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## IVb1

**Calcul approché des dalles rectangulaires en béton armé  
pour une charge uniformément répartie ou hydrostatique**

**Näherungsmethode zur Berechnung von rechteckigen Platten  
aus Eisenbeton bei gleichmässig verteilter  
und hydrostatischer Belastung**

**Approximative method of analysis for rectangular reinforced  
concrete plates under uniformly distributed or hydrostatic load**

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### Introduction

As a direct integration of the differential equation of the plate is in general not possible for rectangular plates, supported at the edges, several methods have been developed to cope with this difficulty, using double and single Fourier series, differences equations, etc. Some of these methods lead to sufficiently accurate results. They are, however, rather laborious, especially if all kinds of boundary conditions have to be taken into account, whilst no usable general formulae for bending moments, etc. are obtained. It is true Marcus<sup>(1)</sup> gave relatively simple formulae for rectangular plates with uniformly distributed load, but these have not been derived directly. They have been composed in such a manner that they approximate as much as possible the results of his more accurate calculation<sup>(2)</sup>, which leads only to numerical results.

At the other hand our method<sup>(3)</sup> is rather simple and gives a clear insight in the way in which the plates carry the load. Moreover it leads

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<sup>(1)</sup> MARCUS, *Die vereinfachte Berechnung biegsamer Platten*, Springer, Berlin, 1925.

<sup>(2)</sup> MARCUS, *Die Theorie elastischer Gewebe und ihre Anwendung auf die Berechnung biegsamer Platten*, Springer, Berlin, 1924.

<sup>(3)</sup> BIJLAARD, *De Ingenieur*, n° 26, 1934, n° 23, 1935; *De Ingenieur in Ned. Indië*, n° 12, 1935; *Proc. Third Engineering Congress*, Tokio, 1936.

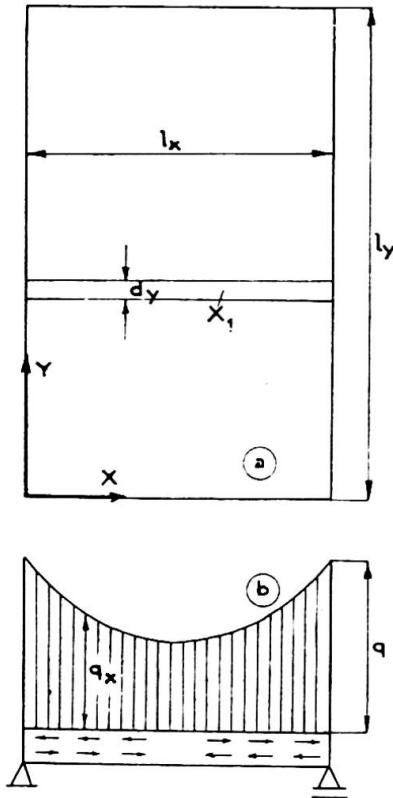


Fig. 1.

to simple general formulae for the maximum bending moments as functions of the ratio  $\varepsilon = l_y/l_x$  of the sides, which are similarly built for all boundary conditions, for uniformly distributed as well as for hydrostatic load. Furthermore our formulae are more accurate than those of Marcus.

### Simply supported plates

Let us consider first a rectangular plate (fig. 1a) with simply supported edges, carrying an arbitrary load  $q = f(x, y)$ . An arbitrary X-strip,  $X_1$ , of this plate, having a breadth  $dy$ , will only have to carry a vertical load  $q_x$ , the remainder of the load  $q$  being carried by the vertical shearing forces acting on its sides. The bending moment per unit breadth of the strip, due to this vertical load  $q_x$ , we denote by  $M_{xv}$ .

As, however, also twisting moments  $M_{yx}$  and  $M_{yx} + \frac{\partial M_{yx}}{\partial y} dy$  are acting on the sides of the strip, which cause bending moments  $M_{x\tau}$  per unit breadth of its cross sections, the real bending moment per unit breadth of the strip amounts to

$$M_x = M_{xv} + M_{x\tau} \quad (1)$$

so that

$$M_{xv} = M_x - M_{x\tau} \quad (2)$$

We know that <sup>(4)</sup>

$$M_x = -D \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \quad (3)$$

<sup>(4)</sup> TIMOSHENKO, *Theory of Plates and Shells*, 1940, p. 88.

whilst

$$M_{y,x} = -D(1 - \nu) \frac{\partial^2 w}{\partial x \partial y}$$

acting on strip  $X_1$  as indicated in fig. 1b. Consequently the total moment exercised per unit length of the strip by the twisting moments  $M_{y,x}$  and  $M_{y,x} + \frac{\partial M_{y,x}}{\partial y} dy$  is

$$\frac{\partial M_{y,x}}{\partial y} dy = -D(1 - \nu) \frac{\partial^3 w}{\partial x \partial y^2} dy$$

acting on the strip in the opposite direction of the arrows in fig 1b. Hence the bending moment  $M_{x,x}$  per unit breadth of the strip, caused by the twisting moments, amounts to

$$M_{x,x} = -D(1 - \nu) \int_x^{l_x} \frac{\partial^3 w}{\partial x \partial y^2} dx = -D(1 - \nu) \left[ \frac{\partial^2 w}{\partial y^2} \right]_x^{l_x} = D(1 - \nu) \frac{\partial^2 w}{\partial y^2} \quad (4)$$

a similar result being already obtained by Marcus.

But with eqs. (3) and (4) it follows now from eq. (2) that

$$M_{xv} = -D \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \quad (5)$$

whence we obtain, by changing  $x$  and  $y$

$$M_{yv} = -D \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \quad (6)$$

so that we draw the conclusion that

$$M_{xv} = M_{yv} \quad (7)$$

Hence at any point of a rectangular plate with simply supported edges and arbitrary load, the bending moment  $M_{xv}$  per unit breadth, that would occur in an X-strip, if it had to carry its total vertical load  $q_x$  as a simple beam, without being discharged by the twisting moments  $M_{y,x}$ , is equal to the bending moment  $M_{yv}$ , occurring at the same point in an Y-strip, if it would have to carry its total vertical load  $q_y$  as a simple beam, whilst of course

$$q_x + q_y = q \quad (8)$$

At the other hand we know that, according to eq. (3), whence  $M_y$  follows by changing  $x$  and  $y$ , we have

$$M_x + M_y = -D(1 + \nu) \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \quad (9)$$

so that it follows from eqs. (5) and (6) that

$$M_x + M_y = (1 + \nu) M_{xv} = (1 + \nu) M_{yv} \quad (10)$$

With structures in reinforced concrete, where Poisson's ratio is usually equated to zero, this yields

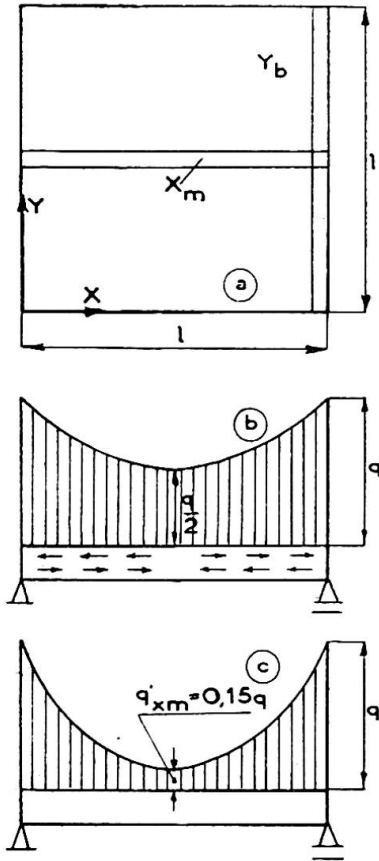


Fig. 2.

$$M_x + M_y = M_{xv} = M_{yv} \tag{11}$$

or  $M_x = M_{xv} - M_y \tag{12}$

and  $M_y = M_{yv} - M_x \tag{13}$

Furthermore it follows from eq. (4) and also by comparison of eqs. (1) and (12) that, with  $\nu = 0$ ,

$$M_{xv} = -M_y \tag{14} \quad \text{and} \quad M_{yv} = -M_x \tag{15}$$

the latter equation following from the first one by changing  $x$  and  $y$ .

For example we shall use eq. (11) in order to find the bending moment in the centre of a square plate with sides  $l$  which carries a uniform load  $q = \text{const.}$  (fig. 2 a). As at the edges  $M_x = M_y = 0$  according to eq. (10) in any section of a boundary strip  $Y_b$  the moment  $M_{yv}$  will be zero too, so that for these strips  $q_y$  must be zero. Therefore, according to eq. (8), at  $x = 0$  and  $x = l$  the middlemost  $X$ -strip  $X_m$  has to carry a load  $q_x = q$ , whilst at  $x = l/2$ , on account of symmetry,  $q_x = q_y = q/2$ , so that  $q_x$  is distributed according to fig. 2b. Assuming a parabolic limitation of  $q_x$  we find that in the middle of  $X_m$

$$M_{xv} = \frac{1}{8} q l^2 - \frac{5}{48} \frac{q}{2} l^2 = \frac{7}{96} q l^2$$

so that, because in the centre of the plate  $M_x = M_y$ , eq. (11) yields

$$M_x = M_y = \frac{1}{2} M_{xv} = \frac{7}{192} q l^2 = 0.0365 q l^2 .$$

According to Nadai <sup>(5)</sup> the real bending moment with  $\nu = 0$  is  $0.0368 q l^2$ , so that our result is sufficiently accurate.

