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**Autor:** Cowan, Henry J. / Armstrong, Stewart

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# CII 1

## Reinforced concrete in combined bending and torsion

## Le comportement du béton en flexion et torsion combinées

## Stahlbeton unter Biegung mit Verdrehung

HENRY J. COWAN, M.Sc., A.M.I.Struct.E.
University of Sheffield

and

STEWART ARMSTRONG, B.Sc., A.M.I.C.E.
University of Sheffield

INTRODUCTION

In practical reinforced-concrete construction torsion usually occurs as a secondary effect of bending. The only important example of torsion unaccompanied by bending is the screw pile, which is a case of combined torsion and compression, with the torsion as the controlling factor in the design.

Members subject to combined bending and torsion should normally be designed in the same way as sections subject to combined bending and shear, which is a similar problem. The section should be proportioned to resist the bending moment, and additional torsional shear reinforcement should then be introduced if the torsional resistance of the section is inadequate.

In practice, however, this procedure is rarely followed. The only major code of practice in current use known to the authors which makes any provision for the design of sections in torsion is the code issued by the French Ministry of Reconstruction[1] in 1945. In general, engineers seem to regard reinforced concrete as an unsuitable material to resist torsion, and considerable ingenuity is often exercised in the layout of the structural framework to eliminate secondary torsion.

The problem of combined bending and torsion is none the less very commonly met with. It arises essentially out of the monolithic character of reinforced-concrete construction. In a beam and slab floor any asymmetry in the loading of the slab produces torsion in the supporting beams, the extreme case being a continuous beam with alternate spans loaded. The L-beams at the edge of the slab are always subject to torsion. This problem, particularly important in long bridge-girders, has been discussed by Kasarnowsky[2] and Jakobsen.[3] The Waterloo Bridge,[4] an example of this case, and the Royal Festival Hall[5] are the only two major reinforced-concrete structures in Great Britain known to the authors in which torsional stresses were considered in the design; both have shallow box-girders.

[1] For references see end of paper.

In rectangular rigid space-frames the end moments of a loaded beam give rise to bending in the columns and torsion in the beams which are connected to the same joint at right angles to the loaded beams. This problem has been discussed by Andersen,[6] Matheson [7] and Chronowicz.[8] It has been pointed out [9] that the torsional moments are never likely to be significant in the case of steel structures, but that they could be appreciable in reinforced-concrete construction.

## SURVEY OF PREVIOUS TEST RESULTS

Tests on the strength of reinforced concrete in bending are too numerous and well known to require mention. The experimental evidence on the torsional strength of concrete and reinforced concrete is more scanty. The early investigations by Mörsch, Bach and Graf [10, 11] are still the most comprehensive on record. They showed that plain concrete specimens fail with a helical fracture following the lines of the principal tensile stress. The strength of concrete in torsion is determined by its tensile strength; for plain concrete it is therefore low, and failure occurs suddenly without warning. No appreciable improvement can be achieved by adding either longitudinal bars or vertical stirrups alone, because the principal stress makes an angle of 45° with the axis. A combination of longitudinal bars and stirrups or spirals produces substantial increases in ultimate strength; failure is gradual and accompanied by considerable cracking and deformation. A theory for the strength of circular reinforced-concrete sections based on the German experiments was proposed by Rausch.[12]

Experiments on T- and L-shaped sections showed that the ultimate torsional strength of concrete could not be satisfactorily explained in terms of the elastic theory. It has been noted by Bach[10] and Gilkey[13] that the diagonal tensile strength in torsion computed on the basis of the elastic theory was lower than the direct tensile strength of the concrete. This discrepancy increased as the section departed from the circular shape.[14, 15] From an analysis of earlier work Marshall [16] showed that agreement between the results of a wide range of experiments could be obtained if concrete was treated as a plastic material. This result was later confirmed by Nylander [17] by direct experiment.

