$ sont continus aux limites communes à deux intervalles. On se rend compte qu'on obtient bien ainsi autant d'équations qu'il y a d'inconnues à déterminer.

Des équations (4) et (2) on déduit aisément l'expression de M,

$$M = - \frac{Q}{K^2} \left| A e^{Kx} + B e^{-Kx} + \frac{p'}{Q'} - \frac{p}{Q} \right| \tag{5}$$

puis celle de l'effort tranchant T dans la poutre de rigidité :

$$T = \frac{dM}{dx} = - \frac{Q}{K} [A e^{Kx} - B e^{-Kx}] \tag{6}$$

§ 2. Les résultats du paragraphe précédent montrent que tout le problème du calcul de la poutre de rigidité revient à la détermination de l'accroissement de la poussée Q dû aux surcharges et à la température. Rigoureusement on obtiendrait Q en écrivant que la variation de longueur du câble due à la déformation v, compte tenu des déplacements de ses extrémités, est égale à la variation de longueur due à l'allongement élastique et à la température. On obtiendrait ainsi une équation qui conduirait à des calculs fort longs, ne pouvant être exécutés que par des méthodes d'intégration numérique approchée.

On peut simplifier cette équation de bien des façons en négligeant des quantités très petites. La méthode exposée ci-après consiste à utiliser une relation due à M. l'Inspecteur général des Ponts et Chaussées Mabilleau (*Annales des Ponts et Chaussées*, 1937). Sous forme différentielle cette relation (dite équation des déplacements orthogonaux) s'écrit :

$$\frac{du}{dx} = -\frac{dy}{dx}\cdot\frac{dv}{dx} + \frac{Q}{E'S}\left(\frac{ds}{dx}\right)^3 + \tau\left(\frac{ds}{dx}\right)^2$$

s désigne l'abscisse curviligne du câble; E′ et S son module d'élasticité et sa section; τ la variation unitaire de longueur due à la température; enfin u est le déplacement horizontal d'un point du câble.

On peut intégrer la relation précédente, et lui donner la forme intégrale suivante :

$$u_1 - u_0 = -\frac{8f}{l^2}\int_0^l v\,dx + \frac{Q}{E'S}L_s + \tau L_t\,. \qquad (7)$$

dans laquelle u_0 et u_1 sont les déplacements horizontaux de l'origine et de l'extrémité du câble, et L_s et L_t les intégrales

$$L_s = \int_0^l \left(\frac{ds}{dx}\right)^3 dx \qquad L_t = \int_0^l \left(\frac{ds}{dx}\right)^2 dx\,. \qquad (8)$$

L'équation (7) est en quelque sorte l'analogue de l'équation de Bresse relative aux déformations des poutres à fibre moyenne que l'on utilise pour la détermination de la poussée d'un arc à 2 articulations.

§ 3. On applique aisément les résultats précédents au calcul d'un pont suspendu à n travées, sans câbles de tête. Nous supposons que le câble n'exerce aucune réaction horizontale sur le sommet des pylônes, et que les poutres de rigidité sont indépendantes.

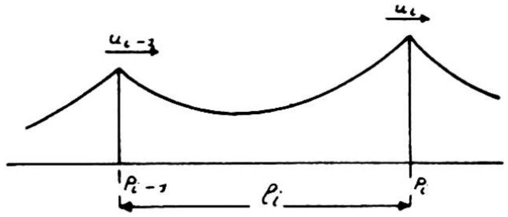

Fig. 2.

Nous utiliserons les notations définies précédemment, en affectant de l'indice i celles afférentes à la i^e travée, nous désignerons en outre par u_i, le déplacement du câble au droit du pylône p_i.

L'équilibre, sans surcharges à la température de réglage exige que l'on ait

$$Q' = p_i'\frac{l_i^2}{8f_i} \quad \text{ou} \quad \frac{8f_i}{l_i^2} = \frac{p_i}{Q'} \quad (i = 1,\ 2\ \dots\ n). \qquad (9)$$

Lorsque les travées sont complètement ou partiellement surchargées, ou si la température est différente de la température de réglage, la poussée devient $Q' + Q$. On détermine Q en écrivant pour chaque travée et pour les câbles de retenue la relation (7); (en effet la relation (7) s'applique aux câbles de retenue s'ils sont suffisamment tendus, en faisant $v = 0$). On ajoute les équations obtenues afin d'éliminer les u_i et l'on obtient l'équation

$$\frac{Q}{E'S}\mathcal{L}_s + \tau\mathcal{L}_t = \sum_{i=1}^{i=n} \frac{8 f_i}{l_i^2}\int_0^{l_i} v_i\, dx \,. \tag{10}$$

$\mathcal{L}_s$ et $\mathcal{L}_t$ désignant les intégrales (8) calculées d'une extrémité à l'autre du câble (d'ancrage à ancrage).

Si les câbles de retenue étaient longs, ou trop peu tendus, il suffirait d'ajouter au premier membre de cette équation un terme correctif dépendant de Q, mais petit; ceci ne modifierait en rien les transformations effectuées par la suite.

Il est alors naturel de remplacer v_i par sa valeur déduite de (4); on parvient ainsi à l'équation :

$$Q = \frac{\displaystyle\sum_i \frac{8 f_i}{l_i^2}\int_0^{l_i}\left(\mu_i - \frac{p_i}{K_i^2}\right)dx - \tau\,(Q' + Q)\,\mathcal{L}_t}{\displaystyle\sum_i \frac{8 f_i}{l_i^2}\left[\frac{2}{3} f_i l_i - \frac{8 f_i}{l_i K_i^2} - \frac{1}{K_i^2}\int_0^{l_i}(A_i e^{K_i x} + B_i e^{-K_i x})\,dx\right] + \frac{Q'+Q}{E'S}\mathcal{L}_s.}$$

C'est exactement l'équation utilisée par les ingénieurs américains pour le calcul du pont de Philadelphie, et obtenue en utilisant le principe des travaux virtuels.

Cette équation contient Q au second membre car les (A_i, B_i) et les K_i dépendent de Q. On pourrait la résoudre par itération, la substitution d'une valeur approchée de Q dans le second membre fournissant une nouvelle valeur de Q plus approchée. Mais les calculs auxquels on est ainsi conduit sont fort longs, car il faut à chaque approximation calculer toutes les constantes (A_i, B_i), et il faut ne pas perdre de vue qu'il y a, dans chaque travée autant de couples de constantes qu'il y a d'intervalles de surcharges, au sens du paragraphe 1, dans cette travée.

§ 4. Mais on peut donner à l'équation (11) une autre forme dans laquelle ne figurent plus les (A_i, B_i). On évitera ainsi dans les approximations successives le calcul fastidieux et long de ces constantes. Les calculs de transformation étant assez compliqués, nous nous bornerons à énoncer les résultats obtenus.

On définit tout d'abord dans chaque travée une fonction $\mathcal{F}(x)$; l'abscisse x étant comptée à partir de l'extrémité de gauche de la travée :

$$\mathcal{F}(x) = \frac{1}{12}(3\,lx^2 - 2\,x^3) - \frac{x}{K^2} + \frac{1}{K^3}\,\frac{e^{Kx} - e^{K(l-x)}}{e^{Kl} + 1} \tag{12}$$

dont la dérivée est

$$\mathcal{F}'(x) = \frac{x\,(l - x)}{2} - \frac{1}{K^2} + \frac{1}{K^2}\,\frac{e^{Kx} + e^{K(l-x)}}{e^{Kl} + 1}\,. \tag{13}$$

Supposons alors que la travée considérée supporte des surcharges concentrées P_j aux abscisses α_j et des surcharges d'intensité constante p_j réparties entre les points d'abscisses β_{j-1} et β_j, on pourra définir une expression $\mathcal{F}$ par la relation

$$\mathcal{F} = \sum_j P_j \mathcal{F}'(\alpha_j) + \sum_j p_j\,[\mathcal{F}(\beta_j) - \mathcal{F}(\beta_{j-1})] \tag{14}$$

Cette formule (14) peut du reste encore s'écrire

$$\mathcal{F} = \int_0^l p(x)\, \mathcal{F}'(x)\, dx \tag{14'}$$

$p(x)$ désignant la fonction de charge, $p(x)$ peut du reste parfaitement ne pas être continue et même comprendre des charges concentrées : dans le cas, par exemple où $p(x)$ se réduit à la seule charge concentrée P à l'abscisse α, $p(x)$ sera nulle en tout point de l'intervalle $(0,l)$, sauf pour $x = \alpha$, où $p(x)$ tend vers l'infini de façon que l'intégrale de $p(x)$ prise dans un intervalle contenant la valeur α soit égale à P; dans ces conditions, on aura, $\varphi(x)$ étant une fonction continue

$$\int_0^x p(x)\cdot\varphi(x)\, dx = \begin{cases} 0 & \text{si } x < \alpha \\ P\varphi(\alpha) & \text{si } x > \alpha \end{cases}$$

Ceci posé, $\mathcal{F}_i$ désignant l'expression $\mathcal{F}$ ainsi définie relative à la i^e travée on peut transformer l'équation (11) de façon à la mettre sous la forme :

$$Q = \frac{\sum_i \frac{8 f_i}{l_i^2} \mathcal{F}_i - \tau(Q' + Q)\mathcal{L}_t}{\sum_i \frac{8 f_i}{l_i^2}\left[\frac{2}{3} f_i l_i - \frac{8 f_i}{l_i K_i^2} + \frac{2}{K_i^3}\left(\frac{p_i'}{Q'}\right)\frac{e^{K_i l_i} - 1}{e^{K_i l_i} + 1}\right] + \frac{Q' + Q}{E'S} L_s} \tag{15}$$

Telle est l'équation que nous avions en vue; elle s'écrit directement sans passer par l'intermédiaire des constantes (A_i, B_i). En outre, elle a l'avantage sur l'équation (11) de conduire à des approximations successives beaucoup plus rapidement convergentes; il suffit en pratique de faire deux ou au plus trois approximations successives, de sorte que la résolution de l'équation (13) ne demande guère plus d'une demi-heure pour un projeteur entraîné disposant d'une table de la fonction e^x et d'une machine à calculer.

On observera que le dénominateur de l'équation (15) ne dépend des surcharges que par l'intermédiaire des K_i, c'est-à-dire de Q; ce dénominateur varie peu d'un cas de surcharges à un autre. Cette remarque permet de trouver la disposition des surcharges conduisant à un effet maximum (par exemple, moment fléchissant maximum en un point de la poutre de rigidité) au moyen de la considération de pseudo-lignes d'influences; en effet, on connaît toujours une valeur approchée de Q correspondant au cas de surcharges et de température envisagé; en substituant cette valeur dans le second membre de l'équation (15), on peut définir une pseudo-ligne d'influence de la poussée dont l'équation est, la charge unité se trouvant dans la i^e travé à l'abscisse α :

$$Q = \frac{\frac{8 f_i}{l_i^2}\cdot \mathcal{F}_i'(\alpha)}{\sum_i \frac{8 f_i}{l_i^2}\left[\frac{2}{3} f_i l_i - \frac{8 f_i}{l_i K_i^2} + \frac{2}{K_i^3}\left(\frac{p'_i}{Q'}\right)\frac{e^{K_i l_i} - 1}{e^{K_i l_i} + 1}\right] + \frac{Q' + Q}{E'S} L_s}$$

Il faut noter que le dénominateur est alors une constante, ainsi que la valeur de K_i qui figure dans $\mathcal{F}_i'(\alpha)$. On peut alors à partir de la définition de la pseudo-ligne d'influence de Q définir la pseudo-ligne d'influence de l'effet considéré (flèche, moment fléchissant, effort tranchant en un point, de la poutre de rigidité). On connaîtra ainsi les zones d'ordonnées positives ou négatives de ces lignes et par suite la disposition des surcharges la plus défavorable. La disposition des surcharges étant ainsi déterminée, on calculera Q au moyen de l'équation (15), et l'on pourra si la valeur de Q qui a servi à la détermination des pseudo-lignes d'influence, n'était pas suffisamment approchée, recommencer le tracé de ces lignes.

L'emploi des pseudo-lignes d'influence permet donc de donner une solution théorique au problème de la recherche du maximum d'un effet. On peut encore, dans des cas simples où les surcharges ne dépendent que de un ou de deux paramètres, faire le calcul de l'effet pour quelques valeurs de ces paramètres et déterminer le maximum de l'effet au moyen des formules d'interpolation.

Nous noterons encore la formule suivante qui permet de déterminer le déplacement des câbles sur les pylônes :

$$u_i - u_{i-1} = -\frac{8 f_i}{l_i^2}\left[\frac{\mathcal{F}_i}{Q'+Q} - \frac{Q}{Q'+Q}\left(\frac{2}{3} f_i\, l_i - \frac{8 f_i}{l_i\, K_i^2} + \frac{2}{K_i^3}\cdot\frac{p_i'}{Q}\cdot\frac{e^{K_i l_i}-1}{e^{K_i l_i}+1}\right)\right] + \frac{Q}{E'S} L_{si} + \tau L_{ti} \quad (16)$$

§ 5. Donnons le principe d'une méthode permettant de calculer un couple de constante (A, B) relatif à un intervalle de surcharges donné, sans avoir à calculer tous les couples. Reprenons l'équation différentielle (3), et supposons que le cas de surcharges envisagé résulte de la superposition de n cas de surcharges produisant dans la poutre sur appuis simples de même portée l que la travée considérée des moments fléchissants μ_1, μ_2 ... μ_n. On a évidemment :

$$\mu = \mu_1 + \mu_2 + \ldots + \mu_n$$

L'intégrale de l'équation (3) que nous cherchons est craactérisée par la propriété que v est nul pour $x = 0$ et $x = l$, et que v et $\frac{dv}{dx}$ sont continus dans l'intervalle $(0,l)$.

Considérons alors les équations différentielles

$$\begin{aligned} \frac{d^2 v_0}{dx^2} - K^2\, v_0 &= -\frac{K^2\, Q}{Q'+Q}\, y \\ \frac{d^2 v_j}{dx^2} - K^2\, v_j &= -\frac{K^2}{Q'+Q}\, \mu_j \quad (j = 1,2\ , \ldots\, n) \end{aligned} \quad (17)$$

Et soient v_0 et v_j les intégrales de ces équations jouissant des mêmes propriétés que v, c'est-à-dire s'annulant pour $x = 0$ et $x = l$, et continues ainsi que leurs dérivées premières dans l'intervalle $(0,l)$. Il est évident que l'intégrale cherchée v n'est autre que

$$v = v_0 + v_1 + v_2 + \ldots\ldots + v_n$$

On aboutit ainsi à un principe de superposition permettant de déduire immédiatement tout couple de constantes pour un intervalle quelconque d'un cas de surcharges tel que celui défini au paragraphe 1, de l'étude des deux cas de surcharges simples suivants :

1° Charge concentrée unique à l'abscisse α;

2° Charge répartie d'intensité constante entre les abscisses β et γ.

§ 6. La détermination des efforts dans les poutres de rigidité est ainsi résolue pour les cas de surcharges définis au paragraphe 1. La méthode s'applique encore dans le cas d'une fonction de charge $p(x)$ quelconque. La détermination de la poussée n'est pas changée si l'on définit, dans l'équation (15), les $\mathcal{F}_i$ par la formule (14'). Par contre, dans l'intégration de l'équation (3), A et B deviennent des fonctions de x que l'on peut calculer au moyen de la méthode classique de variation des constantes.

On peut enfin, comme l'a montré M. Blaise, ingénieur des Ponts et Chaussées, dans les *Annales des Ponts et Chaussées* de mars-avril 1946, aménager la méthode exposée ci-dessus de façon à pouvoir l'appliquer à des poutres de rigidité d'inertie variable; il faut alors renoncer à l'intégration formelle de l'équation (3), et employer les méthodes de calcul aux différences finies, les points de division correspondant aux suspentes; les calculs deviennent longs, mais très systématiques.

§ 7. On peut examiner les hypothèses qui ont servi de base à cette théorie. En particulier, l'équation (7) n'est qu'approchée; pour se rendre compte de l'erreur commise, il suffit de refaire le calcul au moyen de la méthode indiquée au début du paragraphe 2. L'expérience montre alors que l'erreur relative sur la poussée est de l'ordre de 10^{-3}, et l'erreur relative sur les moments fléchissants de $2{,}5 \cdot 10^{-3}$. L'emploi de l'équation (7) est donc parfaitement justifié.

L'allongement des suspentes sous l'action des surcharges étant parfaitement négligeable, il subsiste le point suivant : au cours de la déformation, les points d'attache des suspentes sur le câble subissent outre un déplacement vertical, un déplacement horizontal qui a été négligé; il en résulte que les suspentes deviennent obliques, et par suite exercent sur le câble des forces horizontales de rappel qui tendent à diminuer le déplacement vertical; en conséquence, on trouvera dans les poutres de rigidité des efforts un peu plus grands que les efforts réels. Cet effet est du reste faible, si les suspentes courtes ont une longueur suffisante. Nous ne croyons pas du reste qu'aucune théorie des ponts suspendus en ait jamais tenu compte.

Le fait de supposer le moment d'inertie des poutres de rigidité constant introduit également des erreurs lorsque les poutres sont de section variable (le moment d'inertie varie toujours peu le long de la poutre). Aux points où le moment d'inertie est maximum, le moment fléchissant est plus élevé que le moment fléchissant calculé, comme on peut s'en rendre compte, soit en faisant le calcul aux différences finies, soit en divisant la poutre en un certain nombre d'intervalles dans chacun desquels le moment d'inertie peut être considéré comme constant. Dans cette dernière hypothèse, l'équation (3) est valable dans chacun des intervalles, K variant seulement, d'un intervalle à l'autre.