In order to find the equation of the deflection surface we remark that according to eq. (14) the moment  $M_{xv}$  in  $X_m$  is distributed as  $-M_y$ , consequently practically as the negative deflection  $w_x$  of  $X_m$ , so that it may be

<sup>(5)</sup> NADAI, *Elastische Platten*, Berlin, 1925.

assumed to be caused by a fictitious load, being distributed according to the second differential quotient of  $w_x$ , consequently as  $-M_x$ . Assuming again a parabolic limitation of the total fictitious load  $q_x'$  according to fig. 2c, it follows that

$$M_{xm} = \frac{1}{8} q l^2 - \frac{5}{48} (q - q'_{xm}) l^2 = \frac{7}{192} q l^2$$

or  $q'_{xm} = 0.15 q$ . The deflection  $w_x$  of  $X_m$  follows from  $q_x'$  by integrating four times and dividing by  $EI$ . Assuming the deflections of the other X-strips to be proportional to those of  $X_m$ , we obtain in this way the deflection surface of the square plate

$$w = \frac{8}{104\,175} \frac{q l^4}{EJ} (24 \xi - 65 \xi^3 + 75 \xi^4 - 51 \xi^5 + 17 \xi^6) \\ (24 \eta - 65 \eta^3 + 75 \eta^4 - 51 \eta^5 + 17 \eta^6) \quad (16)$$

in which  $\xi = x/l$  and  $\eta = y/l$  and from which the bending and twisting moments at any point may be calculated immediately. These are situated fairly between the values obtained by Marcus <sup>(2)</sup> and Lewe <sup>(6)</sup>, as was shown in our third paper in footnote 3. In a similar way we found for a square plate with a hydrostatic load  $q = xp/l$  the deflection surface

$$w = \frac{8}{2\,187\,675} \frac{\rho l^4}{EJ} (208 \xi - 217 \xi^3 - 42 \xi^5 + 51 \xi^7) \\ (24 \eta - 65 \eta^3 + 75 \eta^4 - 51 \eta^5 + 17 \eta^6) \quad (17)$$

Various boundary conditions

If an X-strip is for example fixed at two sides (fig. 3), the moment  $M_{x\tau} = -M_y$ , that would be caused in it by the twisting stresses  $\tau_{yx}$  if it were simply supported, being  $-M_{ym}$  in the middle of the strip, will cause moments  $\varphi_{xc} M_{ym}$  at the clamped edges (fig. 3b and 3c), by which at the crossing M of the most loaded strips  $X_m$  and  $Y_m$  we get, instead of eqs. (12) and (13)

$$M_{xm} = M_{xv} - \varphi_x M_{ym} \quad (18)$$

and

$$M_{ym} = M_{yv} - \varphi_y M_{xm} \quad (19)$$

As for example along  $X_m$  the moment  $M_{x\tau} = -M_y$  is about proportional to the deflection  $w_x$  of  $X_m$ , values  $\varphi_x$  or  $\varphi_y$  may be calculated if the ratios of the ordinates of  $w_x$  or  $w_y$  are known.

Also to this effect we remark that, by the twisting moments  $M_{xy}$  alone, the Y-strips, that cross  $X_m$ , would obtain deflections  $w_{y\tau}$  according to curve 1 in fig. 3d, being proportional to  $M_{y\tau} = -M_x$ . As at their crossing with  $X_m$  their real deflections coincide with the deflection  $w_x$  of  $X_m$ , given by curve 2, the part  $q_y$  they take there of the total load  $q$  will be about proportional to  $w_x - w_{y\tau}$ , being the distance between curves 1 and 2. Assuming a uniformly distributed load  $q$ , we therefore find the

(6) LEWE, *Pilzdecken*, Berlin, 1926.

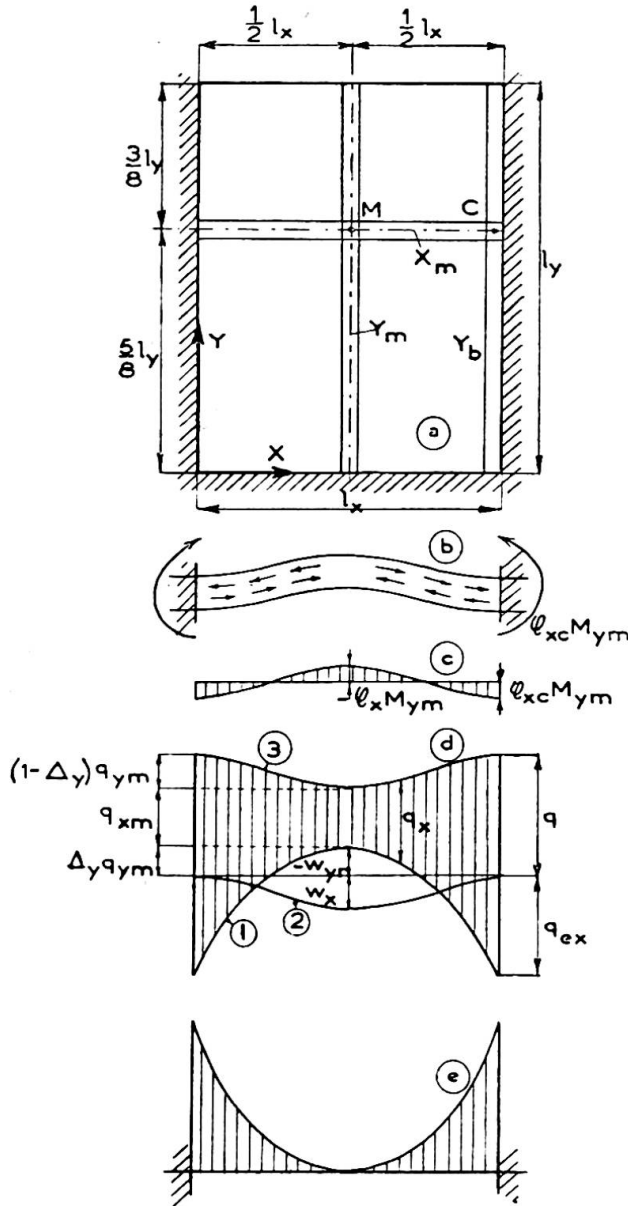


Fig. 3.

vertical load  $q_x$  on  $X_m$  by drawing curve 3 at a distance  $q$  above curve 2, load  $q_x$  being denoted by cross hatching. It follows that near a clamped edge the Y-strips cause an excess load  $q_{ex}$  on  $X_m$  instead of discharging it. Value  $q_{ex}$  was computed from the condition that the load  $q_y$ , taken by the boundary strip  $Y_b$ , being  $q_y = -q_{ex}$  at C, has to cancel the bending moments

$$M_{y\tau} = -M_{xc}$$

caused by the twisting stresses  $\tau_{yx}$ , in which  $M_{xc}$  is the mo-

ment  $M_x$  along the clamped edge. For a square plate with all edges built in  $q_{ex}$  is e.g.  $0.61 q$ .

At the crossing M of the most loaded strips (fig. 3a)

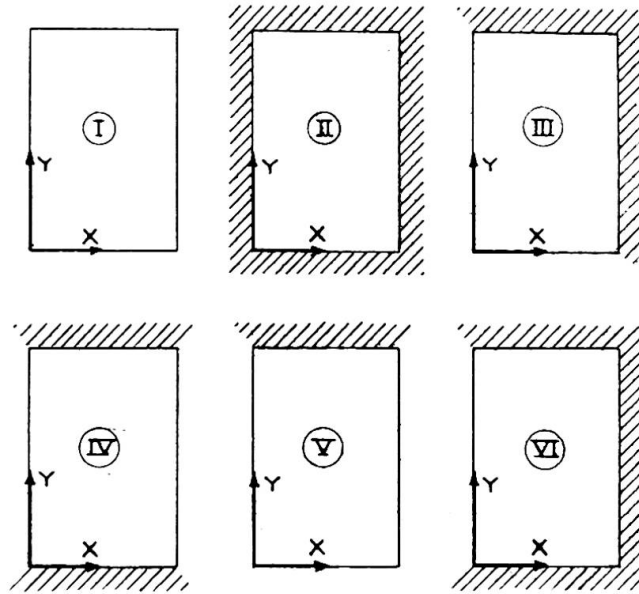
$$q_{xm} + q_{ym} = q \tag{20}$$

whilst the deflections of  $X_m$  and  $Y_m$  in M must be equal. Expressing these deflections in the maximum positive moments  $M_{xm}$  and  $M_{ym}$  and assuming for that the same relations which hold for uniform load, this condition gives us the relation

$$M_{xm} = \mu \varepsilon^2 M_{ym} \tag{21}$$

in which  $\varepsilon = l_y/l_x$ , whilst for the six cases we considered (fig. 4, where a single line denotes a simply supported and cross hatching a clamped edge)  $\mu$  is 1, 1, 1, 0.6,  $32/45$  and  $27/32$  respectively. Furthermore in fig. 3d the deflection  $w_{y\tau}$  of  $Y_m$  is the result of the twisting moments  $M_{xy}$ ,