Miyamato's work [18] provided a measure of the efficacy of the various ways of placing the reinforcement. It demonstrated the superiority of continuous spirals over a combination of longitudinal bars and vertical stirrups. It is, however, not easy to explain this result from theoretical considerations. It seems likely that the difference is due mainly to bond failure. The formation of a crack in the concrete throws the whole of the tensile forces at that point on to the reinforcement. This results in local yielding of the steel, which gives rise to large strains and so produces local bond failure. The loads are applied to the concrete and transmitted to the steel through bond. The effectiveness of the reinforcement depends therefore on an adequate length of bond. This is more easily attained in a continuous spiral. The practical usefulness of spiral reinforcement is, however, almost entirely confined to the design of screw piles, because of the difficulty of producing spirals satisfactorily for non-circular sections.

There are two previous investigations on the strength of concrete in combined bending and torsion. Fisher [19] carried out a series of experiments on plain concrete. The ultimate strength of concrete in both bending and torsion is controlled by either the maximum tensile stress or the maximum tensile strain, and the same criterion applies to combined bending and torsion. The torsional strength of plain concrete is thus *reduced* by the addition of bending. Fisher's experiments were not conclusive, but generally favoured the maximum stress theory.

Nylander's experiments on combined bending and torsion[17] were carried out on beams with longitudinal reinforcement. The strength in bending is consequently determined either by the crushing strength of the concrete in compression or by the yield stress of the steel in tension. Nylander notes that "bending moments exert in general a *favourable* effect on the torsional strength." This observation indicates the fundamental difference between the criteria of failure for plain and reinforced concrete, and it is fully borne out by the present investigation.

## THEORETICAL ANALYSIS

Although general solutions have been obtained for the torsion of composite sections of two isotropic materials, these are not in a form suitable for application to reinforced-concrete design. The substitution of a simpler approximate method is justified because of the very small proportion of the cross-sectional area occupied by the steel.

The torsional resistance moment of a rectangular plain-concrete section within the elastic range is given by [20]:

$$M_{TPE}=\alpha b^2 d f_{max} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

where $b$ and $d$ are the shorter and longer sides of the rectangle respectively, $\alpha$ a constant varying with the ratio $d/b$ (Table I), and $f_{max}$ the maximum shearing stress in the section, which in pure torsion is also equal to the principal tensile stress.

If concrete is treated as a fully plastic material [17] the torsional resistance moment is increased to [21]:

$$M_{TPP}=\tfrac{1}{2}b^2\left(d-\tfrac{1}{3}b\right) f_{max} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

Neglecting the area of concrete displaced, the additional torsional resistance moment due to four longitudinal bars, one in each corner of the section, is [22]:

$$M_{TL}=\tfrac{1}{2}A_s(f_{yz}b'+f_{xz}d') \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

where $A_s$ is the cross-sectional area of all four bars, $b'$ and $d'$ are the distances between the bars (fig. 1) and $f_{xz}$ and $f_{yz}$ are the component shear stresses at the centres of the bars.

The principles underlying the design of shear reinforcement for beams subject to torsion are the same as for beams under the action of shear due to transverse loads. The solution can be obtained by equating the work done by the twisting moment to the strain energy stored in the beam.[23] Neglecting the tensile strength of the concrete, one of the basic assumptions of the British and most other European codes of practice, the twisting moment is given by:

$$M_{TS}=\frac{\lambda}{\sqrt{2}}\, b''\, d''\, \frac{A_v}{p} f_v \sin(\beta+45°) \quad . \; . \; . \quad (4)$$

where $p$ is the pitch of the shear reinforcement, $A_v$ the cross-sectional area of one bar, $f_v$ the maximum stress in the bar, and $\beta$ its angle of inclination to the horizontal. $b''$ and $d''$ are dimensions of the reinforcing cage (fig. 1), and $\lambda$ is a constant varying with the ratio $d''/b''$.

