AI 3

Plastic analysis and design of steel-framed structures

Analyse plastique et calcul des ouvrages métalliques en cadres

Plastizitäts-Untersuchung und -Berechnung von Rahmenkonstruktionen aus Stahl

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INTRODUCTION

The methods presented in this paper for the analysis and design of rigid structures are purely mathematical in character; that is, techniques are formulated on the basis of certain fundamental assumptions. These assumptions may or may not be true for any particular structure; for example, the instability of axially loaded stanchions is ignored, as is the lateral instability of beams subjected to terminal bending moments. While for some simple structures under particular conditions of loading these effects may be relatively unimportant, recent work by Neal (1950a) and Horne (1950) has shown that the problem may in fact be critical. In addition, it will be seen below that an "ideal" plastic material is assumed. Structural mild steel approximates to such an ideal material, but a highly redundant frame will experience strain-hardening which may invalidate the calculations. The techniques presented here, in short, in no sense form a practical design method; however, it is felt that they are of sufficient interest to warrant a description of some of the more important results.

The characteristic ideally plastic behaviour of a beam in pure bending is shown in fig. 1. From O to A increase of bending moment is accompanied by purely elastic (linear) increase of curvature. Between A and B, increase of bending moment is accompanied by a greater increase of curvature, until at the point B the full plastic moment M_0 is attained. At this moment the curvature can increase indefinitely, and "collapse" occurs.

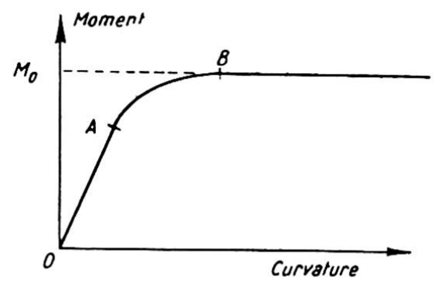

Fig. 1

In a general plane structural frame, a section at which the bending moment has the value M_0 is called a plastic hinge, and has the property that rotation at the hinge can occur freely under constant bending moment. From the definition of the full plastic moment, the moments in the frame can nowhere exceed M_0; if the component members of a frame have different sizes, it must be understood of course that M_0 refers to the particular member under consideration.

Collapse of a frame is said to occur when a sufficient number of plastic hinges are formed to turn whole or part of the frame into a mechanism of one degree of freedom; in general, the number of hinges exceeds by one the number of redundancies of that part of the frame concerned in the collapse. For example, the simple rectangular portal frame, of constant section throughout, subjected to loads V and H as shown in fig. 2(*a*), may fail in any one of the three basic modes shown in figs. 2(*b*), (*c*) and (*d*). The actual mode is determined by the values of the two loads.

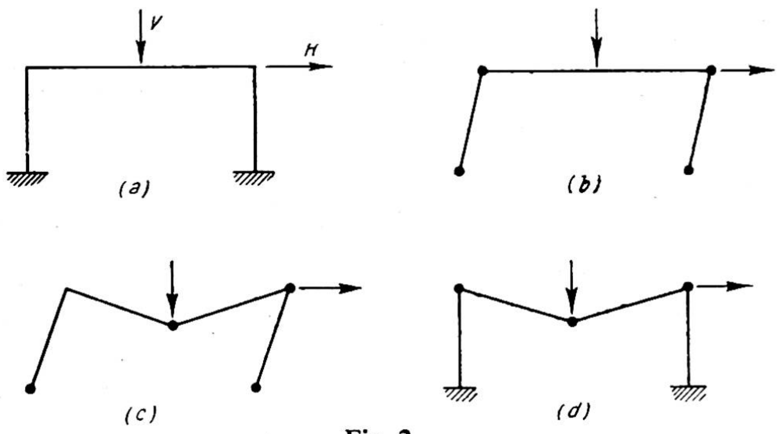

Fig. 2

The first part of this paper deals with methods for the exact determination of the quantities required (location of the hinges, values of collapse loads, etc.); the second part presents methods for determining upper and lower bounds on the loads, it being possible to make these bounds as close as is considered necessary. The third part applies the ideas to space frames, where hinges are formed under the combined action of bending and torsion.

Exact methods

The use of inequalities in the solution of structural problems was first introduced by Neal and Symonds (1950), who used a method due to Dines (1918). The very simple example shown in fig. 3 will be used to illustrate the solution of linear sets of inequalities.

(a) *Collapse analysis under fixed loads*

Suppose in fig. 3 that the two spans of the continuous beam are of length l, and that the fixed loads P_1 and P_2 act at the centres of the spans. The full plastic moment of the beam will be taken as M_0, and it is required to find the minimum value of M_0 in order that collapse shall just occur. (P_1 and P_2 may be taken to incorporate a suitable load factor.)

The general equilibrium state of a frame of n redundancies can be expressed as the sum of one arbitrary equilibrium state and n arbitrary independent residual states.