Fig. 4.



which cause in M a moment  $-\varphi_y M_{xm}$  in  $Y_m$ . At the other hand  $w_x$  may be considered to be the result of the actual bending moment  $M_y$  in  $Y_m$ , being  $M_{ym}$  in M, so that, according to eq. (21) and fig. 3d we have e.g.

$$\Delta_y = \frac{-w_{y\tau}}{w_x - w_{y\tau}} = \frac{\varphi_y M_{xm}}{M_{ym} + \varphi_y M_{xm}} = \frac{\mu \varepsilon^2 \varphi_y}{1 + \mu \varepsilon^2 \varphi_y} \tag{22}$$

We could show <sup>(3)</sup> that for values  $\varepsilon$  between 0.5 and 2 it is sufficiently accurate to assume curve 1 in fig. 3d, the ordinates of which are proportional to  $-M_x$ , as a parabola, if  $M_{xm}$  and  $M_{ym}$  are considered as the maximum positive moments in the strips and not always as those in M. Then also curve 3, being proportional to  $w_x$ , may be easily found. For we showed above here that the effect of the twisting stresses on  $X_m$  is equivalent to that of a fictitious load proportional to  $-M_x$ , so that the deflection  $w_x$  is between that by a uniform and that by a parabolic load, the latter being indicated in fig. 3e. With known shapes of  $w_x$  and  $w_y$  values  $\varphi_x$ ,  $\varphi_{xc}$ ,  $\varphi_y$  and  $\varphi_{yc}$  can also be computed. The only unknown values in fig. 3d being now  $q_{xm}$  and  $q_{ym}$ , we may calculate

$$M_{xv} = m(q_{xm} + eq_{ym} + fq_{cx})l_x^2 \tag{23}$$

whilst in the same way we find

$$M_{yv} = n(q_{ym} + gq_{xm} + hq_{cy})l_y^2. \tag{24}$$

Consequently we have six equations, (18), (19), (20), (21), (23) and (24), with six unknown values,  $q_{xm}$ ,  $q_{ym}$ ,  $M_{xv}$ ,  $M_{yv}$ ,  $M_{xm}$  and  $M_{ym}$ . In this way we got the maximum positive moments  $M_{xm}$  and  $M_{ym}$ , whence we obtain also the clamping moments

$$M_{xc} = M_{xvc} + \varphi_{xc} M_{ym} \tag{25}$$

and 
$$M_{yc} = M_{yvc} + \varphi_{yc} M_{xm}. \tag{26}$$

For values  $\varepsilon = l_y/l_x$  between 0.5 and 2 we got

$$\left. \begin{aligned} &M_{xm} = mq_x' l_x^2 \quad \text{and} \quad M_{ym} = nq_y' l_y^2 \\ \text{with} \quad &q_x' = \beta \frac{\varepsilon^4}{\varepsilon^4 + \gamma \varepsilon^2 + \alpha'} q, \quad q_y' = \beta \frac{\alpha}{\varepsilon^4 + \gamma \varepsilon^2 + \alpha'} q \end{aligned} \right\} \tag{27}$$



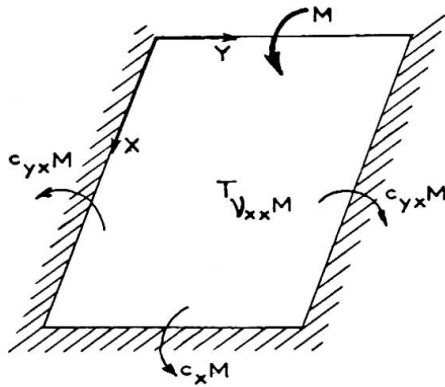


Fig. 5.

$$\left. \begin{aligned} & M_{xc} = m_c q_{xc}' l_x^2 \quad \text{and} \quad M_{yc} = n_c q_{yc}' l_y^2 \\ \text{with} \quad & q_{xc}' = \lambda_x \frac{\varepsilon^4 + \omega_x \varepsilon^2 + \rho_x}{\varepsilon^4 + \gamma \varepsilon^2 + \alpha'} q \leq q, \quad q_{yc}' = \lambda_y \frac{1 + \omega_y \varepsilon^2 + \rho_y \varepsilon^4}{\varepsilon^4 + \gamma \varepsilon^2 + \alpha'} q \leq q \end{aligned} \right\} \quad (28)$$

If we put  $q_{ex}$  and  $q_{ey}$  equal to their values for  $\varepsilon=1$ , values  $\beta$ ,  $\gamma$ ,  $\alpha'$ , etc. are constants<sup>(7)</sup>. Somewhat more accurate values could be obtained, however, by taking into account the variation of  $q_{ex}$  and  $q_{ey}$  with  $\varepsilon$ . In order to avoid that by this  $\beta$ ,  $\gamma$ ,  $\alpha'$ , etc. would become functions of  $\varepsilon$ , we expressed  $q_{ex}$  and  $q_{ey}$  in  $q_{xm}$  and  $q_{ym}$ , by which  $\beta$ ,  $\gamma$ ,  $\alpha'$ , etc. remained constant in the intervals  $\varepsilon=0.5-1$  and  $\varepsilon=1-2$ <sup>(8)</sup>. In table I, where cases I<sub>u</sub>-VI<sub>u</sub> refer to the 6 cases in fig. 4 and to uniform load, all values of eqs (27) and (28) are given.

Only for the moments along the short clamped edges it is not allowed to assume curve 1 in fig. 3d as a parabola. These clamping moments we derived as follows.  $M_{yc}$  for case II<sub>u</sub> we found by superimposing on the deflection of a square plate, with sides  $l_x$ , for which  $M_{yc}$  is known (see also under here), a deflection of one of the clamped edges according to that of the middle strip X<sub>m</sub> in case  $\varepsilon=2$ , yielding  $M_{yc} = -0.057 ql_x^2$ . Subsequently we found  $M_{yc}$  for case IV<sub>u</sub> and  $\varepsilon=2$  by remarking that here the X-strips, with lengths  $l_x$ , have the same rigidity as if they were clamped with lengths  $l_x' = l_x \sqrt[4]{5}$ , so that  $M_{yc}$  has about the same value as for case II<sub>u</sub> and  $\varepsilon = \frac{2}{\sqrt[4]{5}} = 1.34$ , being

$$M_{yc} = -0.057 ql_x'^2 = -0.127 ql_x^2.$$

As, however, here  $q_{ey} = 1.22 q$  instead of  $1.32 q$  for case II<sub>u</sub>, we have to multiply this value with about  $2.22/2.32$ , yielding  $M_{yc} = -0.120 ql_x^2$ . Using the carry-over factors, mentioned under here, we computed from these values  $M_{xc}$  and  $M_{yc}$  for  $\varepsilon=0.5$  and 2 for the other cases (fig. 4). Demanding that  $dM_c/d\varepsilon$  is continuous at  $\varepsilon=1$  and zero at  $\varepsilon=0.5$  and 2, we obtain for  $\varepsilon < 1$  and  $\varepsilon > 1$  respectively

$$M_{xc} = [-A + B(\varepsilon - 0.5)^p] ql_x^2 \quad \text{and} \quad M_{yc} = [-C + D(2 - \varepsilon)^p] ql_x^2 \quad (29)$$

<sup>(7)</sup> BIJLAARD, *De Ingenieur*, n° 23, 1935.

<sup>(8)</sup> BIJLAARD, *De Ingenieur in Ned. Indië*, n° 12, 1935.

in which A, B, C, D and  $p$  are given in table II. For case VI and  $\varepsilon < 1$  we got

$$M_{xc} = [-0.057 - 0.06(\varepsilon - 0.5)^2 + 0.35(\varepsilon - 0.5)^5] ql_y^2. \quad (30)$$

In the same way we examined cases I and II (fig. 4) for hydrostatic load  $q = xp/l_x$ . As we considered both cases separately and disposed already of the data for uniform load, we could approximate the shapes of curves 1 and 3 (cf. fig. 3d), for which in case II we used curves of the fifth and seventh degree respectively, still better than before. Eqs. (27) and (28) hold here as well, the coefficients being given in table III (cases  $I_h$  and  $II_h$ ), in which  $M_{xcu}$  and  $M_{xcl}$  refer to the upper ( $x=0$ ) and lower edge ( $x=l_x$ ). For the clamping moments at the shorter edges we got for case  $II_h$  and  $\varepsilon < 1$

$$M_{xcu} = 0.002(9 - 40\varepsilon + 22\varepsilon^2)pl_y^2 \quad \text{and} \quad M_{xcl} = 0.0066(4\varepsilon - 9)pl_y^2 \quad (31)$$

whilst for  $\varepsilon > 1$  the maximum clamping moment  $M_{yc} = -0.028 pl_x^2$ . The better approximation appears from the fact, that with the values in table III it follows from case  $II_h$  and  $\varepsilon = 1$  a clamping moment for uniform load  $p = q$ , being the sum of  $M_{xcu}$  and  $M_{xcl}$ , the value  $M_{xc} = -0.0507 ql_x^2$ , whilst with our values for case  $II_u$  in table I we find directly  $M_{xc} = -0.0529 ql_x^2$ , the accurate value being  $M_{xc} = 0.0513 ql_x^2$  <sup>(9)</sup>. In the same way other cases may be examined. It is, however, easier to determine from our values for uniform load the carry-over factors  $c$  according to the Cross method of moment distribution (fig. 5). It is for example obvious that, with reference to the moments at the middles of the edges,

$$c_x = \frac{M_{xcVI} - M_{xcII}}{M_{xcII}} \quad \text{and} \quad c_{yx} = \frac{1}{2}(1 - c_x) \frac{M_{ycIV} - M_{ycII}}{M_{xcII}}.$$