Nous avons constaté sur plusieurs exemples que les erreurs dues à l'hypothèse de la verticalité des suspentes et à l'hypothèse du moment d'inertie constant se compensaient sensiblement.

§ 8. Nous avons eu l'occasion d'appliquer la méthode précédente dans de nombreux projets. Permettant d'effectuer sans difficulté le calcul d'un pont à plusieurs travées, cette méthode a augmenté la tendance consistant à substituer toutes les fois qu'on le peut, au pont suspendu à travée unique, le pont suspendu à trois travées symétriques, à câbles porteurs continus, sans câbles de tête. On obtient ainsi des ouvrages dont l'aspect est très satisfaisant et qui sont très économiques. En effet, les câbles sont beaucoup moins importants car ils sont moins longs et leur tension est beaucoup plus faible; les poutres de rigidité sont plus légères; la dépense supplémentaire due aux piles est en général compensée par la diminution des massifs d'ancrage.

L'expérience montre que la portée de la travée centrale doit être environ les 6/10 de l'ouverture totale; on obtient ainsi des efforts du même ordre dans les poutres de rigidité de la travée centrale des travées de rive.

La méthode exposée permet également la détermination des déplacements horizontaux des câbles en tête des pylônes; ces déplacements sont en général assez faibles pour qu'on puisse supprimer les appareils de dilatation à rouleaux lourds et coûteux placés sous les selles d'appui des câbles; les pylônes encastrés à leur base, subissent alors des flexions du fait du déplacement de leur sommet imposé par les câbles; mais on peut montrer que la flexibilité des pylônes est accrue par la forte charge verticale qu'ils supportent, et que la réaction horizontale qu'ils exercent sur les câbles est du même ordre de grandeur que celle due au frottement de roulement dans le cas d'appareils à rouleaux de dilatation bien entretenus; il est donc logique de négliger cette réaction dans le calcul de l'ouvrage. Il arrive souvent que les déplacements des pylônes vers le milieu du pont sont un peu plus grands que les déplacements vers la rive; on peut alors compenser les efforts dans les pylônes en imposant au moment du réglage du pont, c'est-à-dire en l'absence de surcharges et à température moyenne, un léger déplacement du sommet du pylône vers la rive.

Dans les petits ouvrages, des pylônes flexibles paraîtraient trop grêles; il est alors préférable d'articuler les pylônes à leur base, plutôt que de prévoir des appareils à rouleaux.

Les pylônes en béton armé sont plus économiques que les pylônes métalliques. L'articulation de base, lorsqu'elle est prévue, est du type préconisé par M. Freyssinet; on obtient ainsi une articulation très économique qui, en raison des faibles rotations qui lui sont imposées, peut supporter des réactions élevées par unité de longueur. Dans le cas de pylônes flexibles, on arrive aisément à éliminer tout effort de traction dans le béton.

Les poutres de rigidité des petits ouvrages (120 m de portée environ) sont en général à âme pleine, pour les ouvrages moyens (200 m de portée environ), nous préférons employer une poutre de rigidité à triangulation Warren double à membrures à caissons, dont les nœuds de charge sont les points de croisement des diagonales; le tablier est placé à mi-hauteur des poutres, les pièces de pont étant assemblées sur les poutres par l'intermédiaire des montants reliant les nœuds centraux à la membrure inférieure; les suspentes, situées dans l'axe des poutres, sont directement attachées sur le nœud central; il en résulte des avantages pour le montage, les pièces de pont étant d'abord suspendues avant l'exécution des poutres.

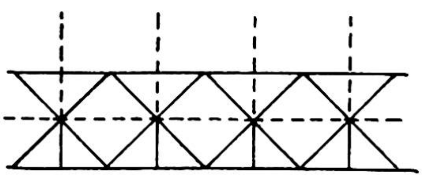

Fig. 3.

La dalle en béton armé du tablier, qui constitue l'âme de la poutre au vent, est rendue solidaire de longerons métalliques latéraux assemblés sur les nœuds centraux; les membrures des poutres de rigidité peuvent ainsi jouer le rôle de membrures de la poutre au vent, sans que les efforts de compression ou de traction qui leur sont transmis par la dalle par l'intermédiaire des nœuds centraux n'entraînent d'efforts secondaires de flexion dans la triangulation.

§ 9. Il est possible d'appliquer la théorie précédente au calcul des ponts à câble de tête, comme l'a montré M. Avenas, ingénieur des Ponts et Chaussées (*Annales des Ponts et Chaussées*, 1947). Nous désignerons par Q_i' la composante horizontale de la traction du câble porteur, et par q_i' la composante horizontale de la traction du câble de tête dans la i^e travée, en l'absence de surcharges et à la température de réglage; l'équilibre dans ces conditions exige que $Q_i' + q_i'$ soit constant.

Lorsque le pont est surchargé, ces composantes horizontales deviennent respectivement $Q_i' + Q_i$ et $q_i' + q_i$, et comme l'on néglige les réactions horizontales exercées par les pylônes sur les câbles, la somme $Q = Q_i + q_i$ ne varie pas d'une travée à l'autre.

L'expression $u_i - u_{i-1}$ est alors donnée, lorsque l'on considère le câble porteur associé à la poutre de rigidité par la relation (16), et lorsque l'on considère le câble de tête par la relation suivante :

$$u_i - u_{i-1} = \frac{q_i}{E'S_i'} L_{si}' + \tau L_{ti}' + R_i \ . \tag{17}$$

dans laquelle S_i' désigne la section du câble de tête, L_{si} et L_{ti} les intégrales (8) pour le câble de tête et R_i un terme correctif petit égal à :

$$R_i = \frac{P_i^2 \, l_i \cos^2 \theta_i}{2\,H} \left[\frac{1}{q'^2_i} - \frac{1}{(q_i' + q_i)^2} \right]$$

θ_i étant l'angle avec l'horizontale de la corde joignant les extrémités du câble de tête de la i^e travée et P_i *le* poids total du câble de tête dans cette travée.

Supposons connues des valeurs de départ approchées des Q_i et des q_i, on pourra calculer des valeurs plus approchées de la façon suivante : en donnant à Q_i et q_i ces valeurs de départ dans les formules (16) et (17), on aura :

$$u_i - u_{i-1} = C_i\,Q_i + D_i = c_i\,q_i + d_i$$

C_i, D_i, c_i, d_i ayant des valeurs numériques connues. En tenant compte de ce que $q_i + Q_i = Q$, on en déduira :

$$Q_i = \alpha_i\,Q + \beta_i \qquad\qquad q_i = \alpha_i'\,Q + \beta_i' \tag{18}$$

$$u_i - u_{i-1} = \gamma_i\,Q + \delta_i \tag{19}$$

La condition $\Sigma(u_i - u_{i-1}) = 0$ fournira alors la valeur de Q.

$$Q = -\frac{\Sigma\delta_i}{\Sigma\gamma_i} \ .$$

Les formules (18) et (19) feront alors connaître des valeurs plus approchées pour Q_i, q_i et $u_i - u_{i-1}$.

Deux ou trois approximations au plus suffisent si l'on choisit de la façon suivante les valeurs de départ des Q_i et q_i : on prend $q_i = 0$, ce qui revient à négliger le terme R_i de la formule (17) dans la première approximation; et l'on détermine les Q_i de départ en faisant $u_i = 0$; en effet, les Q_i exacts différeront assez peu de ces valeurs, car une légère variation de la distance des appuis des câbles porteurs d'une travée ne donne qu'une faible variation de la poussée des câbles porteurs.

§ 10. L'équation des déplacements orthogonaux du paragraphe 2 peut également se mettre sous la forme suivante :

$$u_i - u_{i-1} = \int_0^l v_i \frac{d^2 y_i}{dx^2} dx + \frac{Q}{E'S'} L_{s_i} + \tau L_{t_i}, \qquad (20)$$

qui se prête bien au calcul des ponts suspendus auto-ancrés, dans lesquels le câble est fixé à ses extrémités sur la poutre de rigidité.

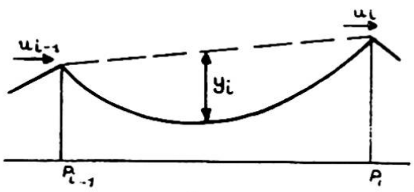

Fig. 4.

Considérons donc un pont suspendu auto-ancré à n travées articulées en leurs extrémités, et dont la poutre de rigidité rectiligne a pour module d'élasticité E, pour section S et pour longueur totale L. La figure d'équilibre du câble dans la i^e travée, sans surcharges et à la température de réglage est, comme dans un pont suspendu ordinaire, une parabole lorsque la charge permanente p_i' est d'intensité constante; s'il n'en était pas ainsi, y_i serait un funiculaire de p_i', ce qui se traduit par l'équation

$$Q' y_i = \mu_i'$$

μ_i' désignant le moment fléchissant que produirait la charge p_i' dans la poutre sur appuis simples de même portée que la poutre de rigidité.

Lorsque l'ouvrage est surchargé ou que la température est différente de la température de réglage, y_i devient $y_i + v_i$, la poussée augmente de Q, et l'on se rend compte immédiatement que le moment fléchissant M_i dans la poutre de rigidité, compte tenu du moment $(Q' + Q)\, v_i$ dû à la compression de la poutre a l'expression simple

$$M_i = \mu_i - Q y_i. \qquad (21)$$

Il suffit alors pour déterminer Q d'écrire que le déplacement relatif des extrémités du câble est égal au raccourcissement total de la poutre. Compte tenu de ce que

$$\frac{d^2 v_i}{dx^2} = - \frac{M_i}{EI_i}$$

on aboutit à l'équation

$$Q \left[\sum_i \int_0^{l_i} \frac{y_i^2\, dx}{EI_i} + \left(\frac{L}{ES} + \frac{\mathcal{L}_s}{E'S'} \right) \right] = \sum_i \int_0^{l_i} \frac{\mu_i\, y_i\, dx}{EI_i} + \tau (L - \mathcal{L}_t). \qquad (22)$$

Il en résulte que le calcul des ponts suspendus auto-ancrés est beaucoup plus simple que celui des ponts suspendus ordinaires; en particulier on peut tracer des lignes d'influence pour Q, et pour les moments fléchissants dans la poutre de rigidité.

Si la poutre de rigidité n'est pas rectiligne, il suffit dans la théorie précédente de remplacer y_i par $Y_i = y_i + y_i'$, y_i' désignant l'ordonnée de la fibre moyenne de la poutre dans la i^e travée comptée positivement vers le haut à partir de la corde joignant ses extrémités dans cette travée.

Il serait possible de réaliser économiquement des ponts auto-ancrés à trois travées symétriques, en prévoyant des poutres en béton armé associées au hourdis. La portée de la travée centrale pourrait atteindre une centaine de mètres. L'inconvénient de tels ouvrages est le montage qui exige des échafaudages.

§ 11. La méthode exposée dans ces quelques pages n'apporte rien de nouveau en ce qui concerne les résultats numériques, par rapport aux méthodes employées lors de la construction des grands ponts suspendus américains, tels que le pont de Philadelphie, puisque l'équation de la poussée, sous sa première forme (11) est la même, bien qu'obtenue par une autre voie. Par contre, la nouvelle forme de l'équation de la poussée, dont la résolution est très rapide, permet d'abréger d'une façon considérable le temps passé dans les calculs numériques.

Cette équation présente également l'intérêt théorique de montrer dans quelle mesure le principe de superposition est valable pour les ponts suspendus. On peut dire que pour un pont suspendu considéré dans un état initial donné de température et de surcharges, il existe des lignes d'influence sous l'action d'une charge concentrée infiniment petite se déplaçant sur l'ouvrage; mais ces lignes d'influence varient (du reste assez peu) avec l'état initial.

Résumé

La méthode de calcul des ponts suspendus est basée, d'une part sur l'expression des efforts dans la poutre de rigidité en fonction de la composante horizontale (ou poussée) de la traction des câbles, d'autre part, sur une relation approchée donnant en fonction du déplacement vertical des points d'un câble les déplacements horizontaux des extrémités du câble. On retrouve ainsi, pour déterminer la variation de poussée due aux surcharges et aux variations de température l'équation utilisée par les ingénieurs américains. Mais on a transformé cette équation, dont l'utilisation entraînait de longs calculs, de manière à lui donner une nouvelle forme conduisant à des calculs très rapides, et mettant en évidence les effets de superposition des charges.

La méthode indiquée permet un calcul aisé des ponts à trois travées symétriques qui sont des ouvrages économiques et d'aspect agréable; elle a ainsi réagi sur la conception même des ouvrages, notamment en ce qui concerne les pylônes : suppression des appareils de dilatation au sommet des pylônes qui sont soit encastrés, soit articulés à leur base. On signale également un type de poutre de rigidité bien adapté aux efforts qu'elles ont à supporter du fait des surcharges et du vent.

La note se termine par l'extension de la méthode aux ponts suspendus à travées multiples et à câbles de tête, et aux ponts suspendus auto-ancrés.

Zusammenfassung

Die Berechnungsweise der Hängebrücken stützt sich einerseits auf die Ermittlung der Beanspruchung des Versteifungsträgers durch die Horizontalkomponente (oder Druckkomponente) des Seilzuges, andererseits auf eine Näherungsbeziehung, die die Horizontalverschiebungen der Kabel-

enden in Funktion der vertikalen Verschiebung des Kabels ergibt. Man kommt so auf die durch die amerikanischen Ingenieure angewandte Gleichung zur Berechnung der den Auflasten und Temperaturunterschieden entsprechenden Aenderungen des Druckes. Diese Gleichung, die in ihrer Anwendung lange Berechnungen nötig macht, wurde nun in eine neue Form gebracht, in der sie sich sehr schnell lösen lässt und die die Ueberlagerungswirkungen der Lasten augenscheinlich macht.

Das beschriebene Verfahren gestattet die einfache Berechnung der wirtschaftlich und ästhetisch günstigen Brückenbauten mit drei symmetrisch angeordneten Oeffnungen und war auch von Einfluss auf den Entwurf solcher Bauwerke, besonders hinsichtlich der Pfeilerausbildung. Es führte zum Wegfall der Dilatationseinrichtungen auf der Spitze der entweder eingespannten oder gelenkig gelagerten Pfeiler. Es wird auch eine Bauart des Versteifungsträgers angegeben, die die Beanspruchungen infolge Auflasten und Wind gut aufnimmt.

Der Aufsatz schliesst mit der Erweiterung des Verfahrens auf Hängebrücken mit vielen Oeffnungen und solche mit Schrägkabeln, und auf Hängebrücken mit aufgehobenem Horizontalschub.

Summary

The method of calculating suspension bridges is based, on the one hand, on research as to the stress of the stiffening girder by the horizontal components (or pressure components) of the traction of the cables and, on the other hand, on an approximate relation which gives the horizontal displacements of the cable ends in function of the vertical displacement of the cable. We thus find the equation used by American engineers for calculating the differences in pressure corresponding to differences in load and temperature. This equation, for the use of which long calculations are required, has now been presented in another form in which it is quickly solved and which places clearly on record the effects of superposing of loads.

The process described allows of simple calculation of economically and aesthetically suitable bridges comprising three symmetrically arranged spans and also had an influence on the designing of such structures, particularly as regards placing of the piers. Information is also given concerning a type of stiffening-girder well adapted to cope with load and wind stresses.

The paper ends with an extension of the process to suspension bridges with multiple spans and oblique cables and to self-anchored suspension bridges.

IIIb3

Calcul des ponts suspendus à grande portée

Berechnung der weitgespannten Hängebrücke

Analysis of the long span suspension bridge

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In deriving the governing equation of the stiffened suspension bridge, the following simplifying assumptions are generally made. That the hanger pull is constant along the cable, and the hangers are inextensible, that the cable displacements in each span are purely vertical, and the cable slides over rigid towers, and that the stiffening truss is of uniform rigidity.

Following the lead of Timoshenko and Priester, in recent years the exponential form of the equation has been replaced by a solution in the form of a trigonometric series, which takes account of variable hanger pull. By applying Southwell's Relaxation technique to the series solution, the author presents a method of analysis which does not embody the remaining assumptions. In the following pages the method is developed, and applied to the calculations for a bridge of 3 300 ft span. It is then shewn that for a structure of this magnitude the usual simplifications are legitimate, but that the effects of non-uniform stiffening truss rigidity, and of hanger extension are not negligible.

Derivation of the increment in the horizontal cable tension

The increment H_L in the horizontal component of the cable tension resulting from application of live load, and temperature change, may be derived from considerations of energy, or from the kinematics of the distorted cable. The energy solution assumes that the cable displacements are purely vertical, or if horizontal movements take place, that they are of no significance. It is not intuitively apparent that this is so, and by a kinematic approach the effect of the horizontal movements of the cable can be investigated. One result of these horizontal displacements and the consequential inclination of the hangers and translation of the tower tops, is

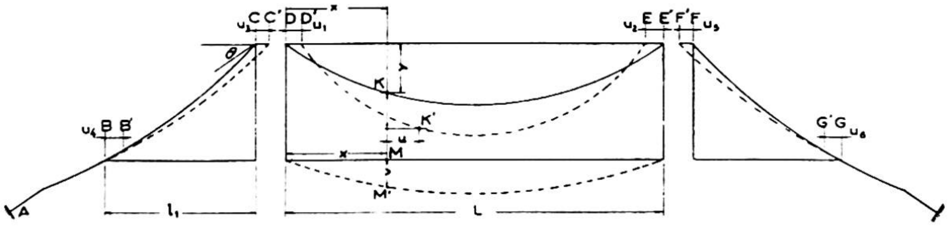

Fig. 1.

that the horizontal component of the cable tension is not constant across the bridge. The unbalance between centre and side spans arising from the stiffness of the towers may be corrected for without difficulty. On page 448 (post) the effect of hanger inclination on the moments in the truss is assessed, and it is found to be small. Neglect is generally on the side of safety. In what follows, the horizontal component of the cable tension is assumed to be constant across the bridge.