By a "state" is meant some bending moment distribution, so that a state in equilibrium with the applied loads is *any* bending moment distribution such that equilibrium is attained. A residual state is a bending moment distribution that satisfies equilibrium conditions when no external loads are applied to the frame. Thus, confining attention to any one cross-section in the frame, the bending moment there may be expressed as

$$M^*+M_1'+M_2'+\ldots+M_n' \quad \ldots\ldots \quad (1)$$

where M^* is the equilibrium bending moment at the section and $M_1', M_2', \ldots M_n'$ are the bending moments, at the section considered, corresponding to n arbitrary residual states. Suppose that the full plastic moment at the section (as yet undetermined) is M_0. Then

$$-M_0 \leqslant M^*+M_1'+M_2'+\ldots+M_n' \leqslant M_0 \quad \ldots \quad (2)$$

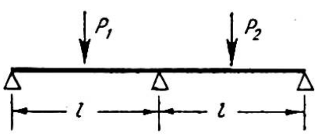

Fig. 3

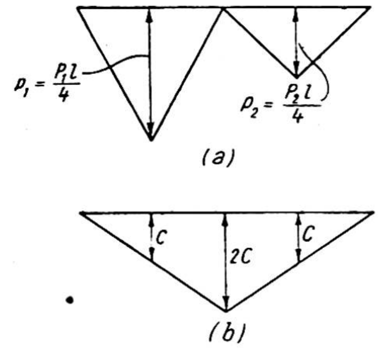

Fig. 4

Since the continuous beam system under consideration has one redundancy, the plastic behaviour can be represented as the sum of an equilibrium state and one residual state, which may be taken as the two bending moment distributions in fig. 4. The continued inequality (2) may be written for the three critical sections:

$$\left.\begin{array}{r} \text{Under the load } P_1,\ -M_0 \leqslant p_1+c \leqslant M_0 \\ \text{At the central support},\ -M_0 \leqslant \quad 2c \leqslant M_0 \\ \text{Under the load } P_2,\ -M_0 \leqslant p_2+c \leqslant M_0 \end{array}\right\} \quad \ldots \quad (3)$$

The set (3) may be rewritten as simple inequalities:

$$\left.\begin{array}{r} c+p_1+\ M_0 \geqslant 0 \\ c \quad +\tfrac{1}{2}M_0 \geqslant 0 \\ c+p_2+\ M_0 \geqslant 0 \\ -c-p_1+\ M_0 \geqslant 0 \\ -c \quad +\tfrac{1}{2}M_0 \geqslant 0 \\ -c-p_2+\ M_0 \geqslant 0 \end{array}\right\} \quad \ldots\ldots\ldots \quad (4)$$

If now every inequality in set (4) which has a coefficient of $+1$ for c is added to every inequality which has a coefficient -1 for c, c will be eliminated, and Dines has shown that the resultant set of inequalities (nine in number in this example) gives necessary and sufficient conditions for the existence of a value of c in order that the original set should be satisfied. This is exactly what is required for the present purposes; the actual value of c is of no interest so long as it is known that a c exists such that at each critical section of the frame the bending moment is less than the full plastic value.

In eliminating c from the set (4), it is found that a large number of the resulting inequalities are redundant, and if it is assumed that $P_1 \geqslant P_2$, the single inequality

$$-p_1 + \tfrac{3}{2}M_0 \geqslant 0 \qquad \text{(5)}$$

is found to be critical. As long as this inequality is satisfied, all the moments in the beam will be less than M_0. For collapse just to occur, the equality sign should be taken in (5), giving $M_0 = \frac{2}{3}p_1$. Now inequality (5) was derived by adding the second and fourth of set (4); substituting this value of M_0 into these two inequalities gives

$$\left.\begin{aligned} c + \tfrac{1}{3}p_1 \geqslant 0 \\ -c - \tfrac{1}{3}p_1 \geqslant 0 \end{aligned}\right\} \qquad \text{(6)}$$

i.e.

$$-\tfrac{1}{3}p_1 \geqslant c \geqslant -\tfrac{1}{3}p_1 \qquad \text{(7)}$$

that is, a unique value of c has been derived. Using this value of c, the bending moment distribution shown in fig. 5 has been derived from the analysis; it will be seen that hinges ($M_0 = \frac{2}{3}p_1$) are formed under the load P_1, and at the central support, forming a mechanism of one degree of freedom for small (really, infinitesimal) displacements.

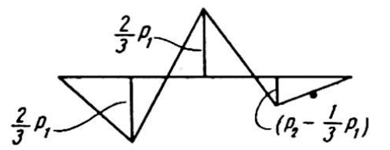

Fig. 5

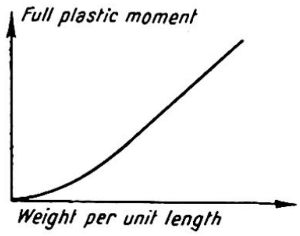

Fig. 6

The type of result obtained in this problem will in general be derived for any more complicated example. For more residual states defined by $c_1, c_2, \ldots c_n$, each parameter c is eliminated successively from the inequalities, and the final inequality, if just satisfied, will generate a unique set of residual states completely defining the collapse configuration.

The method given above may be applied to the analysis of frames collapsing under variable loads; however, this problem will be treated with reference to the slightly more complex condition of minimum weight design.

(b) *Minimum weight design under fixed loads*

The parameters used in order to determine the minimum weight of a structure will be the values of the full plastic moments. If a plot is made for typical structural sections of full plastic moment against weight per unit length, and the points joined by a smooth curve, a non-linear relationship of the type shown in fig. 6 will be obtained. (Owing to the methods used in this paper, the actual relationship is immaterial, but it is of interest to note that a curve given in a British Welding Research Association report (1947) for British structural sections can be approximated by $w = 2{\cdot}7M^{0{\cdot}6}$, where w is the weight in lb./ft. of a beam of full plastic moment M tons ft.) In order to develop suitable methods for design, it will be assumed that a continuous range of sections is available so that a section can be used with any specified full plastic moment.