In a similar way we calculated the influence values  $\nu$  of an edge moment  $M$  (fig. 5) for the positive moments in the slab. Furthermore we calculated the factors with which the clamping moments at the middle of the sides have to be multiplied to get the maximum edge moments. We found for example the coefficients for case  $VI_h$  for hydrostatic load (edge  $x=0$  simply supported), as given in table III, by superimposing the influence of a moment, opposite to  $M_{xcu}$ , on the moments for case  $II_h$ . We have e.g.  $M_{xcIV} = M_{xcII} + c_x M_{xcuII}$ . The clamping moments at the shorter sides are

$$M_{xcl} = [-0.0352(2 - \varepsilon) + 0.072(1 - \varepsilon)^2 - 0.09(1 - \varepsilon)^3]pl_y^2$$

$$\text{and} \quad M_{yc} = [-0.0367 + 0.0074(2 - \varepsilon)^4]pl_x^2. \quad (32)$$

Moreover, using the carry-over factors  $c$ , we are able to calculate continuous floor slabs, starting from case II for all plates, according to the Cross method, whereby we can also allow for the torsional rigidities of the beams. After having found the final edge moments we correct the positive moments in the slabs by means of the partial influence values  $\psi$  of the edge moments, being valid if each edge moment varies separately.

These values  $\psi$  were calculated in another way by Bittner <sup>(10)</sup> (his values  $\mu$ ). From our data for uniform and hydrostatic load we determined

<sup>(9)</sup> TIMOSHENKO, *Theory of Plates and Shells*, 1940, p. 228.

<sup>(10)</sup> BITTNER, *Momententafeln und Einflussflächen für kreuzweise bewehrte Eisenbetonplatten*, Vienne, 1938.

Case	m	n	m <sub>c</sub>	n <sub>c</sub>	α	ε = 1 - 2						ε = 0,5 - 1					
						β	α'	γ	λ <sub>x</sub>	ω <sub>x</sub>	ρ <sub>x</sub>	β	α'	γ	λ <sub>y</sub>	ω <sub>y</sub>	ρ <sub>y</sub>
I <sub>u</sub>	1/8	1/8	—	—	1	7/6	1	2	—	—	—	7/6	1	2	—	—	—
II <sub>u</sub>	1/24	1/24	-1/12	-1/12	1	1.21	0.91	0.86	1.18	0.27	0.22	1.32	1.09	0.94	1.30	0.27	0.22
III <sub>u</sub>	9/128	9/128	-1/8	-1/8	1	1.20	0.94	1.33	1.20	0.37	0.24	1.29	1.07	1.42	1.28	0.37	0.24
IV <sub>u</sub>	1/8	1/24	—	-1/12	5	1.26	5.22	3.08	—	—	—	1.43	6.19	3.34	6.88	0.22	0.07
V <sub>u</sub>	1/8	9/128	—	-1/8	2.5	1.22	2.53	2.64	—	—	—	1.26	2.69	2.72	3.11	0.32	0.12
VI <sub>u</sub>	9/128	1/24	-1/8	-1/12	2	1.23	1.94	1.55	1.23	0.40	0.38	1.36	2.29	1.68	2.66	0.25	0.14

TABLE I

Case	A	B	C	D	p
II <sub>u</sub>	0.057	0.25	0.057	0.004/ε <sup>6</sup>	6
III <sub>u</sub>	0.08	0.19	0.08	0.006/ε <sup>5</sup>	5
IV <sub>u</sub>	—	—	0.12	0.05	2
V <sub>u</sub>	—	—	0.12	0.033	3
VI <sub>u</sub>	—	—	0.08	0.018	4

TABLE II

Case	m	n	m <sub>c</sub>	n <sub>c</sub>	ε = 1 - 2									
					β	β <sub>α</sub>	α'	γ	λ <sub>xu</sub>	ω <sub>xu</sub>	ρ <sub>xu</sub>	λ <sub>xl</sub>	ω <sub>xl</sub>	ρ <sub>xl</sub>
I <sub>h</sub>	1/8	1/8	—	—	0.62	0.52	0.78	1.76	—	—	—	—	—	—
II <sub>h</sub>	1/24	1/24	-1/12	-1/12	0.62	0.58	0.84	0.82	0.50	0.15	0.01	0.71	0.25	0.21
VI <sub>h</sub>	9/128	1/24	-1/8	-1/12	0.47	0.94	1.92	0.80	—	—	—	0.64	0.04	0.60

Case	ε = 0.5 - 1							Φ <sub>1</sub> = 2ε <sup>2</sup> - 4ε + 3	Φ <sub>2</sub> = 2ε <sup>2</sup> - 4ε + 7	Φ <sub>3</sub> = 18ε <sup>2</sup> - 36ε + 23
	β	β <sub>α</sub>	α'	γ	λ <sub>y</sub>	ω <sub>y</sub>	ρ <sub>y</sub>			
I <sub>h</sub>	0.62 Φ <sub>1</sub>	0.52	0.78	1.76	—	—	—	Φ <sub>1</sub> = 2ε <sup>2</sup> - 4ε + 3	Φ <sub>2</sub> = 2ε <sup>2</sup> - 4ε + 7	Φ <sub>3</sub> = 18ε <sup>2</sup> - 36ε + 23
II <sub>h</sub>	0.67 Φ <sub>1</sub>	0.126 Φ <sub>2</sub>	0.97	0.88	0.124 Φ <sub>2</sub>	0.28	0.23			
VI <sub>h</sub>	0.09 Φ <sub>3</sub>	0.18 Φ <sub>2</sub>	1.42	1.14	0.175 Φ <sub>2</sub>	0.31	0.12			

TABLE III

- M <sub>yc</sub>	Timoshenko			Bijlaard			Multiplier
Case	II <sub>u</sub>	IV <sub>u</sub>	V <sub>u</sub>	II <sub>u</sub>	IV <sub>u</sub>	V <sub>u</sub>	
ε = 0.5	0.083	0.084	0.122	0.083	0.083	0.123	ql <sub>y</sub> <sup>2</sup>
ε = 1	0.051	0.070	0.084	0.053	0.070	0.087	ql <sub>y</sub> <sup>2</sup>
ε = 2	0.057	0.119	0.122	0.057	0.120	0.120	ql <sub>x</sub> <sup>2</sup>

TABLE IV

values  $\omega$  and  $\varphi$  according to Bittner too for values  $\varepsilon = 0.5 - 2$ . It is evident that for uniform load e.g.  $\omega_y = -M_{ycv}$  and that  $\varphi_y = \frac{M_{ycv} - M_{ycIV}}{M_{ycIV}}$ , so that, in cases where this is easier, we can calculate a slab in this way too <sup>(11)</sup>.

In table IV we compared the clamping moments  $M_{yc}$  according to our formulae with those according to Timoshenko <sup>(12)</sup> for cases II<sub>u</sub>, IV<sub>u</sub> and V<sub>u</sub> and  $\varepsilon = 0.5, 1$  and  $2$ .

### Résumé

Par une méthode approximative nous obtenons des formules simples et générales nous permettant de déterminer, avec une approximation suffisante, les moments fléchissants maxima positifs et négatifs. Ces moments ainsi déterminés, nous pouvons calculer les coefficients de transmission utilisés pour le calcul des dalles continues selon la méthode de Cross.

### Zusammenfassung

Durch ein Näherungsverfahren werden für die grössten positiven und negativen Biegemomente einfache allgemeine Formeln von genügender Genauigkeit erhalten. Mit den so berechneten Werten können die Uebertragungskoeffizienten bestimmt werden, die es erlauben, durchlaufende Platten auch nach dem Momentenverteilungsverfahren von Cross zu berechnen.

### Summary

By an approximative method simple general formulae have been obtained for the maximum positive and negative bending moments, which give more than sufficiently accurate results. From the data obtained in this way the carry-over factors were calculated, by which the bending moments in continuous floors may also be computed by the Cross method of moment distribution.

<sup>(11)</sup> These values  $\omega_y$  and  $\varphi_y$  have nothing to do with our values  $\omega_y$ ,  $\varphi_y$ , etc.

<sup>(12)</sup> TIMOSHENKO, *Theory of Plates and Shells*, 1940, pp. 228, 206 and 213 resp.