In fig. 1, which represents a bridge over three spans BC, DE and FG, as a result of temperature change and the application of live load, point K on the cable, distant x, y, from D, moves to K′, at $(x+u)$, $(y+v)$. At the towers there will be horizontal displacements U_1 and U_2, at each end of the centre span, and corresponding displacements U_3 and U_5 in the side spans. There will also, in general, be movements indicated by BB′ and GG′, due to the extension of the cable and the anchorage steelwork beyond the side spans.

An element ds of the cable at K will undergo a change in length of

$$(dx^2+dy^2+dv^2+2\,du\cdot dx+2\,dv\cdot dy)^{\frac{1}{2}}-(dx^2+dy^2)^{\frac{1}{2}},$$

or to the first order of small quantities,

$$dx\,(1+y'^2)^{\frac{1}{2}}\left(1+\frac{u'+y'v'+\frac{v'^2}{2}}{1+y'^2}\right)-dx\,(1+y'^2)^{\frac{1}{2}}$$

$$=dx\left(\frac{u'+y'v'+\frac{v'^2}{2}}{s'}\right)=\Delta ds\ (\text{say}),$$

where « dashes » denote differentiation with respect to x, and $s'=(1+y'^2)^{\frac{1}{2}}$.

Δds can also be expressed in terms of the elastic extension of the cable, and the effect of temperature change as

$$\Delta ds=\frac{H+H_L}{EA}\cdot s'^2\left(1+\frac{2\,u'+2\,y'v'+v^{2\prime}}{s'^2}\right)dx-\frac{H}{EA}\cdot s^{2\prime}\,dx$$
$$+\,\omega t\cdot s'\left(1+\frac{u'+y'v}{s}\right)dx$$

where H denotes dead-load horizontal component of cable tension ;
H_L denotes live-load horizontal component of cable tension ;
ω denotes coefficient of expansion of the cable ;
E denotes modulus of elasticity of the cable ;
A denotes cross-sectional area of the cable.

Now H/EA, the dead load extension of the cable is small, H_L/EA is in general much smaller, so noting thats s' will not exceed about 1.1, and neglecting very small quantities,

$$\frac{H_L}{EA}\cdot s'^2\,dx + \omega t\cdot s'\,dx = \left(\frac{u' + y'v' + \frac{v'^2}{2}}{s'}\right)\left(1 - \frac{2\,H}{EA}\right)dx$$

and
$$\frac{H_L}{EA}\cdot s'^3\,dx + \omega t\cdot s'^2 dx = \left(\frac{u' + y'v' + \frac{v'^2}{2}}{s'^2}\right)\left(1 - \frac{2\,H}{EA}\right)dx\,.$$

Integrating between 0 and L,

$$\left(1 - \frac{2\,H}{EA}\right)(U_2 - U_1) = \int_0^L \left\{\frac{H_L}{EA}\cdot s'^3 + \omega t\cdot s'^2\right\}dx + \left(1 - \frac{2\,H}{EA}\right)$$
$$\int_0^L \left(vy'' - \frac{v'^2}{2}\right)dx \qquad (1)$$

Integrating between 0 and x, and neglecting small quantities,

$$u = U_1 + \int_0^x \left\{\frac{H_L}{EA}\cdot s'^3 + \omega t\cdot s'^2\right\}dx - \int_0^x y'v'dx \qquad (2)$$

For the side spans, writing w, and z, in place of v, the centre span deflection, the corresponding equations are

$$\left(1 - \frac{2\,H}{EA}\right)(U_4 - U_3) = \int_0^{l_1}\left\{\frac{H_L}{EA}\cdot s'^3 + \omega t\cdot s'^2\right\}dx + \left(1 - \frac{2\,H}{EA}\right)$$
$$\int_0^{l_1}\left(wy'' - \frac{w'^2}{2}\right)dx \qquad (3)$$

$$u = U_3 + \int_0^x \left\{\frac{H_L}{EA}\cdot s'^3 + \omega t\cdot s'^2\right\}dx - \int_0^x y'w'dx \qquad (4)$$

and similar equations in U_5, U_6, and z, for the second side span.

Then expressing the extension of the cable and anchorage steelwork between A and B as $\frac{H_L}{EA}\cdot l_2 + \omega t l_3$, and the extension between C and D over the tower as $\frac{H_L}{EA}\cdot l_4 + \omega t l_5$, equation (3) can be written

$$\left(1 - \frac{2\,H}{EA}\right)U_1 = \frac{H_L}{EA}\left\{\int_0^{l_1} s'^2\,dx + l_2 + l_4\right\} + \omega t\left\{\int_0^{l_1} s'^2\,dx + l_3 + l_5\right\}$$
$$+ \left(1 - \frac{2\,H}{EA}\right)\int_0^{l_1}\left(\omega y'' - \frac{\omega'^2}{2}\right)dx\,. \qquad (5)$$

Eliminating U_1 and U_2 from equations (1) and (5)

$$\frac{H_L}{EA}\left[\int_0^L s'^3\,dx + 2\left\{\int_0^{l_1} s'^3\,dx + l_2 + l_4\right\}\right]$$
$$+ \omega t\left[\int_0^L s^{2\prime}\,dx + 2\left\{\int_0^{l_1} s^{2\prime}\,dx + l_3 + l_5\right\}\right] =$$
$$= \left(1 - \frac{2\,H}{EA}\right)\left[\int_0^L \left(\frac{v'^2}{2} - vy''\right)dx\right.$$
$$\left. + \int_0^{l_1}\left(\frac{\omega'^2}{2} - \omega y'' + \frac{z'^2}{2} - zy''\right)dx\right] \qquad (6)$$

It is apparent from (6) that the value of H_L, although derived from consideration of the « u » displacements, is independent of them.

The shape of the erected cable is intermediate between catenary and parabola, but closer to the latter. Treatment is simpler with the parabolic form.

Then for the centre span,

$$y = \frac{4\,fx}{L} - \frac{4\,fx^2}{L^2} = \frac{m}{2}\left(x - \frac{x^2}{L}\right) \quad \text{(say)},$$

For the side spans,

$$y = \frac{4\,f_1 x}{l_1} - \frac{4\,f_1 x^2}{l_1^{\,2}} + x \tan\theta = \frac{m_1}{2}\left(x - \frac{x^2}{l_1}\right) + x\tan\theta \quad \text{(say)}$$

Expressing the deflections of centre and side spans by the sine series,

$$\left.\begin{aligned} v &= \Sigma_n V_n \sin n\pi \frac{x}{L} \qquad n = 1,\ 2,\ 3 \\ w &= \Sigma_n W_n \sin n\pi \frac{x}{l_1} \\ z &= \Sigma_n Z_n \sin n\pi \frac{x}{l_1} \end{aligned}\right\} \tag{7}$$

and substituting in (6),

$$\frac{H_L}{EA} \cdot L_s + \omega t \cdot L_t = \left(1 - \frac{2\,H}{EA}\right) \Sigma_n \left[K_n V_n + k_n\,(W_n + Z_n) + \frac{n^2\,V_n^{\,2}}{4\,L} + n^2\,\frac{(W_n^{\,2} + Z_n^{\,2})}{4\,l_1}\right] \tag{8}$$

where

$$L_s = \int_0^L s'^3 dx + 2\left\{\int_0^{l_1} s'^3 dx + l_2 + l_4\right\}$$

$$L_t = \int_0^L s'^2 dx + 2\left\{\int_0^{l_1} s'^2 dx + l_3 + l_5\right\}$$

$$K_n = m\,\frac{(1 - \cos n\pi)}{n\pi} \quad \text{and} \quad k_n = m_1\,\frac{(1 - \cos n\pi)}{n\pi}\,.$$

The term $2\,H/EA$ is for long span bridges of the order of $1/200$. It will however be found that omission of this term from equation (8) changes the value of H_L by very much less than 0.5 %.

Effect of « u » displacements on the partition of load between stiffening-truss and cable

Referring to fig. 1, the load taken by the cable on a length dx at K is originally

$$-H \cdot y''\,.$$

Point K on the cable now moves to K′, and point M on the stiffening-truss to M′, the load taken by the cable becoming

$$-(H+H_L)(y''+v''), \text{ on a length } dx\left(1+\frac{du}{dx}\right).$$

The change in the cable load is to the first order of small quantities,

$$-H_L\cdot y''-(H+H_L)(v''+u'y'')=w_c \qquad \text{(say)},$$

If M_c is the change in the bending moment sustained by the cable, by integrating w_c twice it is found that

$$M_C=H_L\cdot y+(H+H_L)\left(v+y''\int_0^x udx-\frac{xy''}{L}\int_0^L udx\right. \tag{9}$$

since for the parabola, y'' is constant, and $y'''=0$.

The moment sustained by the stiffening-truss, of flexural rigidity B is

$$-B\cdot v''=M_G \qquad \text{(say)}. \tag{10}$$

Then if M_L denote the bending moment from the applied live load and temperature change,

$$M_L=M_C+M_G=H_L\cdot y-B\cdot v''+(H+H_L)v$$
$$+(H+H_L)y''\left[\int_0^x udx-\frac{x}{L}\int_0^L udx\right] \tag{11}$$

Substituting for u and v from equations (2) and (7),

$$M_C=H_L\cdot y+(H+H_L)\Sigma_n V_n\sin\frac{n\pi x}{L}+(H+H_L)\left(\frac{H_L}{EA}+\omega t\right)y$$

$$+(H+H_L)m^2\frac{\Sigma_n V_n}{n\pi}\left[\frac{1-\cos\frac{n\pi x}{L}}{2}\right.$$

$$\left.+\frac{x}{L}\cos\frac{n\pi x}{L}-\frac{2\sin\frac{n\pi x}{L}}{n\pi}-\frac{x}{L}\left(\frac{1+\cos n\pi}{2}\right)\right] \tag{12}$$

Equation (12) can be expressed as the Fourier series,

$$M_C=\Sigma_\lambda(M_C)_\lambda\sin\frac{\lambda\pi x}{L}$$

where

$$(M_C)_\lambda=\frac{2}{L}\int_0^L M_C\sin\left(\frac{\lambda\pi x}{L}\right)dx\,.$$

Then $H_L\cdot y$ gives

$$\frac{2H_L}{L}\int_0^L\left(\frac{4fx}{L}-\frac{4fx^2}{L^2}\right)\sin\left(\frac{\lambda\pi x}{L}\right)dx=\frac{16f}{\lambda^3\pi^3}(1-\cos\lambda\pi)H_L$$
$$=H_L(G)_\lambda \text{ (say)} \tag{13}$$

$$(H+H_L)\Sigma_n V_n\sin\frac{n\pi x}{L}, \quad \text{gives}$$

$$\frac{2\,(H+H_L)}{L}\int_0^L \sin\left(\frac{\lambda\pi x}{L}\right)\Sigma_n V_n \sin\left(\frac{n\pi x}{L}\right)dx = (H+H_L)\,V_\lambda, \quad \text{when } \lambda = n.$$

When $\lambda \neq n$, the integral is zero (14)

$$(H+H_L)\left(\frac{H_L}{EA}+\omega t\right)y \quad \text{gives} \quad H\left(\frac{H_L}{EA}+\omega t\right)(G)_\lambda, \tag{15}$$

as $H_L \cdot y\left(\frac{H_L}{EA}+\omega t\right)$ can be neglected in comparison with $H_L \cdot y$. The last line of equation (12) gives

$$(H+H_L)\,\Sigma_n A_{\lambda,n} V_n$$

where, when

$$\lambda = n, \quad \Sigma_n A_{\lambda,n} V_n = -\frac{m^2 V_\lambda}{2\lambda^2\pi^2}.$$

and, when $\lambda \neq n$,

$$\Sigma_n A_{\lambda,n} V_n = \Sigma_n V_n \frac{m^2 n\,\{1+\cos(\lambda-n)\pi\}}{\lambda\pi^2\,(n^2-\lambda^2)} \tag{16}$$

Expressing M_G also in its harmonic components,

$$(M_G)_\lambda = \frac{2}{L}\,\Sigma_n V_n \frac{n^2\pi^2}{L^2}\int_0^L B \sin\left(\frac{\lambda\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx.$$

When the flexural rigidity B of the stiffening truss is uniform throughout its length,

$$(M_G)_\lambda = \frac{\lambda^2\pi^2 B}{L^2}\,V_\lambda. \tag{17}$$

When B is not uniform, the variation of B along the span can always be expressed as

$$B = B_0 + \Sigma_m B_m \sin\frac{m\pi x}{L} \tag{18}$$

where in any practical design, at the most three odd values of B_m will represent the variation of B along the span.

Then $(M_G)_\lambda = \Sigma_n R_{\lambda,n} V_n$, where,

$$\Sigma_n R_{\lambda,n} V_n = \frac{\lambda^2\pi^2 B_0}{L^2}\,V_\lambda - \Sigma_{m,n} V_n \cdot \frac{4\pi m n^3 \lambda\,\{1-\cos(\lambda-n+m)\pi\}\,B_m}{L^2\,\{(\lambda^2-n^2)^2 - 2m^2(\lambda^2+n^2)+m^4\}} \tag{19}$$

$(\lambda - n)$ is either even or zero.

Now since $M_L = M_C + M_G$ the Fourier series for M_L, and $M_C + M_G$, must be equal term by term, and

$$(M_L)_\lambda - H_L\left(1+\frac{H}{EA}\right)(G)_\lambda - H\omega t\,(G)_\lambda$$

$$= (H+H_L)(\Sigma_n A_{\lambda,n} V_n + V_\lambda) + \Sigma_n R_{\lambda,n} V_n \tag{20}$$

For the side spans, corresponding to equation (20),

$$(M_l)_\lambda - \left\{ H_L \left(1 + \frac{H}{EA} \sec^3 \theta \right) + H\omega t \sec^2 \theta \right\} (g)\,\lambda$$

$$= (H + H_L)(\Sigma_n a_{\lambda,n} W_n + W_\lambda) + \frac{\lambda^2 \pi^2 B_1}{l_1^2} W_\lambda \tag{21}$$

where

$$(g)_\lambda = \frac{16\, f_1\, (1 - \cos \lambda\pi)}{\lambda^3 \pi^3} \tag{22}$$

B_1 is the flexural rigidity of the side span trusses. (It will rarely be necessary to take account of variable stiffness in the side spans), and

$$\Sigma_n a_{\lambda,n} W_n = - \frac{m_1^2\, W_\lambda}{2\, \lambda^2\, \pi^2}, \quad \text{when } \lambda = n\,,$$

when $\lambda \neq n$,

$$\Sigma_n a_{\lambda,n} = \Sigma_n W_n \frac{n\, [m_1^2 \{ 1 + \cos (\lambda - n)\, \pi \} + 2\, m_1 \tan \theta \{ 1 - \cos (\lambda - n)\, \pi \}]}{\lambda \pi^2\, (n^2 - \lambda^2)} \tag{23}$$

The terms in $\frac{H}{EA}$, and $H\omega t$ in equations (20) and (21), included for completeness, can usually be neglected.

For a uniform load extending from $x = a$, to $x = b$,

$$(M_L)_\lambda = \frac{2\, pl^2}{\lambda^3\, \pi^3} \left(\cos \frac{\lambda \pi a}{L} - \cos \frac{\lambda \pi b}{L} \right) \tag{24}$$

For a concentrated load P at $x = a$,

$$(M_L)_\lambda = \frac{2\, PL}{\lambda^2 \pi^2} \sin \frac{\lambda \pi a}{L} \tag{25}$$

It will be appreciated that H_L is an unknown in (20) and (21). A direct solution may be obtained by substituting for H_L from equation (8), and liquidating the deflection coefficients V_n, W_n, Z_n, by Southwell's Relaxation method. Numerical work is much simplified if this substitution is not made, but instead an approximate value of H_L is introduced in equations (20) and (21), and subsequently corrected for. Now very closely equation (8) can be written

$$\frac{H_L}{EA} \cdot L_s + \omega t L_t = \Sigma_n K_n V_n + \Sigma_n k_n (W_n + Z_n)$$

and then approximately

$$\frac{H_L}{EA} \cdot L_s + \omega t L_t = \frac{2\, m V_1}{\pi} + \frac{2\, m_1}{\pi} (W_1 + Z_1) \tag{26}$$

Writing

$$\frac{H_L}{H} = \beta\,, \quad \frac{HL^2}{\pi^2\, B} = \alpha\,, \quad \frac{Hl_1^2}{\pi^2\, B_1} = \alpha_1,$$

$$\beta^2 \left[\frac{HL_s}{EA}(\alpha + 2\,\alpha\alpha_1 + \alpha_1) + \omega t L_t \alpha\alpha_1 + \frac{2\,\alpha\alpha_1}{\pi}(mG_1 + 2\,m_1 g_1)\right]$$
$$+ \beta \left[\frac{HL_s}{EA}(1+\alpha)(1+\alpha_1) + \omega t L_t (\alpha + 2\,\alpha\alpha_1 + a_1)\right.$$
$$\left. + \frac{2}{\pi}\left\{m\alpha(1+\alpha_1)\,G_1 + 2\,m_1\,\alpha_1(1+\alpha)\,g_1\right\} - \frac{2\,m}{\pi}\frac{(M_L)_1}{H}\alpha\alpha_1\right]$$
$$+ \omega t L_t (1+\alpha)(1+\alpha_1) - \frac{2\,m}{\pi}\frac{(M_L)_1}{H}\alpha(1+\alpha_1) = 0\,. \qquad (27)$$

For live load on a side span, and the terms,

$$-\beta\frac{2\,m_1}{\pi}\frac{(M_l)_1}{H}\alpha\alpha_1, \text{ and } -\frac{2\,m_1}{\pi}\frac{(M_l)_1}{H}\alpha_1(1+\alpha)\,.$$

The value of H_L given by equation (27) is substituted in (20) and (21). The « given » moments M_L and M_l are liquidated, and by interpolation corrected values of H_L and V_n are derived.