The assumption is made that the moment–weight curve can be replaced in the region which is significant for any particular problem by a straight line. For a frame built up of N members, each of constant section, the total material consumption will be given by the proportionality

$$W \approx \sum_{i=1}^{N} M_i l_i \qquad \text{(8)}$$

where M_i is the full plastic moment of the ith member of the frame, and l_i is its length.

Considering again the two-span beam shown in fig. 3, suppose that the left-hand span has a full plastic moment M_1, that of the right-hand span being M_2. Since the two spans are of equal length, proportionality (8) may be replaced by the weight parameter

$$X = M_1 + M_2 \qquad \text{(9)}$$

The problem of minimum weight design for this problem is then reduced to choosing values of M_1 and M_2 such that X is made a minimum. The work starts in the same manner as for the collapse analysis given above; set (3) is replaced by

$$\left.\begin{array}{r}-M_1 \leqslant p_1 + c \leqslant M_1 \\ -M_1 \leqslant \quad 2c \leqslant M_1 \\ -M_2 \leqslant \quad 2c \leqslant M_2 \\ -M_2 \leqslant p_2 + c \leqslant M_2\end{array}\right\} \qquad \text{(10)}$$

The two continued inequalities are necessary for the central support since it is not known *a priori* whether $M_1 \gtrless M_2$.

Of the sixteen possible inequalities obtained by the elimination of c from set (3), only five are found to be non-redundant if it be assumed that $P_1 \geqslant P_2$. These are

$$\left.\begin{array}{l}-p_1 \qquad + \tfrac{3}{2}M_1 \qquad\quad \geqslant 0 \\ -p_1 \qquad + M_1 + \tfrac{1}{2}M_2 \geqslant 0 \\ -p_1 + p_2 + M_1 + M_2 \geqslant 0 \\ \quad -p_2 + \tfrac{1}{2}M_1 + M_2 \geqslant 0 \\ \quad -p_2 \qquad\quad + \tfrac{3}{2}M_2 \geqslant 0\end{array}\right\} \qquad \text{(11)}$$

The material consumption parameter X will now be introduced into set (10) by the replacement of M_1 by $(X - M_2)$ from equation (9). Upon slight rearrangement,

$$\left.\begin{array}{l}-M_2 + X - \tfrac{2}{3}p_1 \geqslant 0 \\ -M_2 + 2X - 2p_1 \geqslant 0 \\ \quad M_2 + X - 2p_2 \geqslant 0 \\ \quad M_2 \qquad - \tfrac{2}{3}p_2 \geqslant 0\end{array}\right\} \qquad \text{(12)}$$

together with

$$X \geqslant (p_1 - p_2) \qquad \text{(13)}$$

Now for the problem of determining the minimum value of X, the value of M_2 is not required, and Dines' method may be employed again on set (12) to eliminate M_2. On performing this operation, inequality (13) becomes redundant, and the only significant inequality resulting is

$$X \geqslant p_1 + \tfrac{1}{3}p_2 \qquad \text{(14)}$$

It should be repeated that this single inequality is a necessary and completely sufficient condition that values of M_1, M_2 and c can be found to satisfy the original set (10). Since it is required that X should be as small as possible, the equality sign will be taken in (14), so that

$$X = p_1 + \tfrac{1}{3}p_2 \qquad \text{(15)}$$

Substitution of this value of X back into the previous sets gives the unique values

$$\left.\begin{aligned} M_1&=p_1-\tfrac{1}{3}p_2 \\ M_2&=\tfrac{2}{3}p_2 \\ 2c&=-\tfrac{2}{3}p_2=-M_2 \end{aligned}\right\} \quad \ldots\ldots \quad (16)$$

The bending moment distribution resulting from the analysis is shown in fig. 7, plastic hinges being formed at all three of the critical points.

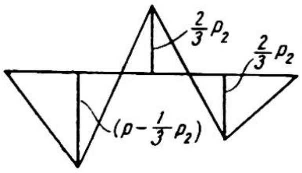

Fig. 7

The method given above for minimum weight design against collapse under fixed loads has been applied by the Author (1950a, 1950b) to the solution of a rectangular portal frame (cf. fig. 2), and also to derive a design method for continuous beams of any number of spans under either concentrated or distributed loads.

(c) *Minimum weight design against collapse under variable loads*

Consider the same beam in fig. 3, but with the loads varying arbitrarily between the limits

$$\left.\begin{aligned} -Q_1 \leqslant P_1 \leqslant Q_1 \\ -Q_2 \leqslant P_2 \leqslant Q_2 \\ Q_1 \geqslant Q_2 \end{aligned}\right\} \quad \ldots\ldots \quad (17)$$

The work proceeds as before up to the derivation of set (11). Now, in this set, the worst values of p_1 and p_2 (i.e. $\pm q_1$, $\pm q_2$) must be inserted in each inequality, giving

$$\left.\begin{aligned} -q_1 \quad\; +\tfrac{3}{2}M_1 \qquad\quad &\geqslant 0 \\ -q_1 \quad\; +\ M_1+\tfrac{1}{2}M_2 &\geqslant 0 \\ -q_1-q_2+\ M_1+\ M_2 &\geqslant 0 \\ -q_2+\tfrac{1}{2}M_1+\ M_2 &\geqslant 0 \\ -q_2 \qquad\quad +\tfrac{3}{2}M_2 &\geqslant 0 \end{aligned}\right\} \quad \ldots\ldots \quad (18)$$