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# IVb2

## Flexion et flambage d'un certain type de plaques continues orthotropes

## Biegung und Beulung eines bestimmten Types von durchlaufenden orthotropen Platten

## Bending and buckling of some types of continuous orthotropic plates

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Gdansk

### Flexion des plaques continues

I. L'équation différentielle connue de flexion d'une plaque orthotrope <sup>(1)</sup> est posée comme suit

$$D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} = p \quad (1)$$

$$D_x = \frac{m_x m_y}{m_x m_y - 1} E_x \frac{h^3}{12} \quad D_y = \frac{m_x m_y}{m_x m_y - 1} E_y \frac{h^3}{12} \quad C = G_0 \frac{h^3}{12}$$

$$2H = \frac{D_x}{m_y} + \frac{D_y}{m_x} + 4C \quad m_x E_x = m_y E_y$$

où  $E_x, E_y$  sont les modules d'élasticité suivant les axes  $x$  et  $y$  ;  
 $m_x, m_y$  sont les nombres de Poisson pour ces directions ;  
 $G_0$  est la constante des matériaux (le corrélatif du module d'élasticité transversale pour plaque isotrope).

Les forces de section sont unies à la flexion dans  $(x, y)$  au moyen des relations suivantes (fig. 1) :

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<sup>(1)</sup> M. T. HUBER : 1. *La théorie générale des hourdis en béton armé* (Czasopismo techniczne, Lwów, 1914); 2. *Teoria pływ*. (Tow. Naukowe, Lwów, 1921); 3. *Probleme der Statik technisch wichtiger orthotropen Platten*, Warszawa, 1929.

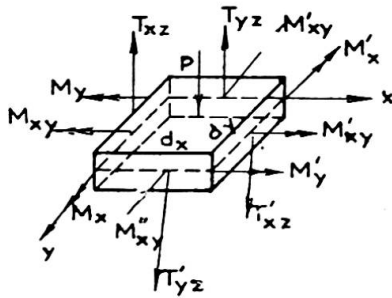


Fig. 1.

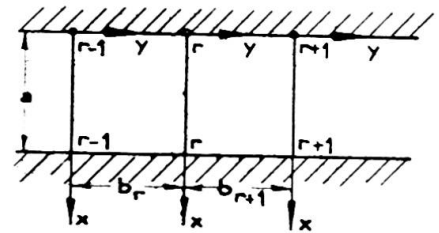


Fig. 2.

$$\begin{aligned}
 M_x &= -D_x \left( \frac{\partial^2 w}{\partial x^2} + \frac{1}{m_y} \frac{\partial^2 w}{\partial y^2} \right) & M_y &= -D_y \left( \frac{\partial^2 w}{\partial y^2} + \frac{1}{m_x} \frac{\partial^2 w}{\partial x^2} \right) \\
 M_{xy} &= -2C \frac{\partial^2 w}{\partial x \partial y} \\
 T_{xz} &= -D_x \frac{\partial^3 w}{\partial x^3} + \left( \frac{D_x}{m_y} + 2C \right) \frac{\partial^3 w}{\partial x \partial y^2} \\
 T_{yz} &= -D_y \frac{\partial^3 w}{\partial y^3} + \left( \frac{D_y}{m_x} + 2C \right) \frac{\partial^3 w}{\partial y \partial x^2}
 \end{aligned} \tag{2}$$

Dans le cas de plaque isotrope

$$D_x = D_y = D \quad m_x = m_y = m \quad C = \frac{1}{2} \frac{m-1}{m} D.$$

II. Considérons l'élément de la plaque continue, limité par les droites  $x=0, x=a$ , ainsi que par les droites  $y=0, y=b$  (fig. 2). Sur les droites d'appui  $y=0, y=b, x=0, x=a$  se produiront les moments fléchissants et de torsion. Les moments fléchissants  $M_y(x, 0)$  sur les lignes d'appui  $y=0, y=b$ , ainsi que les moments  $M_x(0, y)$  sur les lignes d'appui  $x=0, x=a$  seront considérés comme grandeurs hyperstatiques.

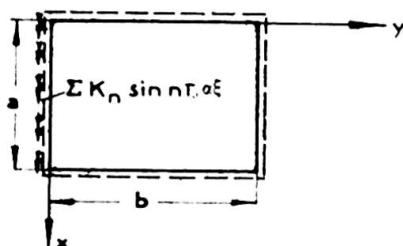
Pour déterminer les moments  $M_y(x, 0)$  et  $M_x(0, y)$  nous allons profiter des conditions de continuité de la plaque sur ces appuis.

Cependant, avant de poser les équations conditionnelles on doit étudier le système fondamental de la plaque à appui libre le long de ses bords, uniformément chargée du poids  $p$  ainsi que par le moment

$$M_y(x, 0) = \sum_n K_n \sin \frac{n\pi x}{a}$$

le long de l'arête  $y=0$ .

a) Charge de la plaque par le moment  $\sum_{n=1,2} K_n \sin \frac{n\pi x}{a}$  (fig. 3).



Pour ce genre de charge aux données

$$\begin{aligned}
 y &= \eta b & x &= \xi b & b &= \alpha a \\
 \epsilon^4 &= \frac{D_x}{D_y} & \rho &= \frac{H}{\sqrt{D_x D_y}}
 \end{aligned}$$

Fig. 3.

nous amenons l'équation (1) à l'expression sans dimension

$$\frac{\partial^4 w}{\partial \eta^4} + 2 \rho \varepsilon^2 \frac{\partial^4 w}{\partial \xi^2 \partial \eta^2} + \varepsilon^4 \frac{\partial^4 w}{\partial \xi^4} = 0. \tag{3}$$

Admettant que

$$w = \sum_{n=1,2}^{\infty} Y_n(\eta) \sin n\pi\alpha\xi$$

nous transformons l'équation (3) en

$$Y_n^{(IV)} - 2 \rho \nu^2 Y_n'' + \nu^4 Y_n = 0 \quad \nu = \varepsilon n \pi \alpha \tag{4}$$

L'équation caractéristique

$$r^4 - 2 \rho \nu^2 r^2 + \nu^4 = 0$$

donne quatre racines.

I	$\rho > 1$	$r_{1,2} = \pm \lambda_1$	$r_{3,4} = \pm \lambda_2$	$\lambda_{1,2} = \nu \sqrt{\rho \pm \sqrt{\rho^2 - 1}}$
II	$\rho = 1$	$r_{1,2} = \nu$	$r_{3,4} = -\nu$	
III	$\rho < 1$	$r_{1,2} = \varphi_1 \pm i\varphi_2$	$r_{3,4} = -(\varphi_1 \pm i\varphi_2)$	$\varphi_{1,2} = \nu \sqrt{\frac{1 \pm \rho}{2}}$

où  $\lambda_{1,2} = \varphi_1 \pm i\varphi_2$ .

Nous allons considérer le premier cas uniquement; nous obtiendrons l'intégrale du deuxième cas au moyen de passage aux limites; le passage du premier cas au troisième s'effectuera par voie de substitution

$$\lambda_{1,2} = \varphi_1 \pm i\varphi_2.$$

Pour  $\rho > 1$  la solution générale de l'équation (4) sera

$$Y_n(\eta) = U_{1n} \cos \lambda_1 \eta + U_{2n} \sin \lambda_1 \eta + U_{3n} \cos \lambda_2 \eta + U_{4n} \sin \lambda_2 \eta. \tag{5}$$

Les constantes d'intégration  $U_{1n}, \dots, U_{4n}$  seront déterminées par les conditions des bords de la plaque

$$Y_n(0) = 0 \quad Y_n(1) = 0 \quad Y''(0) = -\frac{Kn b^2}{D_y} \quad Y''(1) = 0.$$

Nous trouverons

$$U_{1n} = -\frac{Kn b^2}{2 D_y \nu^2} \cdot \frac{1}{\sqrt{\rho^2 - 1}}; \quad U_{2n} = -U_{1n} \operatorname{ctg} \lambda_1$$

$$U_{3n} = -U_{1n} \quad U_{4n} = U_{1n} \operatorname{ctg} \lambda_2. \tag{6}$$

L'inclinaison de la surface de flexion de la plaque le long de l'arête  $y=0$  et de  $y=b$  donne

$$\frac{\partial w}{\partial y} \Big|_{y=0} = \frac{b}{D_y} \sum_{n=1,2}^{\infty} K_n \Phi_n \sin n\pi\alpha\xi \quad \frac{\partial w}{\partial y} \Big|_{y=b} = -\frac{b}{D_y} \sum_{n=1,2}^{\infty} K_n \Psi_n \sin n\pi\alpha\xi \tag{7}$$

$$\Phi_n = \frac{1}{2 \nu^2 \sqrt{\rho^2 - 1}} \cdot \frac{\lambda_1 \cos \lambda_1 \sin \lambda_2 - \lambda_2 \cos \lambda_2 \sin \lambda_1}{\sin \lambda_1 \sin \lambda_2} \tag{8}$$



$$\Psi_n = \frac{1}{2 \nu^2 \sqrt{\rho^2 - 1}} \cdot \frac{\lambda_2 \sin \lambda_1 - \lambda_1 \sin \lambda_2}{\sin \lambda_1 \sin \lambda_2} \quad (9)$$

L'angle d'inclinaison de la surface de flexion aux arêtes  $x=0, x=a$

$$\frac{\partial w}{\partial x} \Big|_{x=0} = \frac{\pi}{a} \sum_{n=1,2}^{\infty} n Y_n(\eta) \quad \frac{\partial w}{\partial x} \Big|_{x=a} = \frac{\pi}{a} \sum_{n=1,2}^{\infty} n (-1)^n Y_n(\eta) \quad (10)$$

sera transformé par le développement de la fonction  $Y_n(\eta)$  en série de Fourier, en série double infinie

$$\begin{aligned} \frac{\partial w}{\partial x} \Big|_{x=0} &= \frac{a}{D_x} \sum_n \sum_i K_n A_{i,n} \sin i \pi \eta \\ \frac{\partial w}{\partial x} \Big|_{x=a} &= \frac{a}{D_x} \sum_n \sum_i K_n A_{i,n} (-1)^n \sin i \pi \eta \end{aligned}$$

où

$$A_{i,n} = \frac{i}{n^3 \pi^2 \alpha^2} \cdot \frac{1}{\left[ 1 + 2\rho \left( \frac{i}{n\alpha\varepsilon} \right)^2 + \left( \frac{i}{n\alpha\varepsilon} \right)^4 \right]} \rho \begin{matrix} \geq \\ \leq \end{matrix} 1. \quad (11)$$

b) Charge par le moment  $M = \sum_n K_n \sin n\pi\alpha\xi$  le long de l'arête  $y=b$  (fig. 4).