The moment M_G taken by the stiffening truss is given by

$$M_G = \Sigma_{\lambda,n} R_{\lambda,n} V_n \sin \lambda\pi \frac{x}{L}$$

This expression may be found to converge too slowly.

Now $M_L = \Sigma_\lambda (M_L)_\lambda \sin \frac{\lambda\pi x}{L}$, and from equation (13),

$$H_L \cdot y = \Sigma_\lambda H_L (G)_\lambda \sin\left(\frac{\lambda\pi x}{L}\right)\;;$$

therefore

$$M_G = M_L - H_L\cdot y - \Sigma_\lambda\left[(M_L)_\lambda - H_L(G)_\lambda - \Sigma_n R_{\lambda,n} V_n\right] \qquad (28)$$

The term in square brackets is rapidly convergent, since, from (20), $(M_L)_\lambda - H_L\cdot(G)_\lambda$ approximates to $\Sigma_n R_{\lambda,n} V_n$ as λ increases.

The effect of the « u » displacements as represented by the parameters $A_{\lambda,n}$, $a_{\lambda,n}$ is extremely small on long span bridges, and equation (28) becomes $M_G = M_L - H_L\cdot y - (H + H_L)\,\Sigma_\lambda V_\lambda \sin\left(\frac{\lambda\pi x}{L}\right)$ (29)

Numerical application (Bridge of 3.300 ft span)

$L = 3\,280$ ft $f = 326$ ft $\frac{f}{L} = 0.09939$ $m = 0.7951$ $L_s = 6\,366$ ft

$l_1 = 1\,000$ ft $f_1 = 30.30$ ft $\frac{f_1}{l_1} = \frac{1}{33}$ $m_1 = 0.2424$ $\tan\theta = 0.3708$

$A = 980$ sq.in $EA = 27.44 \times 10^9$ $H = 58.5 \times 10^6$ lb.

$p = 6\,100$ lb.ft from $a = \frac{3}{16}L$ to $b = \frac{5}{16}L$ on centre span.

$B = 28.51 \times 10^{12}$ (average value) $B_1 = 28.51 \times 10^{12}$ lb.ft²

$\frac{\pi^2 B}{L^2} = 2.615 \times 10^6$ lb (average value) $\frac{\pi^2 B_1}{l^2} = 28\cdot 11 \times 10^6$ lb .

The effective area of the centre span stiffening truss, averaging 260 sq. in, varies from 200 sq. in at the ends, to 290 sq. in at the quarter points, and 240 sq. in at centre span. This variation is closely represented by

$$200.58 + 80 \sin \frac{\pi x}{L} + 40 \sin 3 \frac{\pi x}{L}$$

giving the values of $R_{\lambda,n}$ in table no. I.

In table no. IV, the case of a load of 6 100 lb from $a = 3/16$ L to $b = 5/16$ L is analysed. This is the loading found to give maximum positive moment at the quarter point. « U » effects are not considered in the analysis.

Solving equation (27) for $(M_L)_1 = 1\,168.1 \times 10^6$ lb ft, an approximate value of $H_L = 3.15 \times 10^6$ lb gives $(M_L)_\lambda - H_L(G)$ in line 4. The Relaxation process begins in line no. 5, where the whole of $(M_L)_1$ is liquidated by a value of $V_1 = 108.2/64.28$, which contributes $-R_{31}V_1 = -0.2$ to $(M)_3$, etc. In lines 19 to 25 the residual moments are liquidated by small contributions to $V_1 - V_9$. Account is kept of $V_n K_n + \frac{n^2 V_n^2}{4L}$, in columns 3 to 4 as relaxation proceeds. Lines 27-28 give the contribution to H_L from the side spans, and a value of H_L 2 690 $\times 10^6$ is obtained. It is now apparent that the starting value of H_L was too high, so in lines 32-36 relaxation is continued for an increment in H_L of -0.04×10^6. Finally H_L is interpolated as $3\,114 \times 10^6$ lb. In lines 38-41, the V_λs are corrected for the change in H_L, giving a bending moment in the stiffening truss of 147×10^6 lb. ft. On the assumption of uniform truss rigidity, this moment is found to be 139×10^6 lb. ft, or 5.5 % less. If account is taken in the above analysis of the « u » effect as expressed in the parameters $A_{\lambda,n}$ and $a_{\lambda,n}$, the moment at the quarter point is reduced by only 0.2 %.

Effect of hanger extension

As a result of hanger extension, the stiffening truss is subject to additional moments and shears. The increase is not negligible near the ends of the truss.

Denoting by Δv the hanger extension at any section x, of the centre span, corresponding to a stiffening truss deflection v, and neglecting horizontal displacements,

$$\Delta M_G + M_G = M_L - (H + H_L)(v - \Delta v + \Delta v_0) \qquad (30)$$

where Δv_0 denotes the hanger extension at each end of the span, and ΔM_G the additional moment transferred to the truss by the hanger extension.

Therefore

$$\Delta M_G = (H + H_L)(\Delta v - \Delta v_0). \qquad (31)$$

The hanger pull per unit length is given by

$$q = -\frac{d^2 M_C}{dx^2} = -(H + H_L)\frac{d^2 v}{dx^2} - H_L \cdot y'' \qquad (32)$$

But $M_G = -\frac{B d^2 v}{dx^2}$ therefore

$n \backslash \lambda$	1	2	3	4	5	6	7	8	9	10	11
1	2.63		0.13		— 0.10		— 0.02		— 0.01		
2		11.04		0.104		— 0.49		— 0.118		— 0.051	
3	1.14		23.93		0.04		— 1.17		— 0.305		— 0.139
4		0.415		42.2		— 0.055		— 2.155		— 0.582	
5	— 2.52		0.13		65.7		— 0.20		— 3.44		— 0.95
6		— 4.39		— 0.12		94.4		— 0.36		— 5.00	
7	— 1.03		— 6.39		— 0.37		128.4		— 0.58		— 6.86
8		— 1.38		— 8.62		— 0 67		167.66		— 0.82	
9	— 0.688		— 2.74		— 11.1		— 1.0		212.1		— 1.10
10		— 1.28		— 3.64		— 13.9		— 1.3		261	
11	— 0.527		— 1.87		— 4.6		— 16.9		— 1.66		316
12		— 0.99		— 2.45		— 5.62		— 20.3		— 2.07	
13	0.428		— 1.43		— 3.06		— 6.74		— 23.9		— 2.56
14		— 0.815		— 1.88		— 3.70		— 7.92		— 27.7	
15	— 0.364		— 1.18		— 2.34		— 4.37		— 9.20		— 31.99
16		— 0.693		— 1.51		— 2.80		— 5.09		— 10.5	

$$R_{\lambda, n} = 2.018 \times 10^6 \lambda^2 - \Sigma_n n^2 \lambda \left[\frac{2.050}{(\lambda^2 - n^2)^2 - 2(\lambda^2 + n^2) + 1} + \frac{3.075}{(\lambda^2 - n^2)^2 - 18(\lambda^2 + n^2) + 81} \right] \times 10^6$$

$\lambda = n$	1	2	3	4	5	6	7	8	9	10
$G\lambda$	336.45									
K_n	0.5062		0.1687		0.1009		0.0723		0.0562	
$\lambda^2 \pi^2 B/L^2 \times 10^{-6}$	2.6152	10.46	23.51	41.84	65.38	94.1	128.1	167.4	211.8	262
$H \times 10^{-6}$	58.5									
$H(1 + A\lambda\lambda) 10^{-6}$	56.61	58.03	58.29	58.38	58 42	58.5				
$\lambda^2 \pi^2 B/L^2 + H \times 10^{-6}$	61.12	68.96	82.04	100.34	123.88	152.6	186.6	225.9	270.3	320
$\lambda^2 \pi^2 B^2/L^2 + H(1 + A\lambda\lambda) \times 10^{-6}$	59.23	68.49	81.83	100.22	123.80	152.6	186.6	225.9	270.3	320

$\lambda = n$	1	2	3	4	5	6	7	8	9	
$g\lambda$	31.27									
k_n	0.1513		0.0514		0.0309		0.0220		0.0172	
$\lambda^2 \pi^2 B_1/l^2 \times 10^{-6}$	28.707	114.8	258.4	459	718	1 033	1 407	1 837	2 325	
$H \times 10^{-6}$	58.50		58.5		58.5		58.5			
$H(1 + A\lambda\lambda) \times 10^{-6}$	58.32	58.46								
$\lambda^2 \pi B_1/l_1^2 + H \times 10^{-6}$	87.21	173.3	316.9	517	776.5	1 091	1 465	1 895	2 384	
$\lambda^2 \pi^2 B_1^2/l_1 + H(1 A\lambda\lambda) \times 10^{-6}$	87.03									

Table I

VALUES OF

$R_{\lambda,n} \times 10^{-6}$

FOR CENTRE SPAN

(3 300 ft)

12	13	14	15	16
— 0.027		— 0.016		— 0.011
	— 0.077		— 0.047	
— 0.272		— 0.154		— 0.097
	— 0.452		— 0.259	
— 1.41		— 0.679		— 0.394
	— 1.95		— 0.951	
— 9.02		— 2.59		— 1.27
	— 11.5		— 3.31	
— 1.45		— 14.2		— 4.12
	— 1.8		— 17.3	
376		— 2.19		— 20.5
	442		— 2.55	
— 2.98		513		— 3.12
	— 3.4		587	
— 36.4		— 4.0		669

Table II A

PARAMETERS

FOR CENTER SPAN

(3 300 ft)

11	12	13	14	15	16
0.0460		0.0389		0.0337	
316	377	442	513	588	669
374	435	500	571	646	727
374	435	500	571	646	727

Table II B

PARAMETERS

FOR SIDE SPANS

(1 000 ft)

n \ λ	1	2	3	4	5	6	7	8	9	10	11	12
1	—0.0322		—0.0054		—0.0011		—0.00038		—0.00018		—0.00011	
2		—0.0080		—0.0054		—0.0013		—0.00054		—0.00027		
3	0.0485		—0.0036		—0.0048		—0.0014		—0.0006			
4		0.0215		—0.0020		—0.0013		—0.0013				
5	0.0268		0.0135		—0.0013		—0.0038		—0.0013			
6		0.0121		0.0097		—0.0009		—0.0035				
7	0.0188		0.0075		0.0075		—0.0066		—0.0031			
8		0.0086										
9	0.0145		0.0054									
10		0.0067										
11	0.012											
12												
13	0.010											
14												
15	0.009											

n \ λ	1	2	3	4
1	— 0.0030	— 0.006	— 0.0005	
2	0.024	— 0.00075	— 0.005	— 0.0005
3	0.0041	0.011	— 0.00033	— 0.004
4	0.0096	0.002	0.007	

TABLE III. — VALUES OF $A_{\lambda, n}$

Above : for 3 300 ft centre span
Left : for 1 000 ft side spans

$$q = - H_L \cdot y'' + \frac{H + H_L}{B} \cdot M_G \tag{33}$$

and the hanger extensions are given by

$$\Delta v = \frac{F_C + f - y}{E_h A_h} \left\{ - H_L \cdot y'' \, \frac{(H + H_L)}{B} \, M_G \right\}$$

$$\Delta\, v_0 = \frac{F_C + f}{E_h A_h} (- H_L \cdot y'') \tag{34}$$

where F_c denotes length of centre hanger, E_h denotes hanger modulus of elasticity, and A_H denote area of hangers per unit length.

Then equation (31) can be written

$$\Delta M_G = \frac{H + H_L}{E_h A_h} \left[(F_c + f - y) \, \frac{(H + H_L)}{B} \, M_G + y y'' \cdot H_L \right] \tag{35}$$

and the additional shear ΔS_g is

$$\Delta S_G = \frac{H + H_L}{E_h A_h} \left[\frac{H + H_L}{B} \left\} (F_c + f - y) \, S_G - y' M_G \right\{ + y' y'' H_L \right] \tag{36}$$

BRIDGE
$H = 58.5 \times 10^6$ lb
$A = 980$ sq in
$E = 28 \times 10^6$ lb/in²
$\frac{2H}{EA} = 0.0043$
$\frac{EA}{LS} = 4.3104 \times 10^6$

LOADED LENGTHS
3/16 L — 5/16 L
Centre Span

CENTRE SPAN
$\pi^2 B/L^2 = 2.615 \times 10^6$ lb
$L = 3.280$ ft
$f/L = 0.09939$
$f = 326$ ft
LIVE LOAD PER LIN : FOOT 6 100 lb/ft

SIDE SPANS
$B_1 = 28.71 \times 10^6$ lb
$l_1 = 1\,000$ ft
$f_1/l_1 = 0.0303$
$\tan \theta = 0.3708$
TEMPERATURE CONDITION NORMAL

Col. N°	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Line N°	ΔV_n	V_n	$\Delta V_n K_n + n^2 V_n^2/4L$	$V_n K_n + n^2 V_n^2/4L$	MOMENTS × 10⁶															
					$(M)_1$	$(M)_2$	$(M)_3$	$(M)_4$	$(M)_5$	$(M)_6$	$(M)_7$	$(M)_8$	$(M)_9$	$(M)_{10}$	$(M)_{11}$	$(M)_{12}$	$(M)_{13}$	$(M)_{14}$	$(M)_{15}$	$(M)_{16}$
1	$(H + H_L) + R\lambda\lambda$ for $H_L = 3.15 \times 10^6$				64.28	72.69	85.58	103.8	127.4	156.1	190	229	274	324	378	438	503	574	649	731
2	$(M_L)\lambda$				1168.1	405.0	123.2	—	− 39.9	− 36.3	− 17.2	—	8.05	7.85	3.76	—	− 1.52	− 1.19	− 0.35	—
3	$- H_L (G)\lambda = (G)\lambda \times 3.15 \times 10^6$				− 1059.9		− 39.3		− 8.5		− 3.1		− 1.46		− .80		− .50		− .32	
4					108.2	405.0	− 83.9	—	− 48.4	− 36.3	− 20.3	—	6.6	7.8	3.0	—	− 2.0	− 1.2	− 0.7	—
5	$V_1 = 1.6840$				− 108.2		− 0.2		0.1											
6	$V_2 = 5.576$					− 405.0		− 0.6		2.7		0.7		0.3		0.2		0.1		0.1
7	$V_3 = 0.978$				− 1.1		− 83.7				1.2		0.3		0.1		0.1			
8	$V_4 = -0.0056$	$V_4 = -0.0056$						0.6												
9	$V_5 = -0.379$				− 1.0				48.3		− 0.1		− 1.3		− 0.4		− 0.2		− 0.1	
10	$V_6 = -0.215$					− 0.9		—		33.6		− 0.1		− 1.1		− 0.3		− 0.2		− 0.1
11	$V_7 = -0.101$				− 0.1		− 0.0		—		19.2		− 0.1		− 0.7		− 0.2		− 0.1	
12	$V_8 = 0.0026$	$V_8 = 0.0026$										− 0.6								
13	$V_9 = 0.0201$								0.2		—		− 5.5				0.2		0.1	
14	$V_{10} = 0.0217$	$V_{10} = 0.0217$						0.1		0.3				− 7.0				0.3		0.1
15	$V_{11} = 0.0055$	$V_{11} = 0.0055$	0.0003	0.0003							0.1				− 2.0				0.1	
16	$V_{13} = -0.0042$	$V_{13} = -0.0042$	− 0.0002	0.0001									− 0.1				2.1			
17	$V_{14} = -0.0017$	$V_{14} = -0.0017$																1.0		
18	$V_{15} = -0.0011$	$V_{15} = -0.0011$																	0.7	
19	$V_1 = -0.0312$	$V_1 = 1.6498$	0.8348	0.8349	2.2															
20	$V_2 = -0.0124$	$V_2 = 5.564$	0.0094	0.8443		0.9														
21	$V_3 = -0.007$	$V_3 = 0.970$	0.1638	1.0081			0.6													
22	$V_5 = 0.0016$	$V_5 = -0.377$	− 0.0380	0.9701					− 0.2											
23	$V_6 = 0.0019$	$V_6 = -0.213$								− 0.3										
24	$V_7 = 0.0005$	$V_7 = -0.101$	− 0.0073	0.9628							− 0.1									
25	$V_9 = -0.0004$	$V_9 = 0.0197$	0.0011	0.9639									0.1							
26		Residuals			—	—	—	0.1	—	—	—			—	—	− 0.1	—	—	—	0.1
27	SIDE SPANS																			
28	$H + H_L + \lambda^2 \pi^2 \frac{B_1}{l_1^2}$ for $H_L = 3.15 \times 10^6$				90.36		320													
29		$- 3.15 \times (g)_\lambda \times 10^6$			− 98.4		− 3.6													
30	$W_1 = -1.089$		− 0.1680	− 0.1680	98.4															
31	$W_3 = -0.0113$		− 0.0006	− 0.1686			3.6													
				− 0.3372																
				0.9639																
				0.6267	− 0.0027	0.6240	$H_L = 2.690 \times 10^6$													
32	$\Delta H_L = - 0.04$																			
33	$W_1 = 0.0139$		0.0021	0.0042	1.25															
	MAIN SPAN				13.5		0.5		0.1											
34	$V_1 = 0.210$		0.1061	0.1061	− 13.5															
35	$V_3 = 0.0059$		0.0010	0.0010			− 0.5													
36	$V_5 = 0.0008$		0.0001	0.0001					− 0.1											
				0.7381	− 0.0032	0.7349	$H_L = 3.168 \times 10^6$			3.11										
							2.690			3.15										
							.478			0.04										
							$K = \frac{0.46}{0.518} = 0.89$													
							$H_L = 3.114 \times 10^6$													
37	Interpolated values of V_1, etc																			
38	$V_1 = 0.1870$	1.8368																		
39	$V_3 = 0.0052$	0.9752																		
40	$V_5 = 0.0007$	− 0.3763			V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8	V_9	V_{10}	V_{11}	V_{13}	V_{14}	V_{15}		
41	Final values of V_λ				1.8368	5.564	0.9752	− 0.0056	− 0.3763	− 0.213	− 0.101	0.0026	0.0197	0.0217	0.0055	− 0.0042	− 0.0017	− 0.0011		
42	$\Sigma V\lambda \sin \lambda\pi/4 = 8.1478$																			
	Moment at ¼ point $= 1409.9 \times 10^6 - H_L y - (H + H_L) v$																			
	$= (1409.9 - 761.0 - 502.0) \times 10^6$																			
	$= 147 \times 10^6$ lb ft																			

TABLE IV. — CALCULATION OF STIFFENING TRUSS MOMENTS

For the structure under consideration the maximum end shear is increased by 4 %, and the moment at the 1/16 point by about the same amount. The effect falls off rapidly towards the centre of the span.