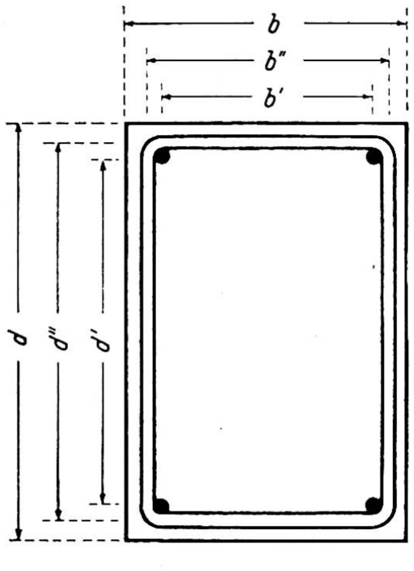



Fig. 1

TABLE I

| $d/b$ or $d''/b''$. | 1·0 | 1·5 | 2·0 | 2·5 | 3·0 |
|---|---|---|---|---|---|
| $\alpha$ | 0·21 | 0·23 | 0·25 | 0·26 | 0·27 |
| $\lambda$ | 1·67 | 1·60 | 1·61 | 1·65 | 1·69 |

Prior to the formation of cracks in the concrete the maximum stress in the shear reinforcement is $m$ times the stress in the surrounding concrete, where $m$ is the modular ratio, i.e. $f_v = mf_{max}$. The total resistance moment of the section is then equal to:

$$M_T = M_{TPE} + M_{TL} + M_{TS} \qquad (5)$$

When the ultimate tensile strength of the concrete is reached at the middle of the longer sides, cracks begin to form at 45° to the axis, and this leads to a very rapid increase in the stress in the shear reinforcement. If the tensile strength of the concrete after cracking is taken as nil, in accordance with European design practice, the torsional resistance moment is provided by the longitudinal reinforcement, and by a combination of the shear reinforcement and the concrete in which the concrete takes the diagonal compression and the steel the diagonal tension. For equilibrium the resultant diagonal compression must equal the resultant diagonal tension, and the total resistance moment is therefore given by:

$$M_T = M_{TL} + 2M_{TS} \qquad (6)$$

The alternative method is based on the American assumption that the shear

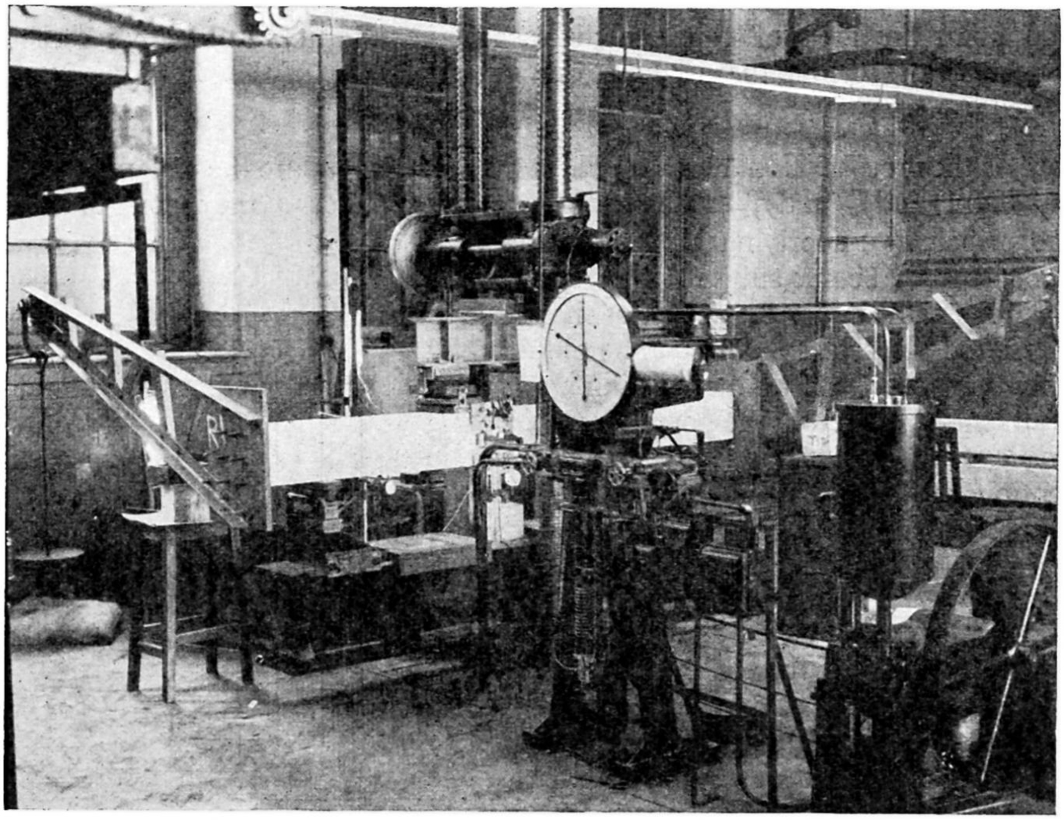
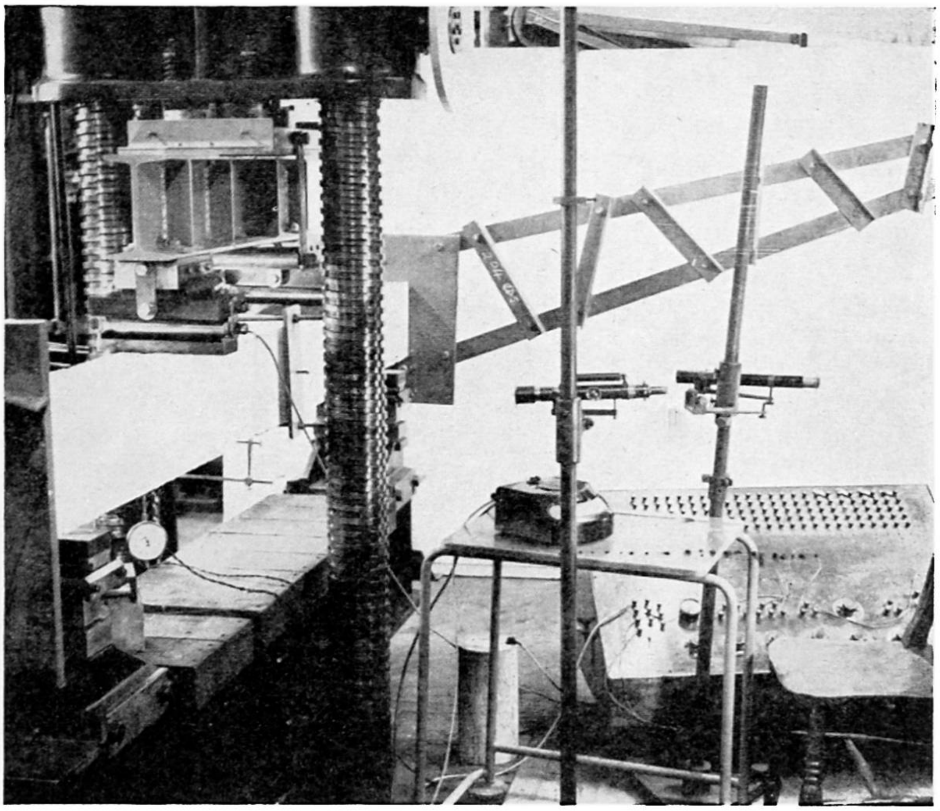
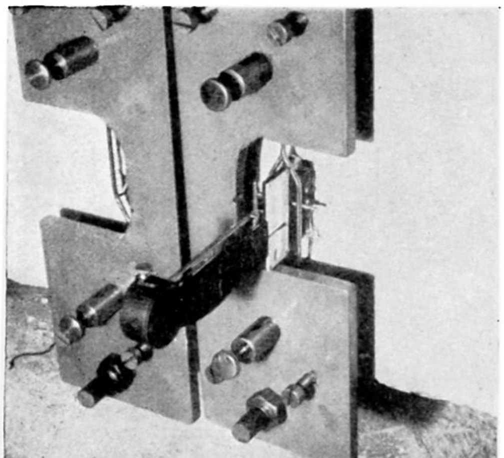