Sans changer les constantes d'intégration  $U_{1n}, \dots, U_{4n}$  (équation 6) il faut dans la fonction  $Y_n(\eta)$  (équation 5) poser  $\eta' = 1 - \eta$  au lieu de  $\eta$ .

En conséquence

$$\begin{aligned} \frac{\partial w}{\partial y} \Big|_{y=0} &= \frac{b}{D_y} \sum_n K_n \Psi_n \sin n\pi\alpha\xi & \frac{\partial w}{\partial y} \Big|_{y=b} &= - \frac{b}{D_y} \sum_n K_n \Phi_n \sin n\pi\alpha\xi \\ \frac{\partial w}{\partial x} \Big|_{x=0} &= - \frac{a}{D_x} \sum_n \sum_i K_n (-1)^i A_{i,n} \sin i \pi \eta \\ \frac{\partial w}{\partial x} \Big|_{x=a} &= - \frac{a}{D_x} \sum_n \sum_i K_n (-1)^{i+n} A_{i,n} \sin i \pi \eta. \end{aligned}$$

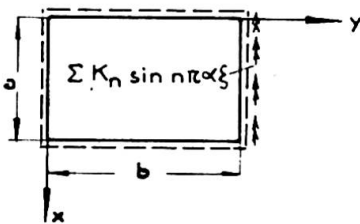


Fig. 4.

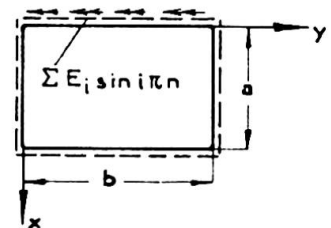


Fig. 5.

c) Le moment  $M = \sum_1^{\infty} E_i \sin i\pi\eta$  agit le long de l'appui  $x = 0$  (fig. 5).

Admettant que  $w = \sum_n^{\infty} \chi_n \sin i\pi\eta$

nous amenons l'équation (3) à l'expression

$$X_i^{IV} - 2\rho\theta^2 X_i'' + \theta^4 X_i = 0 \quad \theta = \frac{i\pi}{\varepsilon}.$$

Pour  $\rho > 1$  nous obtiendrons la solution générale

$$X_i(\xi) = U_{1i} \cos \mu_1 \xi + U_{2i} \sin \mu_1 \xi + U_{3i} \cos \mu_2 \xi + U_{4i} \sin \mu_2 \xi \quad (12)$$

$$\mu_{1,2} = \theta \sqrt{\rho \pm \sqrt{\rho^2 - 1}}.$$

Des conditions des bords

$$X_i(0) = 0 \quad X_i\left(\frac{1}{\alpha}\right) = 0 \quad X_i''(0) = -\frac{E_i a^2}{D_x} \quad X_i''\left(\frac{1}{\alpha}\right) = 0$$

nous obtiendrons

$$U_{1i} = -\frac{E_i a^2}{2 D_x \omega^2} \cdot \frac{1}{\sqrt{\rho^2 - 1}} \quad U_{2i} = U_{1i} \operatorname{ctg} \chi_1 \quad U_{3i} = -U_{1i}$$

$$U_{4i} = U_{1i} \operatorname{ctg} \chi_2 \quad (13)$$

$$\omega = \frac{\theta}{\alpha} \quad \chi_{1,2} = \frac{\mu_{1,2}}{\alpha}.$$

Ensuite

$$\frac{\partial w}{\partial x} \Big|_{x=0} = \frac{a}{D_x} \sum_i^{\infty} E_i \Phi_i \sin i\pi\eta \quad \frac{\partial w}{\partial x} \Big|_{x=a} = -\frac{a}{D_x} \sum_i^{\infty} E_i \Psi_i \sin i\pi\eta$$

où

$$\Phi_i = \frac{1}{2 \omega^2 \sqrt{\rho^2 - 1}} \frac{\chi_1 \cos \xi_1 \sin \chi_2 - \chi_2 \cos \chi_2 \sin \chi_1}{\sin \chi_1 \sin \chi_2};$$

$$\Psi_i = \frac{1}{2 \omega^2 \sqrt{\rho^2 - 1}} \frac{\chi_2 \sin \chi_1 - \chi_1 \sin \chi_2}{\sin \chi_1 \sin \chi_2} \quad (14)$$

Et enfin

$$\frac{\partial w}{\partial y} \Big|_{y=0} = \frac{b}{D_y} \sum_i^{\infty} \sum_n^{\infty} E_i B_{i,n} \sin i\pi\alpha\xi$$

$$\frac{\partial w}{\partial y} \Big|_{y=b} = \frac{b}{D_y} \sum_i^{\infty} \sum_n^{\infty} E_i B_{i,n} (-1)^i \sin i\pi\alpha\xi$$

où

$$B_{i,n} = \frac{n\alpha^2}{\pi^2 i^3 \left[ 1 + 2\rho \left( \frac{n\alpha\varepsilon}{i} \right)^2 + \left( \frac{n\alpha\varepsilon}{i} \right)^4 \right]} \quad \rho \begin{matrix} \geq \\ < \end{matrix} 1. \quad (15)$$

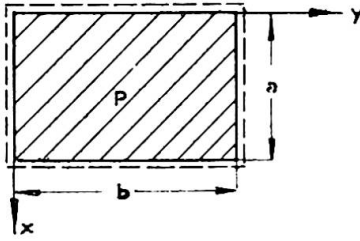


Fig. 6.

d) Charge de la plaque par  $p = \text{const.}$  (fig. 6).

La surface de flexion est traduite par l'équation

$$w = \frac{4pa^4}{D_x} \sum_{n=1,3}^{\infty} \left[ \left( \frac{1}{n\pi} \right)^5 + Y_n(\eta) \right] \sin n\pi\alpha\xi \quad \bar{Y}_n = \left( \frac{1}{n\pi} \right)^5 + Y_n(\eta). \quad (16)$$

Ici la fonction  $Y_n(\eta)$  est identique à la fonction  $Y_n(\eta)$  de l'équation (5).

Nous déterminons les constantes d'intégration des conditions aux limites suivantes

$$\bar{Y}_n(0) = 0, \quad \bar{Y}_n''(0) = 0, \quad \bar{Y}_n(1) = 0, \quad \bar{Y}_n''(1) = 0.$$

D'où pour  $\rho > 1$

$$U_{1n} = \left( \frac{1}{n\pi} \right)^5 \frac{\lambda_2^2}{2\rho^2 \sqrt{\rho^2 - 1}}; \quad U_{2n} = U_{1n} \frac{1 - \cos \lambda_1}{\sin \lambda_1}$$

$$U_{3n} = -U_{1n} \frac{\lambda_1^2}{\lambda_2^2}; \quad U_{4n} = \frac{\lambda_1^2}{\lambda_2^2} \frac{\cos \lambda_2 - 1}{\sin \lambda_2} U_{1n}. \quad (17)$$

L'inclinaison de la plaque aux arêtes  $y=0$  et  $y=b$  sera amenée à l'expression

$$\frac{\partial w}{\partial y} \Big|_{y=0} = - \frac{\partial w}{\partial y} \Big|_{y=b} = \frac{4pb^3}{D_y} \sum_{n=1,3}^{\infty} \Theta_n \sin n\pi\alpha\xi$$

$$\Theta_n = \frac{\varepsilon\alpha^2}{2\rho^5 \sqrt{\rho^2 - 1}} \left[ \lambda_1 \frac{\cos \lambda_2 - 1}{\sin \lambda_2} - \lambda_2 \frac{\cos \lambda_1 - 1}{\sin \lambda_1} \right]. \quad (18)$$

Dans la suite de nos considérations nous allons profiter du développement de la surface de flexion de la plaque en série

$$w = \frac{4pb^4}{D_y} \sum_{i=1,3}^{\infty} \left[ \left( \frac{1}{i\pi} \right)^5 + X_i(\xi) \right] \sin i\pi\eta. \quad (19)$$