There are also increases in the moments and shears at the ends of the truss beyond what are given by (35) and (36) arising from the rigid support afforded to the ends of the truss. In the derivation of equations (35) and (36) it was assumed that the ends of the truss were on hangers.

If Δv_0 is the end hanger extension given by (34), and K is the average hanger reaction for unit extension per unit length, then the additional end shear taken by the truss is

$$\Sigma_n \frac{8\,K\Delta V_0}{n^2\pi^2 + \frac{KL^2}{H} + \frac{KL^4}{n^2\pi^2 B}} \qquad n = 1, 3, 5, \ldots \tag{37}$$

The increase in end shear from this cause is only about 1 %.

The effect of hanger inclination

The importance of the inclination of the hangers discussed on p. 2, in changing the horizontal component of the cable tension must now be considered.

At any section x of the centre span, the hanger inclination is given by $\frac{u}{h_x}$, where h_x is the length of the hanger at x. The increment in horizontal cable tension is $\frac{u \cdot q}{h_x} dx$, where u is given by equation (2), and q by equation (33).

The total increment of horizontal cable tension at x is

$$\int_0^x \frac{uq}{h_x} dx\,. \tag{38}$$

Performing this integration graphically, it is found that at the quarter point the effect of hanger inclination is to increase H_L by 1 %, with a corresponding decrease in M_G of only 0.3 %.

Conclusions

A careful study undertaken for a bridge of 3 300 ft span, indicates that the usual assumptions in the orthodox deflection theory are legitimate when analyzing a structure of this magnitude. On long span bridges, the principal contribution to moments and shears comes from the higher harmonics of the deflection. These harmonics are but little affected by quite large changes in H_L arising from uncertainties as to the exact behaviour of the structure. On the other hand, the primary harmonic V_1, which is highly geared to variation in H_L, contributes only a small amount to the moments and shears taken by the stiffening truss. On the structure discussed in this paper, a 1 % change in H_L, causes the bending moment to change by only 0.2 % on the avérage. Making the assumption of uniform truss rigidity, it is thus possible to simplify the computation very considerably. Equation (8) is replaced by

$$\frac{H_L}{EA} \cdot L_s + \omega t L_t = \Sigma_n K_n + \Sigma_n k_n (W_n + Z_n) . \tag{31}$$

and equations (20) and (28) by

$$M_G = M_L - H_L \cdot y \frac{- \Sigma_\lambda \{ (M_L)_\lambda - H_L (G)_\lambda \}}{1 + \dfrac{\lambda^2 \pi^2 \dfrac{B}{L^2}}{H + H_L}} \cdot \sin \frac{\lambda \pi x}{L} , \tag{40}$$

It will be found that on long span bridges, maximum moments and shears are given by short loaded lengths of the order of 1/8 to 1/4 of the span. It will also be found that the maximum moment at any section from a given loading occurs when the loading extends equally on either side of the section. Work can be systematized and simplified by tabulating multiples sines and cosines of $\frac{\lambda\pi}{32}$, and constructing tables of cos $n\pi a/L$-cos $n\pi b/L$ for several loaded lengths. The author has made up tables of this nature which can be applied without modification to a bridge of any span. By this means, and the use of equations (39) and (40), moments and shears at any section for a given loading can be obtained very rapidly. The effect of changes in make-up giving revised dead load and truss rigidity can be assessed without difficulty.

Résumé

La méthode par approximations successives de Southwell s'applique à la résolution de ponts suspendus à l'aide de solutions trigonométriques. Ce procédé de calcul ainsi obtenu ne nécessite pas la simplification habituelle de suspentes inextensibles à traction constante, un déplacement vertical des câbles, et un coefficient de rigidité constant. Ce procédé est amélioré et étendu à un pont d'une portée de 1 000 mètres. L'influence de ce que l'on néglige les corrections habituelles est étudiée et l'auteur montre que seuls les deux facteurs suivants influencent pour des ponts suspendus de grande portée : rigidité variable des poutres et longueur variable des suspentes. Le mémoire se termine par une représentation simplifiée de la résolution par série suffisante pour la plupart des ouvrages courants.

Zusammenfassung

Die Iterationsmethode von Southwell wird zur Lösung des Hängebrückenproblems mit Hilfe von trigonometrischen Reihen angemeldet. Das sich dabei ergebende Rechnungsverfahren verzichtet auf die üblichen vereinfachenden Annahmen über konstanten Hängezug unelastischer Hänger, vertikale Kabelverschiebungen und konstante Steifigkeit der Versteifungsträger. Das Verfahren wird entwickelt und auf die Berechnung einer Brücke von 1 000 m Spannweite angewendet. Die Einflüsse der üblichen Vernachlässigungen werden untersucht und es wird gezeigt, dass nur die Auswirkungen einer veränderlichen Trägersteifigkeit und der Hängerdehnungen bei Brücken dieser Grössenordnung von Bedeutung sind. Die Arbeit schliesst mit einer vereinfachten Darstellung der Lösung mit Reihen, welche für die meisten Tragwerke genügend ist.

Summary

Southwell's Relaxation technique is applied to the trigonometric series solution of the problem of the stiffened suspension bridge. The resulting method of analysis is free from the usual simplifying assumptions of constant hanger pull, inextensible hangers, vertical cable displacements and uniform truss rigidity. The method is developed and applied to the calculations for a bridge of 3 300 ft span. The results of neglecting the corrections to the orthodox theory are then assessed, and it is shown that only the effects of non-uniform truss rigidity and hanger extension are significant on structures of this magnitude. The paper terminates with a simplified presentation of the series solution, which is sufficiently accurate for most structures.

IIIb4

Reconstruction du pont suspendu de Menai

Umbau der Hängebrücke über die Menai-Strasse

Preservation of the Menai suspension bridge

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The work here described was started about a year before the outbreak of the war and was carried to completion in a period of some difficulty, but before the period of acute stringency in labour and materials which occurred during the latter part of the war had reached its full development.

Before being started the work to be performed had been closely studied in all its aspects and the design was most carefully framed so as to take advantage of the latest ideas and to make use of the best materials procurable. Although the work happened to coincide with the war period there was, therefore, nothing in its composition or execution of inferior quality. It was, in fact, one of the last important public works undertaken in Britain which can be said to have been carried out in the best tradition and unspoiled by the influence of war.

The work was performed under the direction of the British Ministry of Transport, the Roads Department of which nowadays exercises ownership and control of the principal highways in the country.

Sir Alexander Gibb, with whom the author collaborated, was employed by the Ministry to prepare the designs and supervise the work of reconstruction.

The Menai Bridge forms the only road passage across the Menai Straits, a narrow branch of the sea which separates the Island of Anglesey from the Welsh mainland. The principal route to Ireland passes via the bridge across the Island of Anglesey to the Port of Holyhead, and in the early part of the 19th century, in the old coaching days before the railway was built, the road to Holyhead was naturally a route of great importance. During the latter half of the century, the main traffic was by railway and the road fell into partial disuse, but the advent of the motor car at the beginning

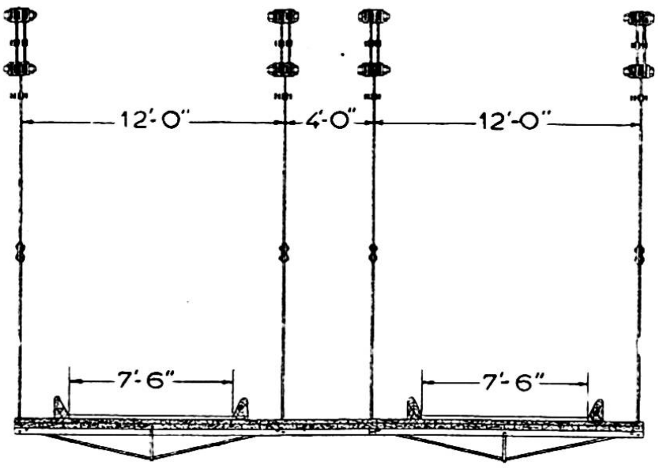

Fig. 1. Cross section of Telford's original deck.

of the present century revived its importance not only as a bearer of local traffic but also as a national highway connecting England and Ireland. Every year up till 1938 saw a steady increase in the volume of mechanically propelled vehicles crossing the Menai Straits and the weak condition of the old suspension bridge, then more than a hundred years old, made it necessary to impose restrictions not only upon the weight and speed of the motor vehicles which passed along its two independent tracks but also a restriction upon the interval between successive vehicles. Such restrictions were irksome to travellers and were found to be difficult of enforcement.

Right from its inception in 1825, the suspension bridge was a toll bridge, every vehicle and for most of the time every pedestrian, having to pay a fare to the collector of tolls posted upon the bridge both by day and by night. These toll collectors were in the tradition of the old Roman publican in that having made an annual lump sum bid to the Government

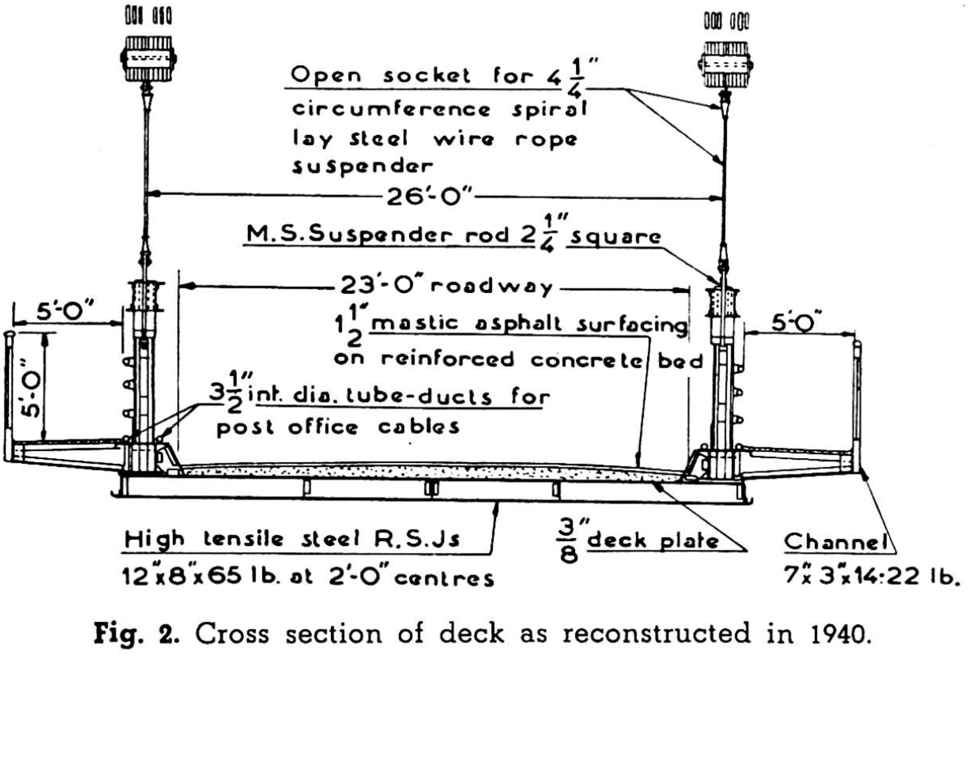

Fig. 2. Cross section of deck as reconstructed in 1940.

for the privilege of farming the tolls and being restricted by Government to the rate of tolls they might charge, they were subsequently dependent upon the volume of traffic to recoup themselves.

After the reconstruction in 1942 the bridge was thrown open to the public for use free of charge and without restriction as to the weight, speed and spacing of vehicles using the route, a benefit which the inhabitants of these islands will no doubt appreciate more amply if and when the free use of motor spirit may permit the resumption of pleasure motoring.

The land approaches and bridge abutments on both sides of the Menai Straits were built in a hard limestone masonry, the blocks of stone having been hewn from local quarries. The work had been very well performed by skilled masons and standing securely upon sound rock foundations the whole of the masonry structure still remains in excellent preservation.

It had originally been designed with an ample margin of safety so that it was found to be capable of carrying without alteration the much heavier loads imposed by the renovated superstructure and by modern transport.

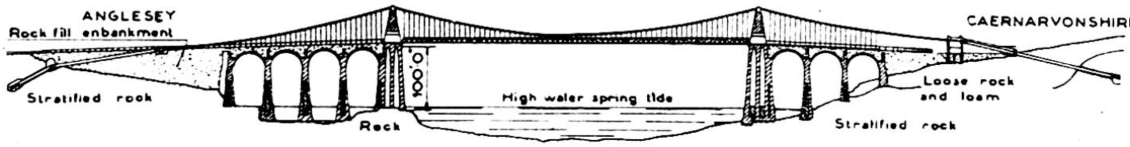

Fig. 3. General longitudinal section.

During the first century of its life the deck of the bridge had been entirely renewed on three occasions and the main suspension chains and hangers had been repaired, numerous defective links of the chains having been taken out and replaced from time to time.

This notwithstanding, the general scheme of chains, hangers, deck, roadway and footways, were found to be totally inadequate for purposes of modern traffic and the whole superstructure of the bridge had to be redesigned on a more liberal scale and in accordance with modern standards, entirely new steelwork being substituted in place of the antique ironwork of the chains and in place of the less antique but still old steelwork of the deck structure.

The work involved in these renovations has several claims to attention which make it of special interest.

There is, firstly, the fact that this bridge, originally built complete in 1826, was one of the earliest iron chain suspension bridges. When first built the clear central span of 580 feet was greater than the spans of any one of the few earlier suspension bridges that had been erected in England, the United States, Germany, Switzerland, and France prior to 1825, indeed, this bridge had then the greatest span ever built. It was also a very handsome bridge and had been designed and built by an Englishman whose reputation as a Civil Engineer surpassed that of any other person so engaged in any other country at that period, and indeed there has probably been no other Civil Engineer in our country since then whose reputation and achievements have equalled those of Thomas Telford, he being one of the founders, and the first President of the British Institution of Civil Engineers.

Apart, therefore, from its utilitarian value, the Menai Suspension Bridge has possessed for British people a unique sentimental attachment standing as a remarkable monument to the industrial revolution of the nineteenth century—a monument moreover which happens in itself and in its natural setting to be a thing of beauty unmarred by the noise, the smoke, the dirt, and the general atmosphere of squalour which has too often been the accompaniment of so many of the other manifestations of the scientific era.

It would be fair to say, therefore, that the principal merit of this bridge during the nineteenth century lay in the fact that it was a spectacular novelty of impressive beauty, full of charm, dignity, interest and utility, and as such that it was an inspiration to the youth of the country.

Apart from such general considerations there are, of course, technical lessons deriving from the long period of its service to be learnt and there are also quite a number of technical features in the design of the restoration work which are of interest to engineers.

Among the latter may be mentioned the ingenious methods which had to be adopted so as to carry out the complete replacement of the suspension chains with their anchorages, their roller bearings and their hangers and also the replacement and widening of the bridge deck and footpaths without any interruption of the traffic crossing the bridge either by day or by night.

Other technical features of interest were the manufacture of the eye bar links of which the chains are composed, and the method of protecting them against rust corrosion. There was also the method which had to be employed for cutting away, underpinning and replacing the central limestone wall between the two roadway tracks where they pass beneath the arches of the principal abutments supporting the suspension chains.

In the space of a short paper such as this it is not possible to go into much detail but some of the main outline of the technical features may be mentioned.

Dealing first with the scheme of operations whereby the whole reconstruction work was effected without stopping the traffic, it would, of course, have been easy to achieve this result by throwing a temporary suspension bridge across the straits and then closing the old bridge to traffic and diverting the latter to the temporary structure. Such procedure would, however, have been very costly.

To understand what was actually done it is first necessary to realise that the old bridge was suspended from four separate chains, one chain on either side and two close together in the middle between the two carriageways. The four chains rested on top of two main piers flanking the main span. Upon each of these two main piers a temporary steel girder or cap was erected and mounted above the old chain seatings and on top of this cap were slung two steel wire rope suspension cables capable temporarily of replacing the outer two sets of the old chains above mentioned.