Operating on set (18) as before to find the minimum value of X, it is found that

$$\left.\begin{aligned} M_1+M_2=X=q_1+q_2 \\ (q_1+\tfrac{1}{3}q_2) \geqslant M_1 \geqslant \tfrac{2}{3}q_1 \\ 2q_2 \geqslant M_2 \geqslant \tfrac{2}{3}q_2 \\ (\tfrac{1}{3}q_1+q_2) \geqslant M_2 \end{aligned}\right\} \quad \ldots\ldots \quad (19)$$

As a specific example, suppose $q_1=q_2=q$. Then

$$\left.\begin{aligned} M_1+M_2=2q \\ \tfrac{4}{3}q \geqslant M_1 \geqslant \tfrac{2}{3}q \\ \tfrac{4}{3}q \geqslant M_2 \geqslant \tfrac{2}{3}q \end{aligned}\right\} \quad \ldots\ldots \quad (20)$$

and any values of M_1 and M_2 satisfying (20) will give a constant material consumption. (It is perhaps of interest to note that for $X=\Sigma(M)^n l$, where $n<1$, the minimum material consumption is given by $M_1=2M_2=\tfrac{4}{3}q$ (or *vice versa*), the worst case occurring for $M_1=M_2=q$. An asymmetrical solution is obtained for what appears to be a completely symmetrical problem. For $n=0{\cdot}6$, the symmetrical solution gives an increase of less than 2% in material consumption compared with the asymmetrical solution.)

INEXACT METHODS

The theorems concerning the existence of upper and lower bounds on the collapse load of a structure were first proved rigorously by Greenberg and Prager (1950). It is assumed that the loads on a structure are all specified in terms of one load, so that when the collapse load is mentioned, this implies the whole system of loads.

An upper bound on the collapse load

Suppose that enough hinges are inserted into a redundant structure in order to turn it into a mechanism of one degree of freedom. Hill (1948) has shown that the stress system is constant during collapse of an ideally plastic body, so that for the frame with one degree of freedom, the equation of virtual work may be written, equating the work done in the hinges to the work done by the external load during a small displacement in the equilibrium state. The work done in a hinge is equal to the full plastic moment multiplied by the absolute value of the change in angle at that hinge (i.e. plastic rotation) and the work done by the load simply the load multiplied by its displacement. There will, of course, be elastic displacements obtaining in the frame, but these do not appear in the equations provided it is assumed that they are small so that the overall geometry of the frame is not disturbed.

For any arrangement of hinges in the frame producing a mechanism of one degree of freedom, the load given by the virtual work equation is either greater than or equal to the true collapse load.

A lower bound on the collapse load

If a state can be found for the structure which nowhere violates the yield condition, and which is an equilibrium state for a given value of the load, then that value is either less than or equal to the value of the true collapse load.

In practice, Greenberg and Prager found it useful to derive a lower bound from the mechanism giving the upper bound. The example will make the ideas clear.

Suppose the values of the loads in fig. 3 are

$$P_1=2P_2=2P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

and that as a first trial the mechanism in fig. 8 is assumed for failure. The rotation at the central hinge is θ, and at the hinge under the load P, 2θ. Hence, by virtual work,

$$P \, . \, \frac{l}{2}\theta=M_0(2\theta)+M_0(\theta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

i.e.

$$p=\tfrac{3}{2}M_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

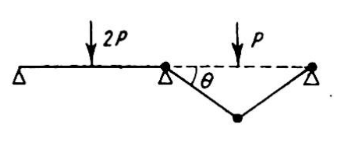

Fig. 8

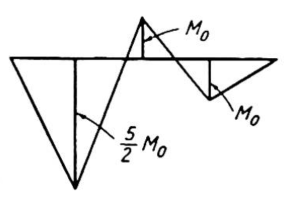

Fig. 9

(It is taken that the beam has the same full plastic moment M_0 in both spans.) By the upper bound theorem, the true value of the collapse load (p_c) is less than $\frac{3}{2}M_0$. The bending moment distribution corresponding to the assumed mechanism and this value of p given in equation (23) is shown in fig. 9, from which it will be seen that the yield condition is exceeded under the load $2P$ in the ratio 5/2. Suppose now that the loads are reduced in the ratio 2/5. Then if the values in fig. 9 are multiplied by 2/5, an equilibrium bending moment distribution is obtained which nowhere violates the yield condition. Hence the load of $\frac{3}{5}M_0$ is a lower bound on the collapse load, i.e.

$$\tfrac{3}{5}M_0 \leqslant p_c \leqslant \tfrac{3}{2}M_0 \qquad \ldots \ldots \ldots \quad (24)$$

It can be shown that removing one of the assumed hinges to the point of maximum moment will improve the bounds on the collapse load; in this example, shifting the hinge from under the load P to under the load $2P$, while retaining the central hinge, immediately gives the correct solution $p_c=\frac{3}{4}M_0$. There is, however, no means at present of choosing which hinge to remove, and in any case the bounds cannot be narrowed indefinitely; either they are separated by a finite amount, which may be quite large for even a relatively redundant frame, or the exact solution will be obtained. Accordingly, Nachbar and the Author (1950) have developed more general methods for obtaining both upper and lower bounds which may be made as close to the true collapse value as is considered necessary.

A general method for the upper bound

Suppose yield hinges are inserted into the frame at any suspected critical sections. In general a frame of N degrees of freedom will result, specified in terms of N deflection parameters. If the equation of virtual work is written, then the corresponding value of the load is an upper bound on the true collapse load. In fact, the virtual work equation is inapplicable, since the system is not an equilibrium system, but it may be shown that the value of the load resulting from this equation is in fact a true upper bound, providing that the mechanism is such that the work done by the loads is positive.