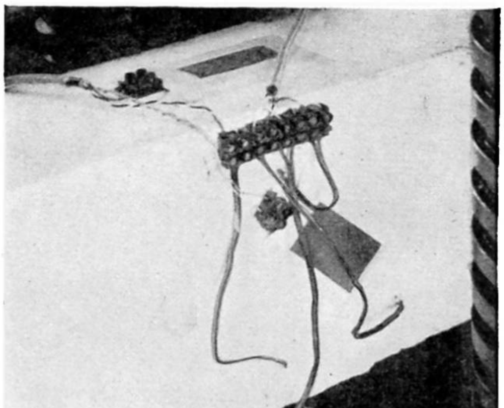
**Fig. 2.** General view of combined bending and torsion machine



(a) General view



(b) Detail of mechanical tensometers



(c) Detail of electrical resistance strain gauges

**Fig. 3.** Arrangement of instruments

reinforcement carries only the difference between the total shear and the shear taken by the concrete. This assumption can be justified on theoretical grounds similar to those advanced by Mylrea for shear due to transverse loads.[24] Analysis of a previous investigation [22] as well as of the experiments of the present investigation shows that the second assumption leads to results which agree well with the experimental data.

Before the concrete cracks, a redistribution of stresses takes place due to inelastic deformation of the concrete. These inelastic deformations in the concrete do not immediately produce plastic strains in the steel, since the yield strain of the steel is about five times as great as the ultimate tensile strain of the concrete. The steel therefore remains elastic almost up to the point of failure. In some cases the beam may fail before the steel yields. The ultimate torsional resistance moment is given by:

$$M_T = M_{TPP} + M_{TL} + M_{TS} \quad \ldots \ldots \quad (7)$$

## Description of experimental work

Fig. 2 illustrates the combined bending and torsion machine set up for the present investigation. Torsion was applied by means of cast-iron weights suspended from frames clamped to the ends of the concrete beam. Bending was applied at two points placed symmetrically about the centre of the beam by an Amsler hydraulic press. It was thus possible to vary the ratio of the bending moment to the twisting moment, both moments being constant over the central portion of the beam. All measurements were made in that portion. The deflection due to bending was measured with dial gauges, and the angle of twist with mirrors and telescopes (fig. 3(*a*)). Electric resistance gauges were used for measuring the strains on the surface of the concrete. Strains in the reinforcement were observed with Huggenberger tensometers in the earlier stages of the investigation (fig. 3(*b*)). They were later replaced by electric resistance gauges so as to avoid the cutting of holes through the concrete cover (fig. 3(*c*)).

All beams were 8 ft. 6 in. long and of rectangular cross-section, with dimensions as shown in fig. 4. The concrete was mixed in the proportions 1:2:2 by weight, and had a mean crushing strength of 8,000 lb./in.² at the time of testing. The steel used for the longitudinal reinforcement and for the stirrups had a yield stress of 48,500 and 20,800 lb./in.² respectively.

Assistance in obtaining materials and general financial support to the experimental work was given by the Cement and Concrete Association.

Strain measurements in the elastic range agreed closely with values calculated on the basis of the theoretical considerations set out in the previous section. The observations of the deflection due to bending and of the angle of twist were of the same order as the theoretical figures.

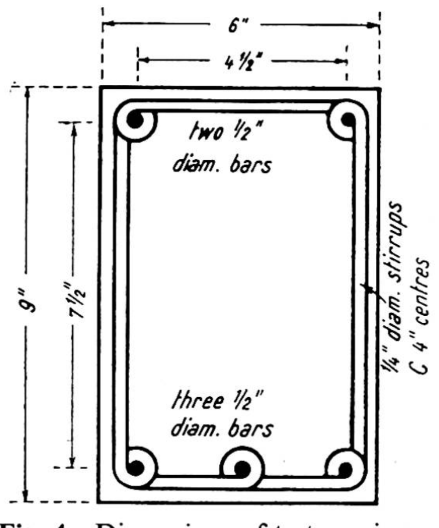
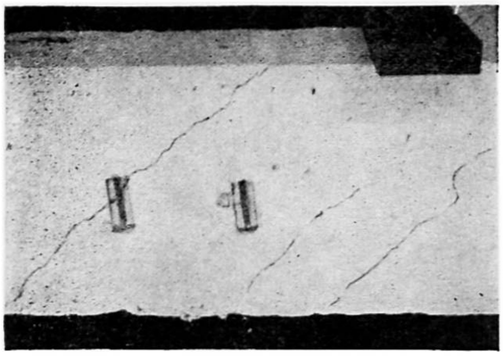
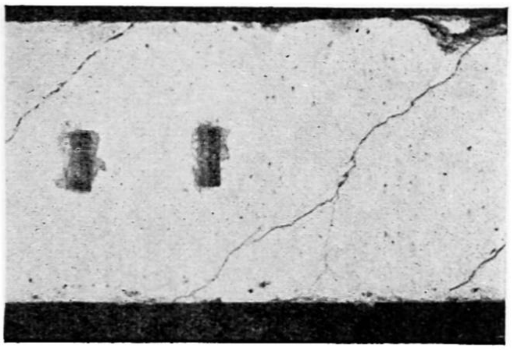