In the reconstructed bridge the four chains each composed of a tier of eyebar links five abreast and four deep were to be replaced by two chains each composed of a tier of six eyebar links two deep and the new chains had perforce to occupy very nearly the same physical position on top of

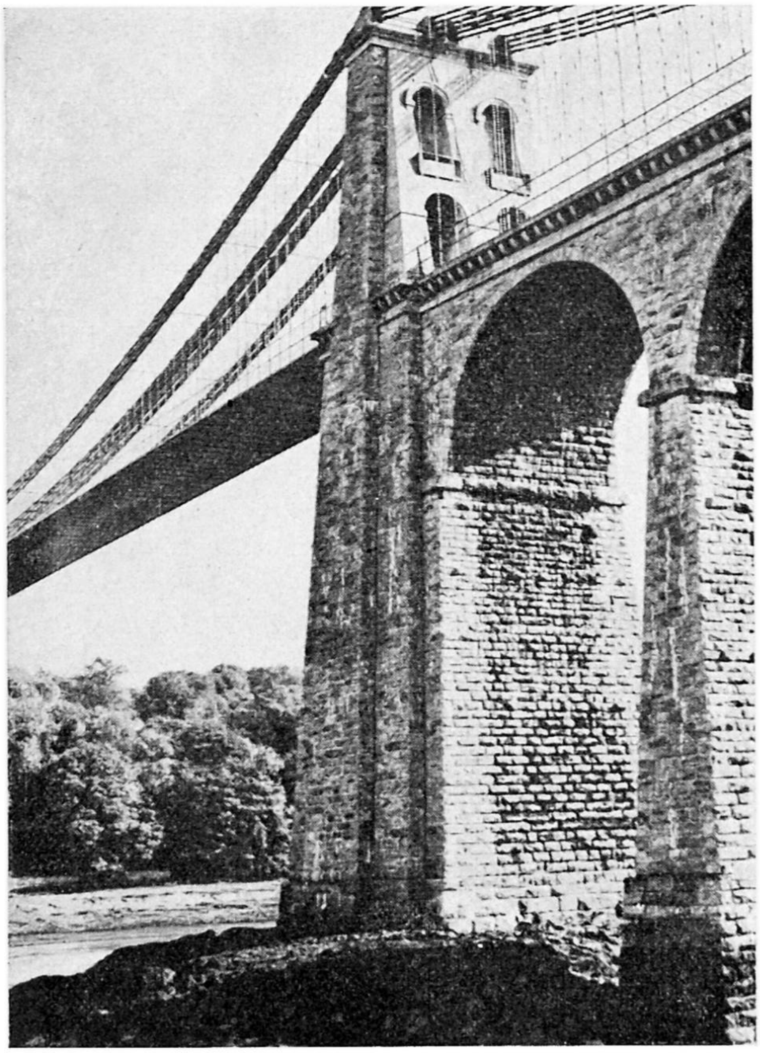

Fig. 3. One of the main masonry piers showing the four chains deck and roadway arches of the original structure.

the piers as had originally been occupied by the two outer tiers of the four original chains. It will be seen, therefore, that it was necessary to remove the two outer tiers of old chains before the new permanent chains could be erected in their place and explains why it was necessary to provide temporary cables for supporting the outer edges of the roadway deck while the work of chain replacement was in progress. The cables were mounted directly above the chains that were to be removed and replaced by new chains and were supported by the temporary cap girders above mentioned. After the cables had been strung across and anchored to the hillsides they were connected by temporary hangers so as to support the outer edges of the old bridge deck and after that the old outer chains were removed and were replaced by the new chains. While this was going on the old inner central tiers of chains were left in service, and traffic continued to use the deck of the bridge without any interruption or interference.

When the new chains had been mounted upon their new roller tracks and carried back through the old anchorage tunnels to deep anchorages in the virgin rock in the hillsides behind, the new suspended deck was erected from the new chains by means of hangers which hung just outside the parapets on either side of the old roadway deck.

The new deck had eventually to occupy the same spatial position as the old deck and as our present state of knowledge does not permit us to make two material objects occupy the same identical spatial position at the same time it was necessary to build the new deck in a position just about 4 feet vertically underneath the old deck which continued to be used by traffic all the time while the new deck was being pieced together.

When the new deck had been built in that position and was itself ready to carry traffic the old deck was demolished, demolition being done in two stages, the old carriageway on one side the bridge being broken up and removed first while the old carriageway on the other side continued to bear traffic. Traffic was then diverted back to run upon the freshly exposed new carriageway at a level four feet below the demolished roadway and the second old carriageway was closed to traffic, broken up and removed in its turn. After this the whole width of the new carriageway was available for the use of traffic which was made to run down temporary wooden ramps on to the new deck. Finally, the new deck had to be raised bodily through a distance of 4 feet so as to bring it up into its proper position and this raising process was done by screwing it up inch by inch on the suspender hangers, all of which had been designed with long screws and nuts for this purpose.

Described in this way the whole operation appears fairly simple but there were a great many minor difficulties and intricacies brought about partly by the rather irregular nature of the old work. The old chains did not lie in one vertical plane but had been made to change direction in plan as well as in elevation and in order to make the new superstructure fit into the old masonry framework the new chains had to be made to do the same. Then there was the stretch of the suspension members, old chains, new chains, and temporary cables which had to be taken into consideration at all times and during all phases of the operations.

The stretch of the wire rope temporary cables supporting the outer edges of the old bridge deck was, of course, quite different to the stretch of the original chains and had to be allowed for. Then the new eyebar link chains had themselves to be supported during erection. On the two side approach spans the new chains were erected on scaffolds supported upon the old masonry arches beneath and there was very little elastic distortion

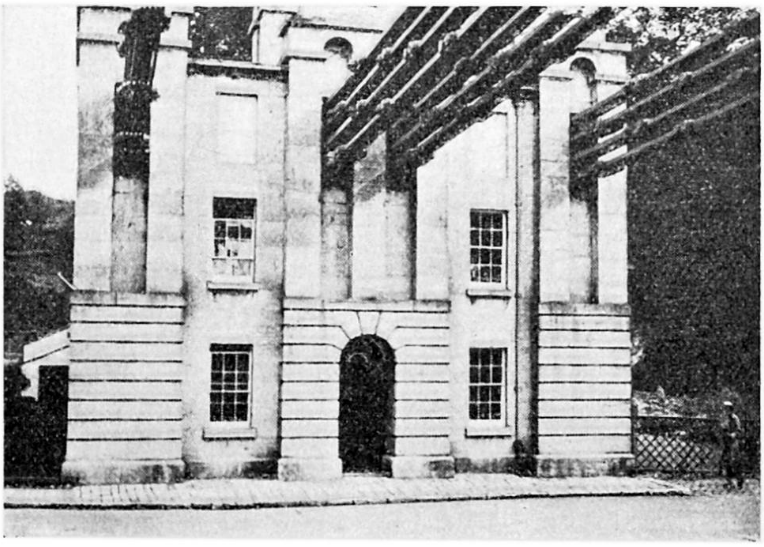

Fig. 4. Old chains entering the anchorage on the Welsh side.

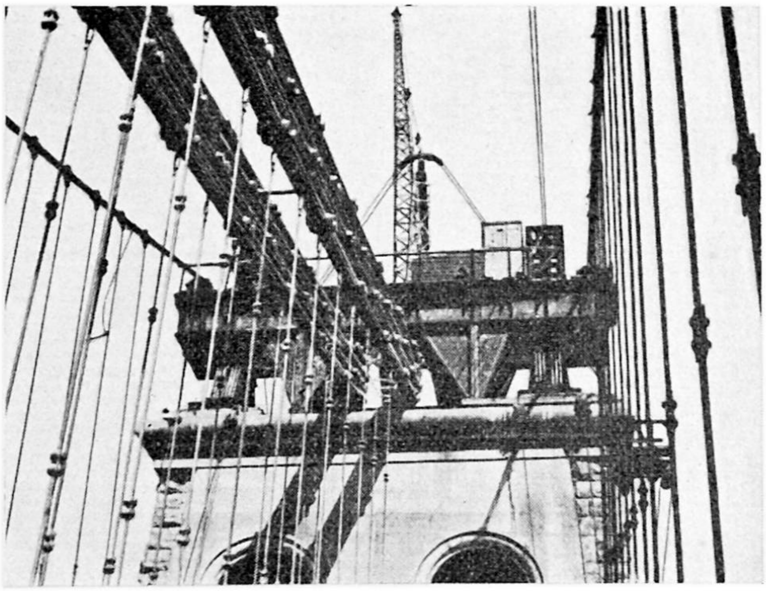

Fig. 5. Top of one of the main piers during reconstruction. The two old central chains are seen in position but the two old outer chains have been removed and the roller bearing saddles for supporting the new chains have been erected on top of the pier. Above the pier is the temporary steel horsehead girder supporting the temporary wire rope cables and suspenders and above them the blondin wires. The suspended walkways for supporting the new chains during erection have not yet been set up.

to be allowed for, but where the two new chains traversed the main span they had to be put together link by link upon two suspended walkways, each of which consisted of five catenary wire ropes hanging side by side and decked over without any stiffening. The stretch of these ropes when the weight of the chain links came upon them was considerable and the sequence in which the chain links were assembled upon the walkway also created distortion away from the catenary. In the stage when the eyebar links were being assembled and before they were connected at both ends to the links on the side spans there was a period of some instability when wire rope walkways tended to twist out of line from under and capsize the load. All these tendencies had to be taken into consideration as the work progressed. The new chains were assembled on the suspended walkways by means of blondins rigged above them.

Perhaps the most instructive technical feature introduced in this design was the method of protecting the new chains against rust. The eyebar links composing the suspension chains had in the original bridge been made of wrought iron—a material which is generally considered to be not very susceptible to rust. They had, moreover, been repainted at regular intervals with the high qualtiy white lead and linseed oil paint which was procurable in those days of plenty. In the course of one hundred years exposure to the salt bearing winds that blow over the Menai Straits, individual links had, however, suffered severely and there were places where a considerable amount of corrosion had occurred, this rusting being usually most marked in or near the eyes of the links, fishplates and hangers where they were pinned together. As the old links were known to contain a number of hidden flaws and as the dead load working stress alone to which they were subject lay between 6 and 7 tons per square inch, the added deterioration due to rust was a serious matter and gave rise to some anxiety. It was decided, therefore, in the new design to do everything possible to eliminate corrosion and with this end in view each link was shot blasted before erection and afterwards spray coated with hot zinc about five thousands of an inch in thickness. Over this was applied a priming coat of red lead followed by three further coats of paint on a linseed oil base. A thick plastic

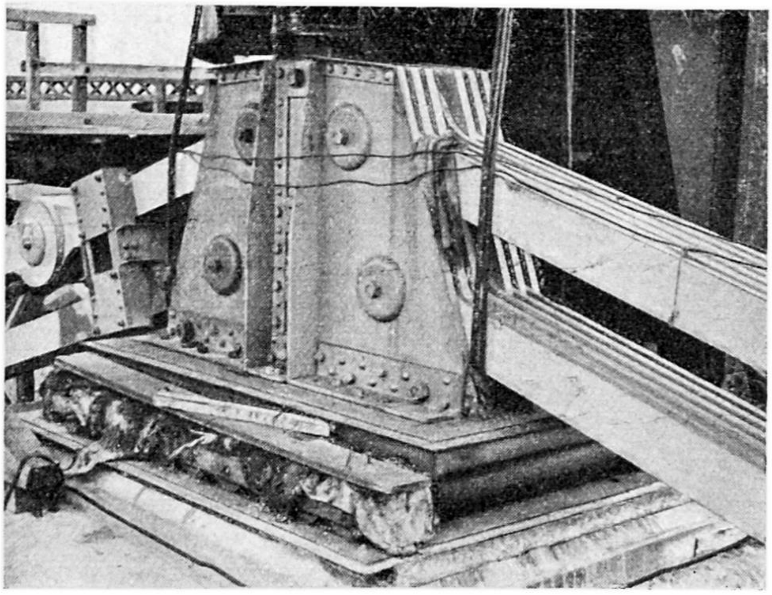

Fig. 6. One of the new roller bearing chain saddles in course of erection on top of the pier and before its enclosure in an oil bath. The new chains have been erected and connected upon the saddle.

paste built up on a petroleum basis was inserted between the meeting faces where the links were pinned together so that the percolation of moisture between links and pins was thereby prevented.

An examination carried out recently, that is about five years after the erection, disclosed no trace of rust anywhere on the eyebar links. While it was apparent that the outer coats of paint had deteriorated in places there was not a trace of rust anywhere so leading to the conclusion that the zinc coating beneath the paint was impervious and was providing perfect immunity against rust.

Owing to the great expense of the shot blasting and zinc coating, this costly process could not be adopted for protecting the steelwork of the stiffening trusses, handrailings and so forth, the protection given here being no more than the ordinary wire brushing, removal of loose rust and mill scale and application of three coat paintwork, the undercoat being red lead paint. In the parts so protected there was considerable evidence of rusting although no more so than could reasonably be expected to occur after five years in such an exposed position.

The underside of the deck structure which consisted of mild steel joists laid as cross beams with continuous steel plating over was in the first instance painted in the ordinary way with three coats of linseed oil and lead paint of good quality.

The lower flanges of the cross beams are, of course, very much exposed to wind and being shielded from sun tend to remain damp and to collect drops of water in misty weather and after five years of this exposure became rusty and some of the paint on the edges of the flanges flaked off. For repainting this part of the structure after the five years it was decided by the Ministry of Transport to employ a black bitumastic paint overcoat, the behaviour of which will be the subject of close observation in future.

While on the subject of corrosion and painting it is interesting to note that Thomas Telford when designing the original wrought iron structure in or about the year 1820 conceived that corrosion of the chains inside the anchorage tunnels would be much more severe than outside in the open air, and he accordingly stiffened up the sizes of the links where they traverse the underground tunnels making them nearly twice as thick as those in use outside so as to allow a margin for rusting. The tunnels consist of narrow passageways cut in the virgin rock where the humidity is considerable and the temperature does not vary very much summer or winter. Telford's conception that rusting in such conditions would be much more active than in the open air has been entirely falsified in the result, experience having proved the exact opposite, there being practically no rusting of the ironwork anywhere in the tunnels during a period of 120 years whereas rusting outside was considerable.

It was observed that the rusting of mild steelwork in the reconstructed bridge after five years exposure was most pronounced in those parts which were most exposed to wind and rain, the worst rusting of all having occurred on the side of the bridge which was exposed to the prevailing south westerly rain bearing wind, and especially was this the case on the outer parapet steel footway railing on that side where the $\frac{5}{8}$ inch diameter upright rods composing the railing were blistered with rust and had the paint stripped off on their exposed western side, the other side of the rods next to the footpath being scarcely blistered at all. It seems as if this question of rust prevention stands today very much where it did fifty or a hundred years ago but with the difference that in earlier times there was always available a sufficient supply of low paid labour to carry out work like painting maintenance whereas today with higher rates of pay and shorter hours of work the onus of maintaining steel and timber structures by means of painting is more than it was, so serious in fact as may actually create some restriction in the use of both steel and timber.

It might have been supposed that the large sums expended in many different countries upon the establishment and maintenance of Building Research Institutions, University Laboratories and the like organisations would have led to a rapid improvement in method and to the solution of problems of this kind, but such does not seem to have occurred. Improvements and solutions when they do appear seem usually to derive in these days, as in earlier times, as the result of the efforts of private traders, firms or individuals, and not as the outcome of the communalised wisdom of State Institutions. The true role of the latter would appear to lie in the

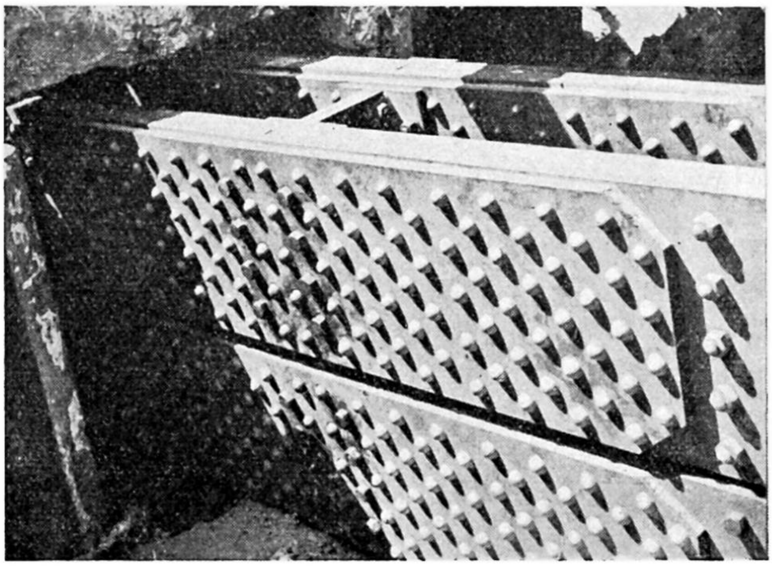

Fig. 7. Steel plate extension of the new eyebar chains which were designed to slide down the anchorage tunnels and to be concreted in solid instead of painted and left exposed as were the underground eyebar extensions in the old structure.

dessemination of knowledge and organisation of improved methods rather than in their origination.

The roadway of the bridge was surfaced with a layer of Trinidad mastic asphalt into the surface of which were rolled clean $\frac{5}{8}$ inch cubical granite chippings applied hot. A special white mastic line $4\frac{1}{2}$ inches wide and in three foot lengths laid at nine foot intervals was embedded in the original asphalt surface to mark the centre of the roadway on the main span.