For the general mechanism in fig. 10,

$$2P\,.\,\frac{l}{2}\theta_1+P\,.\,\frac{l}{2}\theta_2=M_0(|2\theta_1|+|\theta_1+\theta_2|+|2\theta_2|)$$

i.e.

$$p=M_0\left(\frac{|2\theta_1|+|\theta_1+\theta_2|+|2\theta_2|}{4\theta_1+2\theta_2}\right) \qquad \ldots \ldots \quad (25)$$

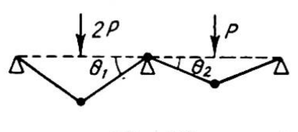

Fig. 10

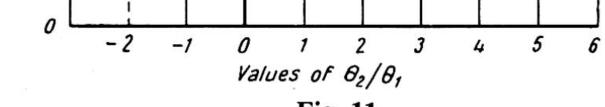

Fig. 11

In equation (25), values of θ_1 and θ_2 must be chosen to give the minimum value of p; since p is always an upper bound on p_c, the minimum value will be equal to p_c. A plot of equation (25) is given in fig. 11, from which it will be seen that $p_c=\frac{3}{4}M_0$ corresponds to $\theta_2=0$. The minimum is not a stationary value, since equation (25) is a ratio of two linear expressions. Nachbar has shown that equations of this type containing absolute values can be reduced by rational successive steps, and the method has been applied to mechanisms with a large number of parameters necessary for their specification.

A general method for the lower bound

Suppose the members of a redundant structure are cut in such a way that a number of separate redundant or statically determinate structures are formed. If the collapse loads are calculated for each of these resulting structures, then the lowest value of these loads is less than the collapse load of the structure as a whole. The proof of this theorem follows immediately from the special lower bound theorem above. An immediate corollary is that if a cut portion of the structure carries no load, then that portion can be ignored in the derivation of the lower bound. In order to make the theorem of practical use, an additional lemma is needed. The collapse load of a structure is unaffected by any initial system of residual stresses (moments, shear forces). That is, at a cut, equal and opposite longitudinal forces, shear forces, and moments may be introduced in an attempt to raise the lower bound.

Fig. 12

Fig. 13

Suppose the beam in the previous example is cut at the central support; then the two separate beams shown in fig. 12 will be obtained. The collapse loads of the right- and left-hand halves are respectively $p=M_0$ and $p=\frac{1}{2}M_0$, i.e.

$$p_c \geqslant \tfrac{1}{2}M_0 \qquad (26)$$

Now if a central moment is introduced (fig. 13), it is easy to show that the collapse loads are respectively

$$p=\frac{M+2M_0}{2} \quad \text{and} \quad p=\frac{M+2M_0}{4} \qquad (27)$$

The maximum value which M can take is, of course, M_0, and hence from (27)

$$p_c \geqslant \tfrac{3}{4}M_0 \qquad (28)$$

and the problem has been completed. For other more complicated examples (a two-storey, two-bay portal frame has been solved under both concentrated and distributed loads), it is found that shear and longitudinal forces as well as bending moments must be introduced at the cuts.

Space frames

The type of space frame considered has members which lie all in the same plane, all loads acting perpendicularly to this plane. Thus bending moments whose axes lie perpendicular to the plane and shear forces in the plane are zero. Any member of the frame is then acted upon by shear forces parallel to the applied loads and by two moments whose axes lie in the plane, that is, a bending moment (M) and a torque (T). For ideal plasticity, hinges will be formed in exactly the same way as for plane frames; the breakdown criterion will be some such expression as

$$g(M, T)=g(M_0, 0)=\text{const.} \qquad (29)$$

where M_0 is the full plastic moment in pure bending, as before. At any one hinge, the maximum work principle of Hill (1948) shows that the moment and torque will be constant during collapse, and that the rate at which work is done at a hinge will be a maximum. If β and θ are the incremental changes in angle in bending and twisting respectively during a displacement in the equilibrium collapse configuration, then the rate at which work is done is

$$M\beta + T\theta \qquad (30)$$

For a maximum,

$$\beta \,.\, \delta M + \theta \,.\, \delta T = 0 \qquad (31)$$

Now the breakdown criterion, equation (29), gives

$$\frac{\partial g}{\partial M} \,.\, \delta M + \frac{\partial g}{\partial T} \,.\, \delta T = 0 \qquad (32)$$

that is,

$$\frac{\beta}{\theta} = \frac{\dfrac{\partial g}{\partial M}}{\dfrac{\partial g}{\partial T}} \qquad (33)$$

This flow relationship may be solved simultaneously with the breakdown criterion to give the moment and torque acting at a hinge during any collapse displacement.

The author (1951) has shown that for a box section, equation (29) becomes

$$M^2 + \tfrac{3}{4}T^2 = M_0^2 \qquad (34)$$

For the present purposes, the circular breakdown criterion

$$M^2 + T^2 = M_0^2 \qquad (35)$$

will be used for the sake of simplicity. The restriction in no way affects the generality of the methods proposed for the solution of space frames.