L'inclinaison de la plaque aux arêtes  $x=0$  ainsi que  $x=a$  donne pour  $\rho > 1$

$$\frac{\partial w}{\partial x} \Big|_{x=0} = - \frac{\partial w}{\partial x} \Big|_{x=a} = \frac{4pa^3}{D_x} \sum_{i=1,3}^{\infty} \Theta_i \sin i\pi\eta$$

où

$$\Theta_i = \frac{1}{2\varepsilon\alpha \sqrt{\rho^2 - 1} \omega^5} \left[ \lambda_1 \frac{\cos \lambda_2 - 1}{\sin \lambda_2} - \lambda_2 \frac{\cos \lambda_1 - 1}{\sin \lambda_1} \right] \quad (20)$$

et les constantes d'intégration

$$\begin{aligned}
 U_{1i} &= \left(\frac{1}{i\pi}\right)^5 \frac{\chi_2^2}{2\omega^2\sqrt{\rho^2-1}}; & U_{2i} &= U_{1i} \frac{1-\cos\chi_1}{\sin\chi_1}; \\
 U_{3i} &= -U_{1i} \frac{\chi_1^2}{\chi_2^2} & U_{4i} &= U_{1i} \frac{\chi_1^2}{\chi_2^2} \frac{\cos\chi_2-1}{\sin\chi_2}
 \end{aligned} \tag{21}$$

III. Considérons les deux aires contiguës de la plaque continue; l'aire  $r$  à la caractéristique d'orthotropie  $\rho_r$  et chargée de  $p_r$ , ainsi que l'aire  $r+1$  à la caractéristique d'orthotropie  $\rho_{r+1}$  et chargée de  $p_{r+1}$ .

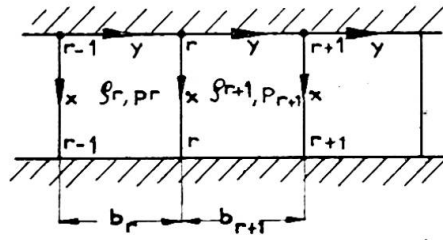


Fig. 7.

Nous nous bornerons à une plaque continue aux arêtes  $x=0, x=a$  rigidement encastree.

Désignons par  $M^{r-1}, M^r, M^{r+1}$  les moments d'appui  $M_y$  en droites  $r-1, r, r+1$ , en outre  $M^r = \sum_n K_n^r \sin n\pi\alpha\xi$  et par  $M^r, M^{r+1}$  les moments  $M_x$  aux arêtes  $x=0, x=a$  de l'aire  $r$  ainsi que de l'aire  $r+1$ ; avec cela

$$M^r = \sum_i E_i^r \sin i\pi\eta.$$

La condition d'encastrement rigide de la plaque à l'arête  $x=a$  ou bien  $x=0$  de l'aire  $r$  de la plaque donne

$$\sum \frac{\partial w^{r+1}}{\partial x} \Big|_{x=0} = 0$$

d'où

$$\begin{aligned}
 \frac{i}{\pi^2 \alpha^2_{r+1}} \sum_n \frac{K_n^r - K_n^{r+1} (-1)^i}{n^3 \left[ 1 + 2\rho \left(\frac{i}{n\alpha\varepsilon}\right)^2 + \left(\frac{i}{n\alpha\varepsilon}\right)^4 \right]_{r+1}} \\
 + E_i^{r+1} (\Phi_i^{r+1} + \Psi_i^{r+1}) + 4 p_{r+1} a^2 \Theta_i^{r+1} = 0 \tag{22} \\
 (i=1, 2, \dots), \quad (n=1, 3, 5, \dots).
 \end{aligned}$$

La condition de continuité de la plaque sur l'appui  $r-r$  conduit à l'équation

$$-\frac{\partial w^r}{\partial y} \Big|_{y=b_r} + \frac{\partial w^{r+1}}{\partial y} \Big|_{y=0} = 0$$

ou bien à

$$\frac{n}{\pi^2} \left\{ \beta^{r+1} \alpha_{r+1}^2 \sum_i^\infty \frac{2 E_i^{r+1}}{i^3 \left[ 1 + 2 \varphi \left( \frac{n \alpha \varepsilon}{i} \right)^2 + \left( \frac{n \alpha \varepsilon}{i} \right)^4 \right]_{r+1}} \right. \\ \left. - \beta^r \alpha_r^2 \sum_i^\infty \frac{2 E_i^r (-1)^i}{i^3 \left[ 1 + 2 \varphi \left( \frac{n \alpha \varepsilon}{i} \right)^2 + \left( \frac{n \alpha \varepsilon}{i} \right)^4 \right]_r} \right\} \\ + K_n^{r-1} \beta^r \Psi_n^r + K_n^r (\beta^r \Phi_n^r + \beta^{r+1} \Phi_n^{r+1}) + K_n^{r+1} \Psi_n^{r+1} \beta^{r+1} \\ + 4 (p_r b_r^2 \beta^r \Theta_n^r + p_{r+1} b_{r+1}^2 \beta^{r+1} \Theta_n^{r+1}) = 0 \quad (23) \\ (n=1, 3, 5, \dots), \quad (i=1, 2, \dots).$$

$\beta^r = \frac{D_y^0, b_r}{D_y^r b_0}$  où  $b_0$  est la longueur comparative de la travée, et  $D_y^0$  la caractéristique comparative de flexion de la plaque.

Dans le cas d'encastrement rigide de la plaque continue en droite  $x=0$  et d'appui libre en droite  $x=a$ , il faut dans l'équation (22) poser  $\Psi_i^{r+1}=0$  et dans l'équation (23) au lieu de  $2 E_i^{r+1}$ ,  $2 E_i^r$  la valeur seule  $E_i^{r+1}$ ,  $E_i^r$ .

Les grandeurs  $i$ ,  $n$  prennent les valeurs successives 1, 2, 3, ...

Enfin dans le cas de plaque continue librement appuyée le long des arêtes  $x=a$ ,  $x=0$  l'équation (22) n'est pas applicable et dans l'équation (23) il y a lieu de poser  $E_i=0$ . Nous obtiendrons de la sorte un système simple d'équations

$$K_n^{r-1} \beta^r \Psi_n^r + K_n^r (\beta^r \Phi_n^r + \beta^{r+1} \Phi_n^{r+1}) \\ + K_n^{r+1} \beta^{r+1} \Psi_n^{r+1} + 4 (p_r b_r^2 \beta^r \Theta_n^r + p_{r+1} b_{r+1}^2 \beta^{r+1} \Theta_n^{r+1}) = 0 \quad (24) \\ (n=1, 3, 5, \dots).$$

Pour une plaque rigidement encadrée le long des quatre arêtes (aire  $r+1$ )  $K_n^r = K_n^{r+1} = K_n$ ,  $\beta^r = 0$  et dans le cas particulier de plaque carrée  $a=b$ ,  $\alpha=1$ , de même  $K_n = E_n$ .

Il en résulte le système d'équations

$$\frac{2n}{\pi^2} \sum_i^\infty \frac{E_i}{i^3 \left[ 1 + 2 \varphi \left( \frac{n \alpha \varepsilon}{i} \right)^2 + \left( \frac{n \alpha \varepsilon}{i} \right)^4 \right]} + E_n (\Phi_n + \Psi_n) + 4 p a^2 \Theta_n = 0 \\ (n=1, 3, 5, \dots), \quad (i=1, 3, 5, \dots).$$

Enfin pour plaque continue à libre appui le long des droites  $x=0$ ,  $x=a$ , nous obtiendrons aux mêmes indices géométriques et d'élasticité ainsi qu'à la même charge  $p = \text{const.}$  des aires, l'équation

$$K_n^{r-1} + 2 c_n K_n^r + K_n^{r+1} = W_n, \quad c_n = \Phi_n / \Psi_n \\ (r=1, 2, \dots, z-1) \\ W_n = \frac{8 p b^2 \Theta_n}{\Psi_n}$$

que nous pouvons considérer comme équation aux différences non homogène du second ordre. Et voici la solution de cette équation

$$K_n^r = - \frac{W_n}{2(1+c_n)} \left\{ (-1)^r \left[ \operatorname{tg} \frac{\omega z}{2} \sin \omega_r - \cos \omega_r \right] + 1 \right\}$$

$$\omega = \ln \frac{1}{\sigma} \quad \sigma = -c_n + \sqrt{c_n^2 - 1}$$

pour

$$z \rightarrow \infty \quad K_n^r \approx - \frac{W_n}{2(1+c_n)} = - \frac{4pb^2 \Theta_n}{\Psi_n + \Phi_n}.$$

Nous remarquons que dans chaque cas particulier le nombre d'équations est conforme au nombre d'inconnues. Les grandeurs  $K, E$  étant connues nous pourrions déterminer la surface de flexion de la plaque, et par conséquent les valeurs des forces de section des équations (2).

Flambage d'une plaque continue orthotrope à appui libre sur les arêtes

$$x = 0, \quad x = a.$$

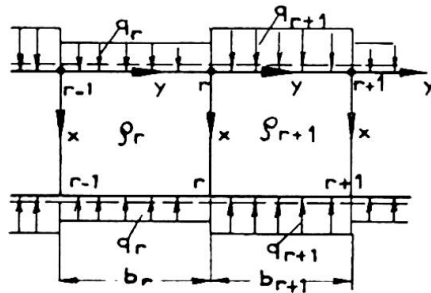


Fig. 8.