**Fig. 4.** Dimensions of test specimens

For a ratio of bending moment/twisting moment less than 2, beams showed the 45° diagonal cleavage-fracture characteristic of primary torsion-failure (figs. 5 (*a*) and (*b*)). Above that ratio the primary failure was in bending. Since the beams were under-reinforced, the failure was initiated by yielding of the steel accompanied by vertical cracks near the tension face; final collapse resulted from shear fracture of the



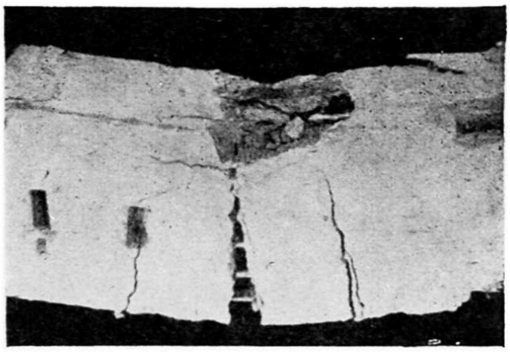
(*a*) Pure torsion



(*b*) $\frac{\text{Bending moment}}{\text{Twisting moment}}=2{\cdot}0$

Primary torsion failures

(*c*) $\frac{\text{Bending moment}}{\text{Twisting moment}}=2{\cdot}5$



(*d*) Pure bending

Primary bending failures

**Fig. 5.** Typical failures of test specimens

concrete near the compression face as shown by the formation of small debris similar to that produced in a cube crushing-test (fig. 5(*d*)).

The detailed experimental results will be published elsewhere.

## A THEORY FOR THE STRENGTH OF REINFORCED CONCRETE UNDER THE ACTION OF COMBINED STRESSES

The load/strain and load/deflection diagrams obtained from the experiments can be divided into two distinct parts with very marked difference of slope; at the time when this change of slope occurs pronounced cracking is usually observed. These points of discontinuity represent the bending moments and twisting moments at the breakdown of elastic action. The points marking the elastic limit are plotted in fig. 6. They fall on two separate lines representing the two types of failure of reinforced concrete.

The two theories most widely used to account for the cleavage failure of a brittle material are the maximum principal stress theory due to Rankine, and the maximum principal strain theory due to St. Venant.[21] Although on theoretical grounds there is much in favour of a maximum strain criterion, the results of this investigation, like those of the earlier tests by Fisher [19] and Nylander,[17] agree more closely with the maximum stress theory.

The work of Richart and Brandtzaeg [25] and of Balmer [26] has shown that the shear