The asphalt $1\frac{1}{2}$ inches thick was laid on a plain screeded concrete base six inches thick and the concrete base beneath was rather heavily reinforced with a steel mesh and was itself laid directly upon the steel plating of the deck structure. The traffic surface obtained in this way appears to be in almost perfect condition after six years use.

Where the main chains rest upon the piers roller bearings permit of horizontal movement and the ten steel rollers each ten inches in diameter and seven feet long in each of the four main bearings are submerged in an enclosing oil bath from which air is totally excluded so as to prevent water from condensing inside the bath.

The vertical suspenders connecting the deck of the spans to the chains are formed of steel spirally laid wire ropes about $1\frac{1}{4}$ inches in diameter socketed at both ends and each fitted with a turnbuckle tension adjustment. These suspenders have a very clean light appearance and being very flexible are believed to be secure against the deterioration caused by wind vibration.

The reconstruction of Menai Suspension Bridge has naturally given rise from time to time to a number of criticisms and suggestions from engineers, from public bodies and individuals.

It has been pointed out that the high tensile steel eyebar links made up 30 per cent of the total suspended weight in the main span and that a considerable saving in weight, and possibly some saving in cost, could have been effected by the use of wire cables instead of eyebar links, also that cables when wrapped round with protective sheating material are less vulnerable to attack by rust. The reason why the eyebar links were made use of was because these links albeit of different size and arrangement and of different material to that used in the original links were nevertheless similar in conception and appearance to the original design, the character of which it was most earnestly sought to preserve.

Another criticism related to the disposition of the steel lattice stiffening girders which flank to roadway on either side where it traverses the main span of the bridge. The criticism has not been levelled against these stiffening girders based on the aesthetic ground that they spoil the externally viewed appearance of the structure because actually they do not have this effect, but the criticism is that the girders obstruct the view of passengers in motor vehicles crossing the main span. The suggestion has been made that the girders might have been disposed partly or wholly below deck level and it must be admitted that a slight lowering of the girders might have been effected advantageously without trenching seriously upon the navigational headroom, but whether it would have been possible to lower the girders sufficiently to permit unrestricted lateral vision to motorists without detracting from the external appearance is more doubtful.

It has also been suggested that the retention of the two narrow carriageway openings in the masonry of the two main piers ought to have been avoided and it is actually a fact that the designers prepared a project for substituting a single archway opening of the full roadway width in place of the two arches. The reasons against the adoption of the single archway project were, firstly, that it involved a very expensive rebuilding and widening of the old masonry piers, and, secondly, that it destroyed the rather quaint effect and architectural character of the old roadway viewed from the approaches.

There were others who considered that the reconditioning of the old structure was altogether a mistake and that the old structure should rather have been entirely demolished and replaced by a wide modern bridge supported upon an arch span, and it cannot be denied that the locality lends itself to arch treatment and that a very fine and impressive design, either in steel or in reinforced concrete, could no doubt have been produced on these lines at or near the site of the old suspension bridge. In Britain, however, we prefer not to destroy old institutions unless and until the proposed replacement has established an undeniable superiority which in this case was hardly proved.

There is every indication that the reconstructed bridge can meet all reasonable traffic requirements at the present time and unless there be a further melancholy increase in the density of population, also in the future.

Résumé

Description de la reconstruction du célèbre pont suspendu de Menai, construit en 1826 par Thomas Telford, l'un des ingénieurs les plus remarquables de son époque. Il est à noter que le remplacement du système porteur fut réalisé sans interruption du trafic. Le vieux pont comprenait quatre chaînes en fer forgé; le remplacement des deux chaînes extérieures nécessita l'emploi provisoire de deux câbles porteurs au-dessus des anciennes chaînes jusqu'à la mise en place des nouvelles chaînes. Les chaînes intérieures furent remplacées en second lieu. Le nouveau tablier fut construit à 10 cm en dessous de son niveau définitif puis remonté au fur et à mesure de la démolition de l'ancien tablier. L'auteur décrit les mesures prises pour éviter la rouille des chaînes.

Zusammenfassung

Es wird der Ersatz der berühmten 1826 von Thomas Telford, einem der hervorragendsten Ingenieure seiner Zeit vollendeten Kettenhängebrücke über die Menai-Strasse beschrieben. Bemerkenswert ist, dass dieser Umbau mit vollständigem Ersatz der alten Tragkonstruktion ohne Verkehrsunterbruch durchgeführt wurde. Die alte Brücke besass vier Ketten aus Schmiedeeisen; über den beiden äussern Ketten wurden, um diese zu ersetzen, zwei provisorische Drahtkabel verlegt, bis die zwei neuen Ketten eingebaut waren. Nachher konnte auch das innere Kettenpaar entfernt werden. Die neue Fahrbahn wurde um 4′ unter ihrer endgültigen Lage unter der bestehenden Fahrbahn montiert und nach deren Abbruch sukzessive durch Hochschrauben an den Hängern in ihre endgültige Lage gehoben. Besondere Massnahmen für den Rostschutz der neuen Kette werden beschrieben.

Summary

A description is given of the substitute of the famous chain suspension bridge built in 1826 by Thomas Telford, one of the most brilliant engineers of his time.

It is worthy of note that this reconstruction of the old supporting structure was carried out without any interruption of traffic. The old bridge had four chains of wrought iron. Over the two outer chains, and for the replacement of them, two temporary wire cables were laid, until the two new chains were built in. After that, they were able to remove the inner pair of chains. The new track was erected 4′ below its ultimate level, beneath the existing track, and after breaking down the latter it was successively raised by jacks to its suspension rods, into its final position. Particular measures for prevention of rust to the new chain are described.

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Recherches expérimentales sur la stabilité aérodynamique des ponts suspendus

Experimentelle Untersuchung über die aerodynamische Stabilität der Hängebrücken

An experimental investigation of the aerodynamic stability of suspension bridges

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Introductory

The collapse of the Tacoma Narrows Bridge in 1940 provided a dramatic illustration of aerodynamic oscillations on suspension bridges, and stressed the importance of research on the subject. Since 1940 the problem has been studied particularly extensively in the U. S. A., by F. B. Farquharson, D. B. Steinman, Theodore von Kármán, Louis G. Dunn, Hans Reissner and F. Bleich and others (1). In many cases these investigations have

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(1) F. B. FARQUHARSON, *General Discussion of the Torsional Stability of Suspension Bridges under wind action* (*Report No. 17 of the Structural Research Laboratory*, Unisersity of Washington).

F. B. FARQUHARSON, *Prototype prediction based on section model tests of configuration LXXVI tested at N = 40 c. p. m. with various degrees of damping* (*Report No. 18 of the Structural Research Laboratory*, University of Washington).

F. B. FARQUHARSON, *Prototype prediction based on full model tests at configuration LXXVI tested at two values of truss stiffness* (*Report No. 19 of the Structural Research Laboratory*, University of Washington).

F. B. FARQUHARSON, *Lessons in Bridge Design Taught by Aerodynamic Studies* (*Civil Engineering*, August, 1946).

D. B. STEINMAN, *Rigidity and Aerodynamic Stability of Suspension Bridges* (*Proceedings of the American Society of Civil Engineers*, November 1943).

D. B. STEINMAN, *Design of Bridges against Wind* (*Civil Engineering*, October, November and December issues 1945).

D. B. STEINMAN, *Wind tunnel tests yield aerodynamically stable bridge sections* (*Civil Engineering*, December 1947).

Hens REISSNER, *Oscillations of Suspension Bridges* (*Journal of Applied Mechanics*, March 1943).

Theodore von KÁRMÁN, with the co-operation of Louis G. DUNN, *Aerodynamic Investigations for the Design of the Tacoma Narrows Bridge* (*Report submitted to the Board of Consulting Engineers*, May 1942).

been mainly concerned with the stability of specific bridges, and these are well exemplified by the model experiments made in connection with the failure of the Tacoma Narrows Bridge by F. B. Farquharson at the University of Washington and by Louis G. Dunn at the California Institute of Technology. Although much valuable knowledge has been gained, there is as yet no reliable purely theoretical basis for the prediction of the stability characteristics of a preferred design. This, in fact, is not surprising since with many conventional types of suspended structure the airflow can be extremely complex. As will be shown later in this paper, the form of small structural details such as handrails and roadway stringers have a marked effect on the stability. It appears therefore that, with the present state of knowledge, the most reliable assessment of the aerodynamic stability of a proposed bridge is to be obtained by tests on oscillatory models in wind tunnels.

The investigation described in this paper is specifically intended to provide guidance in the design of the proposed Severn Suspension Bridge, and is being carried out in close collaboration with the Consulting Engineers associated with that project. However, although this particular application has been in view, the tests have so far included comparisons between a fairly wide range of structural forms, and have led to certain conclusions which should hold good for truss-stiffened types generally. On the other hand, as the investigation has a specific limited objective, time has not been available for a systematic fundamental study.

The following two complementary experimental techniques are being used :—

1) *Tests of Sectional Models.* These involve tests of oscillatory behaviour in a single degree of freedom. For this purpose a rigid model of a representative length of the suspended structure is mounted in a wind tunnel in such a way that it is free to oscillate against a spring constraint, either in a vertical translatory motion or in a rotational motion about a spanwise axis. These motions may be taken to correspond respectively to those in vertical flexure and in torsion occurring on the prototype structure.

2) *Tests of Full Models.* Tests of sectional models, in the simple form adopted for the present investigation, give no information on the possible influence of oscillation form or on the effects of couplings between the natural modes of oscillation : nor can they be carried out satisfactorily in a quartering wind without the use of a large wind tunnel and special apparatus. Since sufficient evidence on the importance of these factors is not yet available it is necessary to verify the stability of a proposed design by tests on a full model constructed to be dynamically similar to the prototype. For this purpose a large wind tunnel has been constructed by the Ministry of Transport in which models of up to 60 feet in length can be tested at various vertical and horizontal inclinations of the wind. Tests are expected to start in June, 1948.

Louis G. Dunn, *Aerodynamic Investigation of the Bending Oscillations of the Original Tacoma Narrows Bridge (Report submitted to the Board of Consulting Engineers*, April 1943).

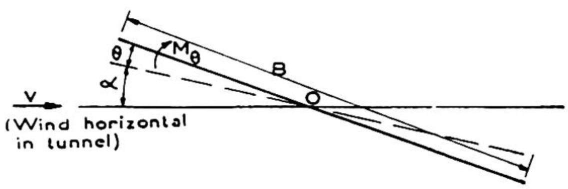
Louis G. Dunn, *Experimental Investigations on the Aerodynamic Characteritics of the suspended Structure of the Tacoma Norrows Bridge* (Appendix VIII of the failure of the Tacoma Narrows Bridge. *Bulletin 78 of the School of Engineering, Texas Engineering Experiment Station*, 1944).

F. Bleich, *Dynamic Instability of Truss Stiffened Suspension Bridges under Wind Action (Report to the Advisory Board on the Investigation of Suspension Bridges*, February, 1947).

The paragraphs which follow refer throughout to experiments with sectional bridge models. In these the aim is to find acceptable forms of suspended structure which show no tendency to oscillate in either vertical or pitching motions up to wind speeds corresponding to 100 miles per hour at vertical inclinations ranging from — 15 to + 15 degrees. Owing to limitations regarding time and availability of wind tunnels, a simple technique of testing had to be adopted. Oscillations are classed equally as unstable whether they appear as steadily maintained non-catastrophic oscillations of limited finite amplitude or whether they show signs of growing to a dangerous amplitude such as might lead to failure. The model is mounted in a 4 ft square wind tunnel and the wind speed in the tunnel is increased until an oscillation started by a small disturbance is just maintained. The value of the reduced velocity V_r is then calculated from the measured values of wind velocity and frequency of oscillation. By this simple procedure a large number of model variations can be rapidly tested and the results used to indicate general design features which tend to produce stability.

Theoretical basis of selectional tests

The following theory relates to the pitching oscillations. With the necessary changes in notation and in expressions for the aerodynamic force it is equally applicable to tests of the vertical oscillations.



a) *Notation*

I_θ = moment of inertia about centre of rotation O per unit spanwise length.
K_θ = structural damping coefficient per unit spanwise length.
e_θ = elastic stiffness per unit spanwise length (2).
α = angle of attack, or mean incidence about which pitching oscillations occur.
θ = angular deviation from mean incidence at any instant of oscillation.
θ_0 = amplitude of simple harmonic oscillations.
B = distance between stiffening trusses.
V = airspeed.
ρ = air densiy (3).
N_θ = natural frequency (cycles/sec) corresponding to inertia I_θ and stiffness e_θ.
V_r = V/NB = « reduced velocity » corresponding to simple harmonic oscillations.
M_θ = increment of aerodynamic pitching moment per unit spanwise length at any instant of oscillation.
$\delta_\theta = \dfrac{K_\theta}{2 I_\theta N_\theta}$ = natural logarithm of the ratio of successive amplitudes of oscillation in still air.

(2) I_θ, K_θ and e_θ assumed to be measured in consistent units, and to be applied uniformly along the unit spanwise length.
(3) Standard value of p in ft lb sec unitis is 0.0765.

b) *Basic formulae*

It is assumed that aerodynamic scale effect is absent (i.e. aerodynamic moments are independent of Reynold's number). In this case, if a simple harmonic oscillation is expressed in the usual complex notation by

$$\theta = \theta_0 \, e^{2\pi N t i} \left(\text{where } i = \sqrt{-1}\right),$$

the aerodynamic moment M_θ takes the form.

$$M_\theta = \rho B^2 V^2 \theta_0 \left[f(V_r) + i g(V_r)\right] e^{2\pi N t i} \tag{1}$$

where $f(V_r)$ and $g(V_r)$ depend on the reduced velocity V_r only.

The conditions for free simple harmonic oscillations are

$$\rho B^2 V^2 g(V_r) = 2\pi N K_\theta \tag{2}$$

and

$$\rho B^2 V^2 f(V_r) = -4\pi^2 N^2 I_\theta + e_\theta . \tag{3}$$

Equations (2) and (3) may be re-written

$$V_r^2 \, g(V_r) = \frac{2\pi K_\theta}{\rho B^4 N} \tag{4}$$

$$V_r^2 f(V_r) = \frac{e_\theta}{\rho B^4 N^2} - \frac{4\pi^2 I_\theta}{\rho B^4} = \frac{4\pi^2 I_\theta}{\rho B^4}\left(\frac{N_\theta^2}{N^2} - 1\right). \tag{5}$$

The possible critical values of V_r for simple harmonic oscillations (i.e. the roots of equations (4) and (5)) will have the same values for prototype and model provided both systems have equal values of

$$\frac{I_\theta}{\rho B^4}, \qquad \frac{K_\theta}{\rho B^4 N}, \qquad \frac{e_\theta}{\rho B^4 N^2}.$$

The practical interpretation of these formulae is greatly facilitated if (as is usually true) the changes of frequency due to the wind are small.

In this case the influence of the term $f(V_r)$ in (5) is negligible, and as an approximation $N = N_\theta$. If this value for N is substituted in (4) the equation becomes

$$V_r^2 g(V_r) = \frac{2\pi K_\theta}{\rho B^4 N_\theta} = \frac{4\pi I_\theta \delta_\theta}{\rho B^4}. \tag{6}$$

The critical values of V_r then depend only on the geometric shape of the structure and the ratio [4] $\dfrac{I_\theta \delta_\theta}{\rho B^4}$.

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[4] This condition is equally valid if the structural damping is represented by a phase lead on the elastic restoring force. The expression for free simple harmonic oscillations is then written

$$I\ddot{\theta} + \sigma e^{i\varepsilon}\theta = \rho B^2 V^2 \theta_0 \left[f(V_r)\right] + i g(V_r)\right] e^{2\pi N T i} \tag{i}$$

where $e_\theta = \sigma \cos \varepsilon$

If $N = N_\theta$

$$V_r^2 g(V_r) = \frac{\sigma \sin \varepsilon}{\rho B^4 N_\theta^2}. \tag{ii}$$

Substituting

$$\delta_\theta = \frac{\sigma \sin \varepsilon}{4\pi I N_\theta^2} \quad \text{in (ii)}$$

$$V_r^2 g(V_r) = \frac{4\pi I_\theta \delta_\theta}{\rho B^4} \tag{6}$$

If accented and unaccented symbols refer to the model and prototype respectively, $\rho' = \rho$ for tests in an atmospheric wind tunnel, and the only conditions necessary for equality of the values of V_r for model and prototype are geometric similarity and

$$\frac{\delta_\theta'}{\delta_\theta} = \left(\frac{I_\theta}{I_\theta'}\right)\left(\frac{B'}{B}\right)^4 . \tag{7}$$

The corresponding expressions for vertical motions is

$$\frac{\delta_z'}{\delta_z} = \left(\frac{I_z}{I_z'}\right)\left(\frac{B'}{B}\right)^4 . \tag{8}$$

where I_z = mass per unit spanwise length.

Model and test conditions

The tests were made on 1/100-scale models of representative types of representative types of suspended structures. Diagrams of the sections of the four truss-stiffened types so far tested are given in fig. 1. All were fitted with stiffening trusses of the single Warren type (see fig. 2) with a full-scale truss panel length of 60 feet and a depth of either 25 or 27.5 feet. A few tests were made with the girder-stiffened sections shown in fig. 3.

A spanwise length of the section to be tested was mounted in a 4 ft square wind tunnel so that it could either oscillate vertically or in pitch about a selected spanwise axis. No attempt was made to reproduce on the models the values of the inertial and elastic coefficients required for full dynamic similarity, as indicated by equations (4) and (5). With all the sections tested it was found that the condition $N = N_0$ was approximately satisfied, so that equation (6) could be used. The values of V_r obtained on the model were directly applicable to a prototype with the following logarithmic decrements due to structural damping :—

$$\delta_\theta = 0.03 \text{ to } 0.04$$
$$\delta_z = 0.07 .$$

The maximum values of V_r obtainable in the tests were approximately 17 and 13 for pitching and vertical oscillations respectively and in both cases corresponded to wind speeds over the prototype of more than 100 miles per hour. The inclination of the wind was in all cases normal to the span but was varied from the horizontal over an angular range from — 15 to 15 degrees.