Equation (33) becomes

$$\frac{\beta}{\theta} = \frac{M}{T} \qquad (36)$$

which, taken with equation (35), gives

$$\left.\begin{aligned} M &= \frac{\beta}{\sqrt{\beta^2+\theta^2}} M_0 \\ T &= \frac{\theta}{\sqrt{\beta^2+\theta^2}} M_0 \end{aligned}\right\} \qquad (37)$$

together with the expression for the work done at the hinge (expression (30))

$$\text{Plastic work} = M_0\sqrt{\beta^2+\theta^2} \qquad (38)$$

Owing to the non-linearity of the breakdown criterion, it is not possible to set up exact systems of linear inequalities to be solved by the Dines' method. However, approximations may be made to the breakdown criterion itself; for example, equation (35) could be replaced by the circumscribed octagon

$$\left.\begin{aligned} M &= \pm M_0 \\ T &= \pm M_0 \\ M \pm T &= \pm\sqrt{2}M_0 \end{aligned}\right\} \qquad (39)$$

and the moment M and torque T at any section constrained to lie within this yield domain.

As will be shown, simple problems are best solved by a direct method; and the systems of linear inequalities corresponding to equations (39) become too complicated for practical use in the solution of highly redundant structures. For the latter, the determination of bounds on the collapse load seems to give the quickest results.

Direct solution

As an example of the direct method, consider the symmetrical two-leg right-angle bent shown in fig. 14. The ends A and D are *encastré* against both torque and moment, and the load P acts at the midpoint B of the leg AC. Suppose failure occurs by the formation of symmetrical hinges at A and D, so that the point C moves vertically downward for a small displacement. It is easy to see that $\beta_A=\beta_D=\theta_A=\theta_D=\theta$, say, so that, from equation (38), the work done in the two hinges is

$$2M_0\sqrt{2}\theta^2 \quad \ldots \ldots \ldots \ldots \quad (40)$$

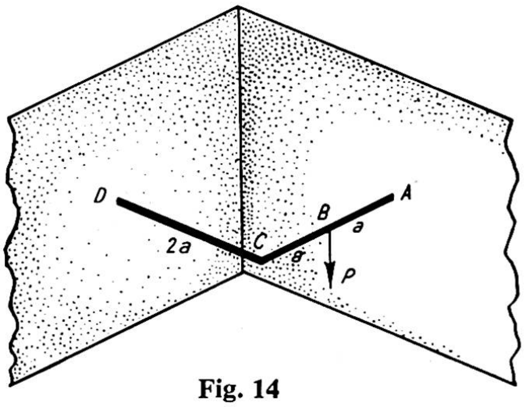

Fig. 14

while the work done by the load P is

$$Pa\theta \quad \ldots \ldots \ldots \ldots \quad (41)$$

Equating these two expressions, and using the upper bound theorem given above,

$$P_c \leqslant P = 2\frac{\sqrt{2}M_0}{a} \quad \ldots \ldots \ldots \quad (42)$$

The frame is, of course, statically determinate in this collapse configuration, and, by using equations (37) to determine the conditions at the hinges, the forces and moments shown in fig. 15 are obtained. The yield criterion is exceeded by the greatest amount at B, where the moment and torque are $\sqrt{2}M_0$ and $\frac{1}{\sqrt{2}}M_0$ respectively, i.e.

$$M_B^2+T_B^2=\tfrac{5}{2}M_0^2 \quad \ldots \ldots \ldots \quad (43)$$

Hence if the load is reduced by a factor $\sqrt{2/5}$, a lower bound will be obtained,

$$\frac{4}{\sqrt{5}}\frac{M_0}{a} \leqslant P_c \leqslant \frac{4}{\sqrt{2}}\frac{M_0}{a} \quad \ldots \ldots \ldots \quad (44)$$

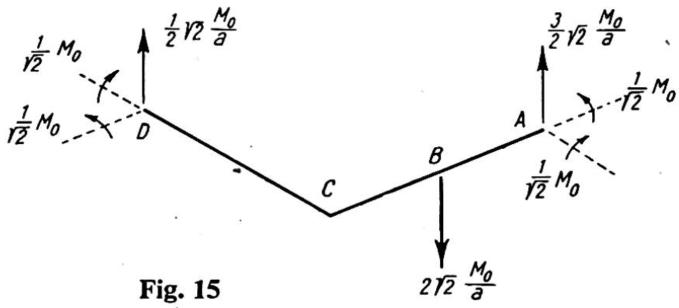

Fig. 15

In order to improve these bounds, a hinge must be inserted at B; but collapse actually occurs with hinges at all three points A, B and D. At first sight this would appear to be a mechanism of three independent degrees of freedom. In fact, owing to the simultaneity of the breakdown and flow criterions (equations (35) and (36)), each hinge as a whole has only one degree of freedom; since a continuity condition is required at each hinge, a space frame of the type considered here may collapse with any number of hinges formed in its members, and an extra hinge may be inserted without actually increasing the number of degrees of freedom.

The general method for the exact solution of a structure with R redundancies may be tabulated as follows:

(1) Construct a mechanism with N hinges.
(2) Specify the mechanism in terms of an arbitrary displacement (one degree of freedom) and $[2N-(R+1)]$ deflection parameters α_j.
(3) $(2N-R)$ equilibrium equations may be formulated in terms of the moments (M_i) and torques (T_i) at the hinges and the applied load.
(4) M_i and T_i at each hinge may be calculated in terms of the α_j from the breakdown and flow criteria.
(5) The load may be eliminated from the $(2N-R)$ equilibrium equations, leaving a set of $(2N-\{R+1\})$ simultaneous equations for the determination of the α_j.
(6) Having determined the α_j, the moments and torques at each hinge may be calculated, and hence the value of the load. This value is an upper bound on the collapse load.
(7) If the yield criterion is violated at any point in the structure, a lower bound may be determined.
(8) If hinges are moved or added to the points where the yield criterion is violated, the whole process can be repeated.