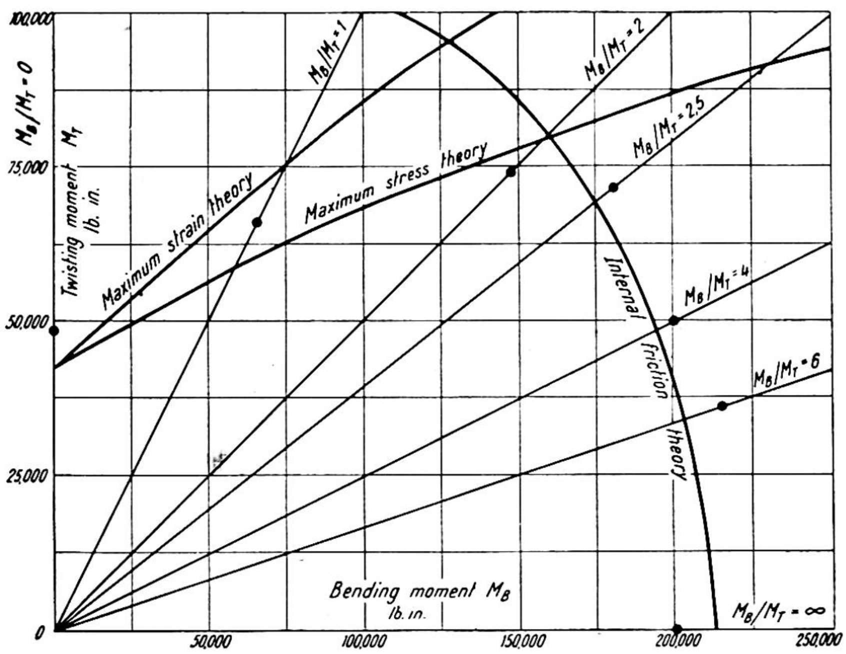



**Fig. 6.** Bending moments and twisting moments at the breakdown of elastic action. Experimental results ●

strength of concrete* is greatly influenced by the magnitude of the minor principal stress in accordance with Mohr's theory. For the purpose of this investigation the simpler internal friction theory of Coulomb [21] is substituted as a first approximation.

The two theories of failure can be represented by the space models shown in figs. 7(*a*) and (*b*). The combined model in fig. 7(*c*) indicates the failure of reinforced concrete under the action of combined stresses. The Rankine surfaces represent the cleavage, or primary torsion, failure, and the Coulomb surfaces the shear, or primary bending, failure.

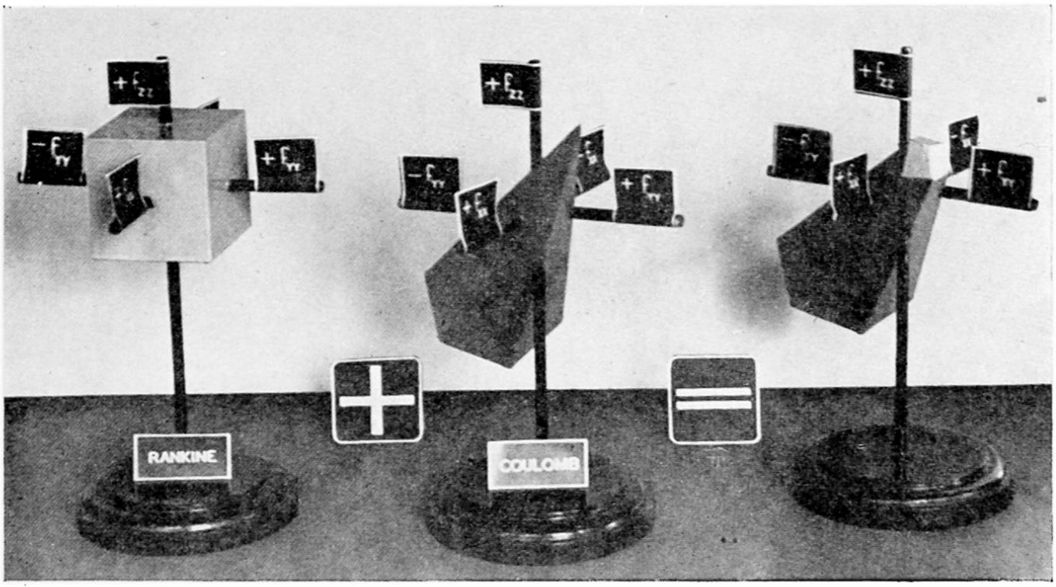
The full lines in fig. 6 correspond to these surfaces. They are computed from the criteria of maximum stress, maximum strain and internal friction, taking Poisson's ratio as 0·20 and the angle of internal friction as 35°. The lines are located on the axes by the values for the strength in pure torsion calculated from equation (5), and the strength in pure bending calculated from the conventional reinforced-concrete theory for the steel at its yield stress. The agreement with the experimental data is remarkably close.