I. Pour  $q > q_k$  ( $q_k =$  charge critique) la plaque fléchira et le long de l'arête  $y=0$  se produiront les moments fléchissants et de torsion.

L'équation différentielle du problème

$$D_x \frac{\partial^4 w}{\partial x^2} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} + q \frac{\partial^2 w}{\partial x^2} = 0 \quad (25)$$

à l'aide des valeurs comme dans l'alinéa Ia sera amenée à l'expression

$$\frac{\partial^4 w}{\partial \eta^4} + 2\varphi \varepsilon^2 \frac{\partial^4 w}{\partial \eta^2 \partial \xi^2} + \varepsilon^4 \frac{\partial^4 w}{\partial \xi^4} + \varepsilon^4 \pi^2 \varphi \frac{\partial^2 w}{\partial \xi^2} = 0 \quad \varphi = \frac{qb^2}{\pi^2 D_x} \quad (25a)$$

Examinons tout d'abord le flambage d'une plaque rectangulaire librement appuyée sur trois arêtes et le long de la quatrième encastree d'une façon élastique (fig. 9).

Admettant que

$$w = \sum_n Y_n(\eta) \sin n\pi \alpha \xi$$

nous obtiendrons

$$Y_n^{IV} - 2\varphi \nu^2 Y_n'' + \nu^2 (\nu^2 - \pi^2 \varphi \varepsilon^2) Y_n = 0. \quad (26)$$

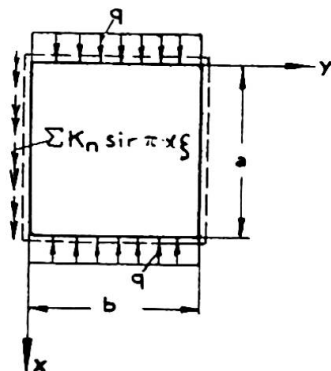


Fig. 9.

L'équation caractéristique

$$r^4 - 2\rho\nu^2 r^2 + \nu^2(\nu^2 - \pi^2\varphi\varepsilon^2) = 0$$

donne quatre racines.

Pour  $\varphi \begin{matrix} > \\ < \end{matrix} 1$  nous obtiendrons deux racines réelles et deux imaginaires

$$r_{1,2} = \pm m_1 \quad r_{3,4} = \pm im_2 \quad m_{1,2} = \nu \sqrt{\nu^2 \rho^2 - 1 + \delta} \pm \rho$$

$$\delta = \left(\frac{\pi\varepsilon}{\nu}\right)^2 \varphi \quad \nu = n\pi a\varepsilon.$$

La solution générale de l'équation (26) sera la suivante

$$Y_n(\eta) = U_{1n} \cos m_1 \eta + U_{2n} \sin m_1 \eta + U_{3n} \cos m_2 \eta + U_{4n} \sin m_2 \eta \quad (27)$$

$$m_1^2 + m_2^2 = 2\nu^2 \sqrt{\nu^2 \rho^2 - 1 + \delta} \quad m_1^2 - m_2^2 = 2\nu^2 \rho. \quad (27a)$$

Les conditions aux limites du problème

$$Y_n(0) = 0 \quad Y_n(1) = 0 \quad Y_n''(0) = -\frac{K_n b^2}{D_y} \quad Y_n''(1) = 0$$

conduisent aux constantes d'intégration

$$U_{1n} = -\frac{K_n b^2}{D_y} \frac{1}{m_1^2 + m_2^2} \quad U_{3n} = -U_{1n}$$

$$U_{2n} = -U_{1n} \operatorname{ctg} m_1 \quad U_{4n} = U_{1n} \operatorname{ctg} m_2.$$

Les équations suivantes déterminent l'inclinaison de la surface de flexion aux arêtes  $y=0$ ,  $y=b$

$$\frac{\partial w}{\partial y} \Big|_{y=0} = \frac{b}{D_y} \sum_n K_n \bar{\Phi}_n \sin n\pi x \xi \quad \bar{\Phi}_n = \frac{m_1 \operatorname{ctg} m_1 - m_2 \operatorname{ctg} m_2}{m_1^2 + m_2^2} \quad (28)$$

$$\frac{\partial w}{\partial y} \Big|_{y=b} = -\frac{b}{D_y} \sum_n K_n \bar{\Psi}_n \sin n\pi x \xi \quad \bar{\Psi}_n = \frac{m_2 \operatorname{coséc} m_2 - m_1 \operatorname{coséc} m_1}{m_1^2 + m_2^2} \quad (29)$$

Remarquons qu'à  $\delta=0$ , c'est-à-dire  $q=0$ , les grandeurs  $\bar{\Phi}_n \rightarrow \Phi_n$ ,  $\bar{\Psi}_n \rightarrow \Psi_n$ .

Pour encastrement rigide de la plaque en l'arête  $y = 0$ , nous obtiendrons de la condition  $\frac{\partial w}{\partial y} \Big|_{y=0} = 0$  l'équation du flambage de la plaque  $\bar{\Phi}_n = 0$ .

Cette équation  $m_1 \operatorname{ctg} m_1 = m_2 \operatorname{ctg} m_2$ , ainsi que les relations (27a) vont nous déterminer  $\infty$  la quantité infinie des racines  $\delta$ .

II. Revenant au flambage de la plaque continue nous adopterons à  $q > q_k$  les moments  $M_y^r(x, 0)$  comme valeurs supplémentaires du système. La condition de continuité de la plaque aux appuis donne

$$-\frac{\partial w^r}{\partial y} \Big|_{y=br} + \frac{\partial w^{r+1}}{\partial y} \Big|_{y=0} = 0$$

Nous obtiendrons le système d'équations homogènes

$$K_n^{r-1} \beta^r \bar{\Psi}_n^r + K_n^r (\beta^r \bar{\Phi}_n^r + \beta^{r+1} \bar{\Phi}_n^{r+1}) + K_n^{r+1} \beta^{r+1} \bar{\Psi}_n^{r+1} = 0$$

$$(r = 1, 2, \dots, z - 1), \quad (n = 1, 2, \dots, \infty). \quad (30)$$

$$\beta^r = \frac{D_y^0 b_r}{D_y^r b^0}.$$

Nous poserons autant d'équations (30) qu'il y a de grandeurs inconnues des moments d'appui. Ce système sera non contradictoire, lorsque le déterminant du système d'équations  $\Delta(\delta)$  sera égal à zéro.

Cette dernière condition ainsi que les relations (27a) établissent le critère du flambage de la plaque.

L'équation (30) comprend une série de cas particuliers.

a) Plaque librement appuyée sur ses arêtes  $r - 1, r + 1$

$$K_n^{r-1} = K_n^{r+1} = 0, \quad \Delta(\delta) = \beta^r \bar{\Phi}_n^r + \beta^{r+1} \bar{\Phi}_n^{r+1} = 0.$$

b) Plaque encastree rigide le long des arêtes  $r, r + 1$ .

Dans l'équation (30) il faut poser

$$K_n^{r-1} = K_n^r = K_n, \quad \beta^{r+1} = 0, \quad \Delta(\delta) = \bar{\Phi}_n + \bar{\Psi}_n = 0.$$

Dans le cas particulier  $b \rightarrow \infty$  nous obtiendrons  $q_k = \frac{\pi^2 D_x n^2}{a^2}$ .

c) Plaque continue aux valeurs équivalentes de  $q, b, \rho$  dans toutes les aires avec le nombre d'appuis  $z + 1$ .

En traitant l'équation

$$K_n^{r-1} + \bar{c} K_n^r + K_n^{r+1} = 0$$

$$(r = 1, 2, \dots, z - 1)$$

$$\bar{c} = \frac{\bar{\Phi}_n}{\bar{\Psi}_n}$$

comme équation linéaire aux différences du second ordre avec solution  $K_n^r = A_n \cos ar + B_n \sin ar$ ; en tenant compte des conditions des bords



( $K_n^0 = 0$ ,  $K_n^z = 0$ ) nous amènerons la condition du flambage de la plaque à l'expression

$$\cos \frac{\pi}{z} = \frac{\bar{\Phi}}{\bar{\Psi}}$$

ce qui, pour une quantité infinie de travées, donne  $\bar{\Phi} = \bar{\Psi}$ , ou bien

$$m_1 \operatorname{ctg} \frac{m_1}{2} = m_2 \operatorname{ctg} \frac{m_2}{2} .$$

### Résumé

Ce mémoire présente un critère de flambage d'une plaque continue orthotrope librement appuyée sur son périmètre et sollicitée par une charge uniformément répartie sur ses arêtes  $x = 0$  et  $x = a$ .

### Zusammenfassung

Es wurde für eine gleichmässig verteilte, in der Plattenmittelebene an den Rändern  $x = 0$  und  $x = a$  angreifende Kraft ein allgemeines Beulungskriterium für die orthotrope, durchlaufende, an den gleichen Rändern frei aufliegende Platte ermittelt.

### Summary

This work presents the general solution of a continuous orthotropic plate whose edges ( $x = 0$  and  $x = a$ ) are loaded with  $p = \text{const.}$  and freely supported on its perimeter.