Following these rules, and inserting hinges at A, B and D, the final exact solution is found to be

$$P_c=\frac{8}{\sqrt{10}}\frac{M_0}{a}=2{\cdot}53\frac{M_0}{a} \quad \ldots\ldots\ldots \quad (45)$$

which as a check lies between the previous limits (44).

Bounds on the collapse load

In the method outlined above, it has been tacitly assumed that the theorems on upper and lower bounds may be extended from plane to space frames; this is in fact the case, and indeed Drucker, Greenberg and Prager (1950) have shown that the special theorems may be applied to the problem of the continuum. The general theorem of an upper bound determined from a non-equilibrium mechanism is also valid for space frames, and this gives the quickest method for the solution of such problems.

The advantage of the kinematic method of determining an upper bound on the collapse load is that no reference is made to equilibrium conditions. Suppose, for

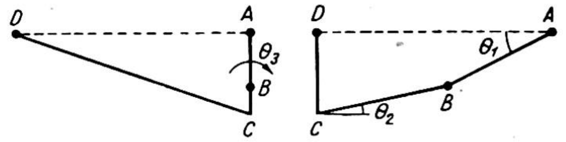

Fig. 16

example, the mechanism in fig. 16 (horizontal projection of frame in fig. 14) is specified by assigning arbitrary deflections to the joints B and C, with hinges occurring at A, B and D. Then an upper bound may be determined simply by equating the work done in the hinges to the work done by the load. By trial of various mechanisms, this bound may be lowered. Alternatively, if, after a trial, the frame is examined statically, it will be found that it is impossible to satisfy equilibrium conditions, the total load at B being either lower or in excess of the value of P determined from the work equation. This implies that an extra (positive or negative) force is required at B in order to produce the originally assumed collapse configuration. The significance of this force is best appreciated by an example.

In fig. 16, take $\delta_B=\delta_C=2a$, say, since the mechanism may be specified in terms of one unknown degree of freedom. The following table gives the conditions at the hinges.

TABLE I

Hinge	β	θ	$\sqrt{\theta^2+\beta^2}$	Moment $(\times M_0)$	Torque $(\times M_0)$
A . . .	2	0·5*	2·062	0·97	0·24
B . . .	2	0·5*	2·062	0·97	0·24
D . . .	1	0	1·000	1·00	0

The asterisked values were chosen to make the torques equal at A and B, as they should be; this is an unnecessary restriction, and improves only slightly the value of the upper bound, and any values of the twist totalling 1·0 could have been used. The work equation gives

$$P\,.\,2a=5{\cdot}124M_0$$

i.e.

$$P_c\leqslant P=2{\cdot}56\frac{M_0}{a} \qquad \text{. . . . . (46)}$$

The statical analysis of the frame is shown in fig. 17. The number in a circle at the joint B gives the actual load required to maintain equilibrium, and it appears that a load of $2{\cdot}91M_0/a$ is required as against the calculated value $2{\cdot}56M_0/a$. Since the equilibrium load is greater than it should be, it is indicated that the assumed deflection of the point B was too large; if this deflection is reduced slightly, a better bound should result. Similarly, a negative load is required at C; the deflection should be increased.

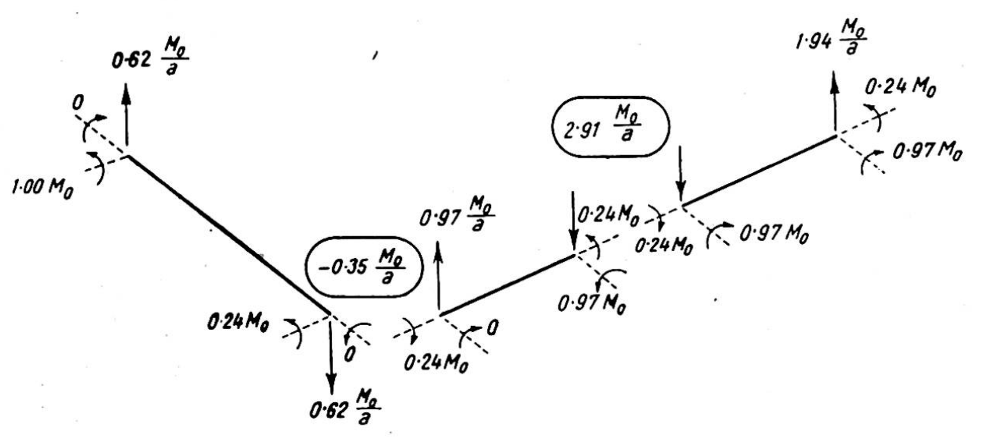

Fig. 17

In working more complicated examples, it is found that the process of adjusting deflections at neighbouring joints bears a marked resemblance to a relaxation process, and that a reduction in the out-of-balance forces at one joint induces increased errors at the ones adjacent. However, the technique is soon mastered, and the Author (1950c) solved, with very little labour, a rectangular grid formed by a set of parallel beams intersecting at right angles another set of 9 beams, loaded transversely at each of the 81 joints, and requiring 108 hinges in the collapse mechanism.