Discussion of results

Notes :

All dimensions quoted have been converted to refer to full-scale;

Unless otherwise specified the results refer generally to all types of section shown in fig. 1;

Except for those described under (*d*) the tests on pitching instability were made with the central rotational axis shown in fig. 6.

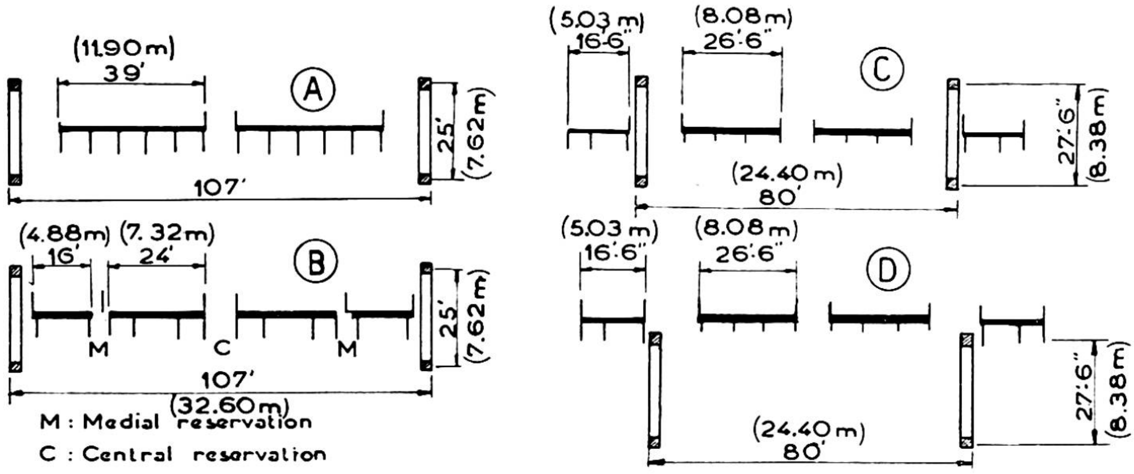


Fig. 1. Diagrams of bridge sections tested.

PITCHING OSCILLATIONS

The effect of various modifications and additions to the model are listed below.

a) *Modifications to Stiffening Trusses*

Width of chords : An increase in the width/depth ratio of the stiffening truss chords improved the stability;

Additional Members : The insertion of extra diagonal members (see fig. 2) improved the stability by moving the instability region to higher angles of incidence;

Depth : Increase of the depth of the truss, the same panel length and general structural form being retained, had no significant effect.

b) *Modifications to Decks*

Width and covering of reservations : The division of the total roadway width into a number of parallel tracks, each separated by an unblocked reservation, (5) was very benificial. Indeed, an unblocked reservation was found to be an essential feature for the attainment of a high degree of stability. Section B had in general better stability characteristics than those of section A. Improvements were also obtained with increase in the width of the unblocked reservations;

Position and type of roadway stringers and of handrailing : The stability of all sections tested was very sensitive to the form of fittings carried

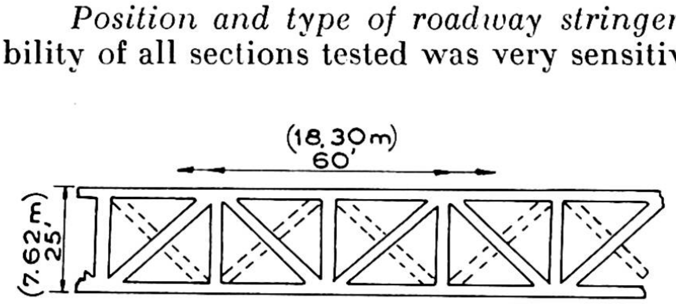


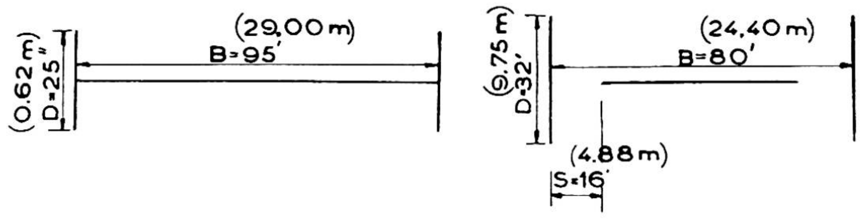
Fig. 2. General arrangement of stiffening truss.

——— Normal truss.

– – – – Additional members.

(5) An unblocked reservation is defined here as a reservation which is completely uncovered or is covered by a grating which permits easy flow of air through it (see fig. 1).

Fig. 3. Line diagrams of girder stiffened sections used in vertical oscillations tests.



Section E $\frac{D}{B} = 0.26$

Section F $\frac{D}{B} = 0.4$ $\frac{S}{B} = 0.2$

by the decks, such as stringers and handrails. Sections with plain decks, without stringers or handrails were unstable at both negative and positive angles of incidence but were stable at zero incidence. The sections were also unstable if either the handrails or the outermost stringers were of solid plate construction. Substantial improvements were effected by fitting designs of handrails and stringers which acted as aerodynamic spoilers by shedding eddies of mixed frequencies along the span. Such spoilers on the deck upper surfaces tended to suppress instability at positive angles of incidence while those on the lower surfaces improved stability at negative angles. By suitable choices of handrail and stringer design, and with unblocked reservations, it was found possible to obtain a high degree of stability for all the sections of fig. 1. Truss type deck stringers, if sufficiently deep [6], proved to be fairly effective aerodynamic spoilers and helped considerably to promote stability. By comparison with truss stringers, plate stringers were ineffective and models with all decks fitted with plate stringers were unstable. However, the stability characteristics of section C with truss stringers under the sidetracks and shallow plate stringers under the carriageways were not much less favourable than when all the decks were fitted with truss stringers. The use of paling handrails (fig. 4*a*) was beneficial but their effectiveness as spoilers was much augmented by the addition of solid castellations to provide larger-scale break up of the airflow. Two forms of castellated handrail are shown in fig. 4; the most effective of which was the single spaced castellated handrailing (fig. 4*b*). By the omission of alternate castellations a handrail of much better appear-

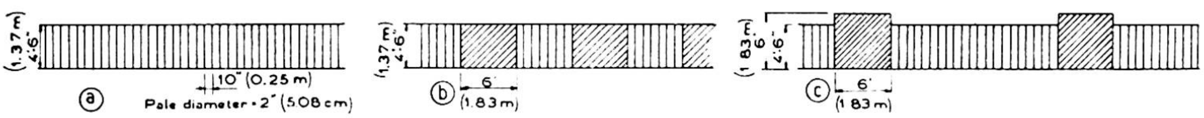


Fig. 4. Types of handrailing :
a. Paling handrailing; **b.** Single-spaced castellated handrailing; **c.** Modified castellated handrailing.

ance (fig. 4*c*) was produced. Although this type was not so efficient as a stabilising agent as that of fig. 4*b*, the advantages gained by the use of castellations were still considerable. Further arrangements of castellated handrailing, involving less frontal area and wider spacing of the castellations, were tested with a view to producing a form of handrail which would be satisfactory from both architectural and aerodynamic considerations. Most of these arrangements showed only small improvements

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(6) Truss stringers of depths 6 feet, 5 feet and 3 feet were used in the tests. Those of depth 6 feet and 5 feet were the most satisfactory.

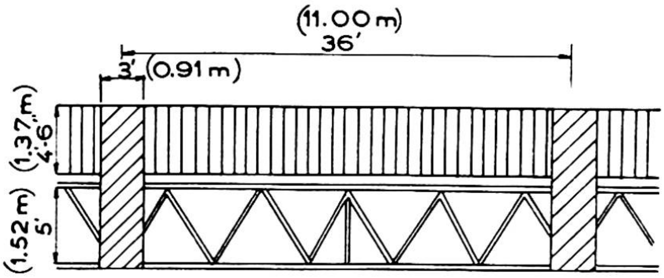

Fig. 5. Castellations extending over stringers and handrails.

on the plain paling handrailing. The castellations were not found to lose their effectiveness when the sets for the various handrails were staggered spanwise relative to each other by various amounts. A further type of castellation, used on sections C and D, was fitted to the outside edge of the sidetrak and extended over the handrails and stringers (see fig. 5). These were effective as stabilisers when placed immediately opposite the vertical posts of the stiffening truss, but were ineffective if displaced spanwise by half the distance between the posts. This result indicates that the stability of a section fitted with these castellations might prove to be sensitive to horizontal wind direction;

Traffic : In view of the marked influence on stability exerted by fittings to the deck surfaces it was desirable to determine the aerodynamic effect of traffic. Several arrangements of models of various types of vehicular traffic were placed on the decks of section B in random order and in different degrees of congestion. None of these had any adverse effect;

Vertical position of the decks : With the central axis of rotation there was little significant difference between the stability characteristics of sections C and D; those of section D were perhaps slightly inferior to those of section C. Some improvement was effected by raising the decks of section D to give a clearance of 2 feet between the bottom edges of the plate stringers used under the carriageways and the plane through the top surfaces of the upper chords of the stiffening trusses.

c) *Type and Position of Lateral Wind Bracing*

Tests on the type and position of the lateral wind bracing were made on model C only. The results indicate that, while the influence of the type of wind bracing located at the level of the stiffening truss bottom chords was not great, lattice type bracing yielded slightly better stability characteristics than the plate girder types. An improvement was obtained by raising the lattice type wind bracing to a position just underneath the roadway stringers. This result is consistent with a spoiling effect in proximity to the decks due to the bracing.

d) *Influence of Axis of Rotation*

The effects of the various modifications to the bridge sections mentioned in the preceding paragraphs were studied by experiments with a spanwise axis of rotation located at an approximately central position with respect to the four stiffening truss chords. A series of experiments was carried out on section B (modified to have stiffening trusses 120 ft apart) in which the pitching axis was moved to the several positions shown in fig. 6*a*. Axes located vertically up, or down, or horizontally upstream from

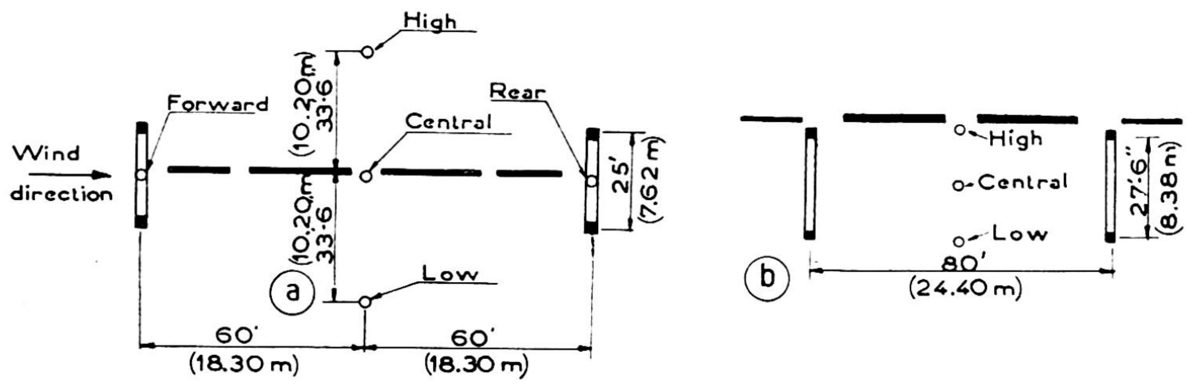

Fig. 6. Location of pitching axes.

the central position proved to be beneficial. A downstream movement of the axis had an adverse effect on stability (7).

Of the three axis positions tested with section D the upper position shown in fig. 6*b* was the least favourable for stability.

Thus for axes lying midway between the stiffening trusses the stability of both sections B and D was least when the axis was located near deck level.

e) *Types of Section*

By incorporating a number of the stabilising features mentioned in the preceding paragraphs high stability in pitch about the central axis was obtained for all sections shown in fig. 1. Sections C and D required the least number of these features and in that respect may be classed as the most stable sections. On a similar basis section B was more stable than section A.

VERTICAL OSCILLATIONS

No vertical oscillations could be excited for any condition of the truss stiffened sections shown in fig. 1. These conditions included that of a solid deck extending over the whole area between the stiffening trusses and fitted with plate handrails of height 4.5 feet and plate girder stringers of depth 6 feet.

Both the plate girder stiffened sections shown in fig. 3 showed instability at low incidences. The instability of section E is not inconsistent with Steinman's criterion for stability $D/B < 0.25$.

General conclusions

There is as yet insufficient knowledge of the problem to enable the aerodynamic stability of a proposed design of suspension bridge to be predicted without recourse to oscillatory tests on models. Both model and

(7) In view of the symmetry of the distribution of mass and elastic stiffness of suspension bridges, any lateral displacement of the effective axis of rotation must arise from aerodynamic couplings between the pure flexural and torsional motions about the axis of symmetry. It is hoped that evidence on this matter will be forthcoming from a study of the modes of oscillation of the full model.

full-scale experience show that the stability of plate girder stiffened bridges compares unfavourably with that of truss-stiffened bridges. The present investigation indicates that if the results of the sectional tests are confirmed by full-model tests, truss stiffened bridges of the types tested are stable in vertical motion and the tendency to instability in torsional oscillations can be corrected by incorporating a number of stabilising features in the design. These include :—

Stiffening truss chords of high width/depth ratio;
Traffic lanes separated from each other by open slots or gratings;
Truss type deck stringers in preference to the plate girder type;
Castellated handrails, or other types of handrailing designed to break up the spanwise continuity of the airflow pattern;
Lattice type wind bracing fitted near deck level;
Sidetracks (e.g. footpaths, cycle tracks) mounted outboard of the stiffening truss.

Acknowledgements

The work described above was carried out in the Aerodynamic Division of the National Physical Laboratory for the Ministry of Transport, by whose permission this paper is published. Valuable help is being received from representatives of the Consulting Engineers, Mssrs. Mott, Hay and Anderson and Messrs. Freeman, Fox and Partners.

To Dr. R. A. Frazer, F. R. S., who is in charge of the aerodynamic investigation, and to other colleagues associated with the work, the author is heavily indebted for valuable advice and close collaboration.

Résumé

Le présent mémoire donne un court aperçu sur les recherches sur la stabilité aérodynamique des ponts suspendus, réalisées dans la division aérodynamique du National Physical Laboratory. Les essais exécutés jusqu'à présent l'ont été sur des éléments de modèles; des travaux sont en cours pour réaliser des essais sur des modèles entiers d'une longueur maximum de 60 pieds dans un tunnel aérodynamique *ad hoc*. Ce mémoire donne les résultats obtenus sur des éléments de modèles et qui ont permis d'établir des détails de construction donnant une stabilité plus favorable.

Zusammenfassung

Die vorliegende Arbeit gibt einen kurzen Ueberblick über eine Untersuchung bezüglich der aerodynamischen Stabilität der Hängebrücken, die in der Aerodynamischen Abteilung des National Physical Laboratory durchgeführt wurde. Bis jetzt wurden die experimentellen Untersuchungen auf Versuche an Teilmodellen beschränkt, doch sind die Vorbereitungen für Versuche an vollständigen Modellen bis zu einer Länge von 60 Fuss in einem speziell für diesen Zweck gebauten Windkanal demnächst beendet. Es wer-

den die bis jetzt erhaltenen Ergebnisse an Teilmodellen angegeben und die zur Erlangung grösserer Stabilität günstigen Konstruktionsdetails beschrieben.

Summary

The paper presents a brief review of an investigation on the aerodynamic stability of suspension bridges which is being carried out in the Aerodynamics Division of the National Physical Laboratory. Experiments have so far been confined to tests of sectional models but preparations are nearly complete for testing full models of lengths up to 60 feet in a large wind tunnel specially constructed for the purpose. The results so far obtained in sectional model tests are discussed, and design features favourable to the promotion of stability are indicated.

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Quelques détails sur le montage des ponts en arc métalliques

Einige Angaben über die Montage stählerner Bogenbrücken

Some details about the erection of steel arch bridges

PROF. IR. A. ROGGEVEEN
Wassenaar (Hollande)

A stiffened flexible arch bridge with the arch above the stiffening girder can be erected in very much the same way as an ordinary beam bridge, i. e. on a large number of jacks, either hydraulic or screw jacks, resting on a centering under the entire length of the bridge. On these supports the stiffening girder and the system of floorbeams are laid out and by jacking the designed camber is obtained.

On top of the stiffening girders the hangers are erected and on these the arches. By means of the jacks care is taken that throughout these operations the right camber, computed for the unstressed condition, is maintained.

Thereupon the holes for the fieldrivets may be reamed in the arches and in the stiffening girders after which the rivets can be driven and when this has been done the bridge is gradually jacked down until it carries its own weight.

For this type of erection a large number of supports has to be placed in the river or whatever the bridge has to cross. In the case of a navigable stream, this may entail closing the fairway to traffic or building over at least part of it an erection-bridge, on the top of which the permanent bridge can be built.

In such an instance a stiffened flexible arch bridge may have advantages, as this type of bridge can be made to work as its own erection bridge, thus avoiding the need for a separate one. By doing so there is an added advantage, as by concentrating the erection supports at a few points the total cost of supports is usually considerably less, due to the fact that it is far easier to obtain sufficient stiffness for one large support than for a number of supports which, together, are supposed to carry the same load as the large one.