## Conclusion

The addition of a small amount of bending to a reinforced-concrete section *increases* its resistance to torsion. This fact, although at first perhaps surprising, is born out by both theory and experiment.

Since torsion is almost invariably a secondary effect in civil engineering structures, very low ratios of bending moment/twisting moment do not occur in practice. If therefore the beam is proportioned in the first place to resist the bending moment, and additional shear reinforcement then introduced independently to resist the

* The phrases "shear failure" and "cleavage failure" are here used in the sense attaching to these terms in the literature on applied mechanics. In the case of reinforced concrete a "cleavage failure," which is easily recognised by the clean appearance of the fracture, is normally the result of diagonal tension due to *shear*. A "shear failure," always accompanied by the formation of debris, is almost invariably caused by the diagonal shear due to *compression*.




(a) (b) (c)

**Fig. 7.** Space models representing theory of failure for reinforced concrete. ($f_{xx}$, $f_{yy}$, and $f_{zz}$ are the three principal stresses, the positive sign denoting tension)

twisting moment, there is a considerable reserve in strength which can be used to reduce the factor of safety.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**Summary**

The paper gives an outline of earlier work on torsion in reinforced concrete. The results of a theory for the torsional strength of reinforced-concrete beams, both in the elastic range and at failure, are stated. The machine developed by the authors to test full-size sections in combined bending and torsion is described.

The results show that the addition of bending *increases* the torsional strength of a section. There is a distinct difference between primary torsion and primary bending failure. A theory of elastic breakdown combining the maximum principal stress criterion of Rankine and the internal friction criterion of Coulomb is advanced; this is in close agreement with the experimental results. The theory is illustrated by a space model.

The authors conclude with recommendations for the design of beams in combined bending and torsion.

**Résumé**

Les auteurs donnent un résumé des travaux précédents sur les poutres en béton armé sollicitées à la torsion. Ils exposent les résultats d'une théorie concernant la résistance à la torsion des poutres en béton armé, tant dans le domaine élastique qu'à la rupture. Ils décrivent une machine mise au point par eux-mêmes pour l'essai des sections en grandeur naturelle sous flexion et torsion combinées.

Les résultats obtenus montrent que l'intervention de contraintes de flexion *accroît* la résistance d'une section à la torsion. On constate une différence très nette entre les processus de rupture dûs essentiellement à la torsion et ceux qui sont dûs essentiellement à la flexion. Les auteurs proposent une théorie de rupture élastique qui combine le critère de contrainte principale maximum de Rankine et le critère de friction interne de Coulomb. Cette théorie est en concordance avec les résultats expérimentaux. Un exemple tri-dimensionnel illustre cette théorie.

Les auteurs donnent, pour terminer, des recommandations pour l'étude des poutres soumises à la flexion et à la torsion combinées.

**Zusammenfassung**

Die Arbeit gibt einen Ueberblick über veröffentlichte Forschungen über die Verdrehung von Stahlbeton. Die Ergebnisse einer Theorie für die Verdrehungsfestigkeit von Stahlbetonträgern, sowohl im elastischen Bereich wie beim Bruchzustand, werden dargelegt. Die Maschine, welche die Verfasser für Untersuchungen an Balken normaler Grösse unter kombinierter Biegung und Verdrehung konstruierten, wird beschrieben. Die Ergebnisse zeigen, dass zusätzliche Biegungsmomente die Verdrehungsfestigkeit *erhöhen.*

Es ergibt sich ein deutlicher Unterschied zwischen Brüchen, die hauptsächlich durch Verdrehung, und Brüchen, die hauptsächlich durch Biegung veranlasst sind. Eine Theorie für die Grenze des elastischen Bereiches wird aufgestellt, welche die Hauptspannungstheorie von Rankine mit der Theorie der Inneren Reibung von Coulomb verbindet, und mit den Ergebnissen der Experimente übereinstimmt. Die Theorie ist durch ein Raum-Modell erläutert.

Die Verfasser schliessen mit Empfehlungen für die Berechnung von Balken unter Biegung mit Verdrehung.