When it is suspected that the upper bound is fairly good, small adjustments in the statical analysis will produce an equilibrium system. For example, in fig. 17, if the torque in CD is increased from 0 to $0{\cdot}35M_0$, the other values remaining unchanged, an equilibrium system results which, however, violates the yield condition at the hinge D in the ratio 1·06. Hence, using the value in equation (46)

$$2{\cdot}41\frac{M_0}{a} \leqslant P_c \leqslant 2{\cdot}56\frac{M_0}{a} \qquad \text{(47)}$$

The general procedure for the solution of space frames may be tabulated as follows:

(1) Insert yield hinges at a large number of points in the frame, producing a mechanism of many degrees of freedom. The hinges should be placed at all the sections at which it is suspected actual hinges might occur in the collapse.

(2) Assign arbitrary (reasonable) deflections to the joints of the grid, and determine the corresponding changes in angle at each hinge. Equating the work dissipated in the hinges to the work done by the external loads gives a value of the load which is in excess of the true collapse load.

(3) Calculate the out-of-balance forces at each joint that are necessary to produce the assumed deflections. If the out-of-balance force acts in the same direction as the actual load at a joint, the deflection of that joint was estimated as too large, and *vice versa*.

(4) Adjust the deflections, and repeat the whole process.

(5) At any stage, if the out-of-balance forces are small, and it is suspected that the upper bound is a good estimate of the collapse load, a statical analysis may be made. Small adjustments are made in the values of the various shear forces and moments in order to produce an equilibrium system, from which a lower bound may be determined.

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References

British Welding Research Association. 1947. Report FE1/2.

Dines, L. L. 1918. *Ann. Math.*, **20**, 191.

Drucker, D. C., Greenberg, H. J., and Prager, W. 1950. Technical Report A18-3 from the Graduate Division of Applied Mathematics, Brown University, to the Office of Naval Research, U.S.

Greenberg, H. J., and Prager, W. 1950. To be published in *Proc. A.S.C.E.*

Heyman, J. 1950a. Technical Report A11–45 from the Graduate Division of Applied Mathematics, Brown University, to the Office of Naval Research, U.S.

1950b. *Quart. J. Appl. Math.*, **8**, No. 4, January 1951.

1950c. Technical Report A11–52 from the Graduate Division of Applied Mathematics, Brown University, to the Office of Naval Research, U.S. To be published in *J. Appl. Mech.*

1951. *J. Appl. Mech.*, **18**, No. 2, June 1951.

HILL, R. 1948. *Quart. J. Mech. and Appl. Math.*, **1**.

HORNE, M. R. 1950. Unpublished.

NACHBAR, W., and HEYMAN, J. 1950. Technical Report A11–54 from the Graduate Division of Applied Mathematics, Brown University, to the Office of Naval Research, U.S. To be published in *J. Appl. Mech.*

NEAL, B. G. 1950a. *Phil. Trans. Roy. Soc. A.*, **846**, 197–242.

NEAL, B. G., and SYMONDS, P. S. 1950. *J. Inst. Civ. Engrs.*, **35**, 20.

Summary

The preparation of this paper forms part of a general investigation into the behaviour of rigid frame structures being carried out at the Cambridge Engineering Laboratory under the direction of Professor J. F. Baker. The paper deals with the mathematical analysis and design of both plane and space frames, and the ideas are presented with reference to very simple examples in order to illustrate the techniques developed. The first part considers methods for the exact determination of conditions at collapse of rigid ideally plastic plane structures. In the second part it is shown that inexact methods lead to upper and lower bounds on the collapse loads, and that these bounds may be made as close as is considered necessary. The various theorems are applied in the third part to the solution of space frames.

Résumé

Le présent mémoire rentre dans le cadre d'une investigation générale portant sur le comportement d'ouvrages en cadres rigides, investigation actuellement en cours au Cambridge Engineering Laboratory, sous la direction du Professeur J. F. Baker. L'auteur traite de l'analyse mathématique et du calcul des cadres, tant en plan que dans l'espace, et son exposé est accompagné d'exemples très simples, qui illustrent les procédés adoptés.

La première partie se rapporte aux méthodes de détermination exacte des conditions qui se manifestent au rupture des ouvrages plans rigides idéalement plastiques. Dans la deuxième partie, l'auteur montre que des méthodes non rigoureuses permettent de fixer des limites supérieures et inférieures aux charges sous lesquelles les ouvrages cèdent; ces limites peuvent d'ailleurs recevoir des valeurs aussi étroites qu'il est jugé nécessaire. Les différents théorèmes sont appliqués, dans la troisième partie, au calcul de cadres à trois dimensions.

Zusammenfassung

Die Arbeiten zum vorliegenden Aufsatz stellen einen Teil der umfassenden Untersuchungen über das Verhalten steifer Rahmenkonstruktionen dar, die am Cambridge Engineering Laboratory unter der Leitung von Professor J. F. Baker durchgeführt werden. Der Verfasser behandelt die mathematische Untersuchung und Bemessung ebener und auch räumlicher Rahmen und entwickelt seine Ueberlegungen an Hand sehr einfacher Beispiele, an denen er die gewählten Verfahren darlegt. Der erste Teil behandelt Methoden zur genauen Bestimmung der Bruch-Verhältnisse steifer, ideal-plastischer ebener Tragwerke. Im zweiten Teil wird gezeigt, dass durch Näherungsmethoden eine obere und untere Grenze der Bruchlast ermittelt werden kann und dass diese Grenzwerte so nahe zusammengebracht werden können, wie es für notwendig erachtet wird. Die verschiedenen Theorien werden im dritten Teil zur Berechnung räumlicher Rahmenwerke angewandt.

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