Design of shells based on the experimental determination of funicular surfaces

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Schalenbemessung durch experimentelle Darstellung der Seilflächen

Dimensionamento das cúpulas a partir do traçado experimental das superfícies funiculares

Dimensionnement des coupoles minces d'après le tracé experimental des surfaces funiculaires

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1 – *Introduction*

In the structural theory of shells, membrane equilibrium is defined as an abstraction of the static equilibrium of a shell obtained exclusively by means of forces contained on a plane tangent in every point to its middle surface, that is, by means of normal forces (compressive and tensile) and shear forces.

Let λ and μ denote two parameters by means of which the position of a point P on the surface of a membrane can be defined so that the equations λ = const. and μ = const. are two families of lines on the surface (Gauss).

The square of the line element is expressed by

$$ds^2 = \alpha^2 \, d\lambda^2 + \beta^2 \, d\mu^2 = ds_1^2 + ds_2^2 \qquad (1)$$

where $\alpha = \alpha(\lambda, \mu)$ and $\beta = \beta(\lambda, \mu)$.

The equations of equilibrium of a surface element of sides ds_1, ds_2, ds_3, ds_4, are obtained by making equal to zero the forces and moments acting on the element considered.

It is established in the Differential Geometry (1) that:

$$
\begin{aligned}
ds_1 &= \beta \, d\mu \\
ds_2 &= \alpha \, d\lambda \\
ds_3 &= \left(\beta + \frac{\partial \beta}{\partial \lambda} d\lambda\right) d\mu \\
ds_4 &= \left(\alpha + \frac{\partial \alpha}{\partial \mu} d\mu\right) d\lambda
\end{aligned}
\tag{2}
$$

Hence angles θ and φ (fig. 1) are:

$$
\begin{aligned}
\theta &= \frac{ds_2 - ds_4}{ds_1} = -\frac{1}{\beta} \frac{\partial \alpha}{\partial \mu} d\lambda \\
\varphi &= \frac{ds_3 - ds_1}{ds_2} = \frac{1}{\alpha} \frac{\partial \beta}{\partial \lambda} d\mu
\end{aligned}
\tag{3}
$$

With the same stress notations as in the theory of plates (2) the internal forces on the edges of the element N_λ, N_μ and $N_{\lambda\mu}$ are in equilibrium with the external forces of components P_x, P_y and P_z per unit of area. Resolution of the forces in the directions of the line element ds_2 yields:

$$
-N_\lambda \beta \, d\mu + N_\lambda \beta \, d\mu + \frac{\partial (N_\lambda \beta)}{\partial \lambda} d\lambda \, d\mu - N_{\lambda\mu} \alpha \, d\lambda + N_{\lambda\mu} \alpha \, d\lambda
$$

$$
+ \frac{\partial (N_{\lambda\mu} \alpha)}{\partial \mu} \partial\lambda \, \partial\mu - N_{\lambda\mu} \theta \beta \, d\mu - N_\mu \varphi \alpha \, d\lambda + P_x \alpha \beta \, d\lambda \, d\mu = 0
$$

since the area of the element can be taken to be $\alpha \beta \, d\lambda \, d\mu$.

A similar equation is obtained for the equilibrium of forces in the direction of the line element ds_1. Elimination of θ and φ by means of equations (3) and simplification of results, yields the equations for shear forces:

$$
\left\{
\begin{aligned}
&\beta \frac{\partial N_\lambda}{\partial \lambda} + \alpha \frac{\partial N_{\lambda\mu}}{\partial \mu} + N_\lambda \frac{\partial \beta}{\partial \lambda} + 2 N_{\lambda\mu} \frac{\partial \alpha}{\partial \mu} - N_\mu \frac{\partial \beta}{\partial \lambda} + \alpha\beta P_x = 0 \\
&\beta \frac{\partial N_{\lambda\mu}}{\partial \lambda} + \alpha \frac{\partial N_\mu}{\partial \mu} - N_\lambda \frac{\partial \alpha}{\partial \mu} + 2 N_{\lambda\mu} \frac{\partial \beta}{\partial \lambda} + N_\mu \frac{\partial \alpha}{\partial \mu} + \alpha\beta P_y = 0
\end{aligned}
\right.
\tag{4}
$$

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(1) W. C. Graustein — «Differential Geometry» — The Macmillan Company, New York, NY 1935.
(2) Timoshenko — Theory of Plates and Shells — Mc Graw-Hill Book Company, Inc. New York, 1940.

Let R_I and R_{II} be the principal radii of curvature of the surface and OXYZ a rectangular system of co-ordinates connected to each point of the membrane so that OZ has the direction of the normal to the lines of principal curvature λ_0, μ_0, Fig. 2.

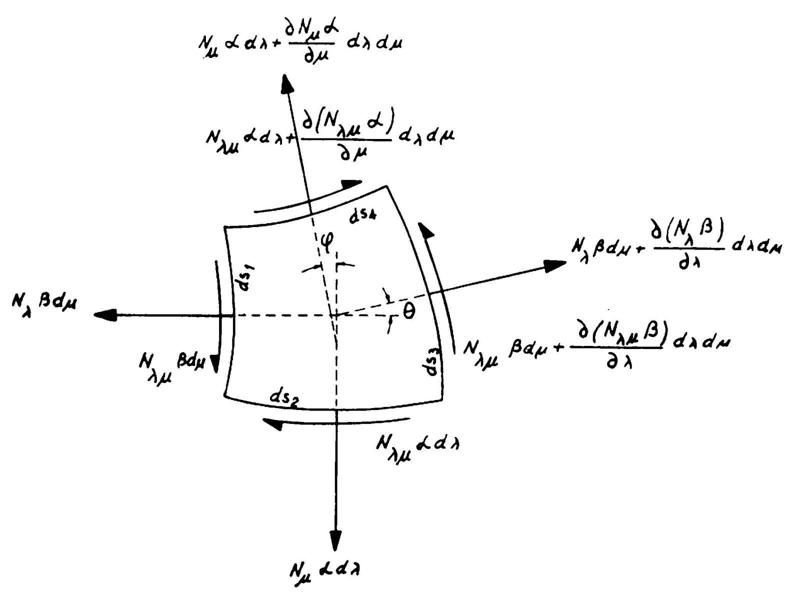

FIG. 1. Membrane equilibrium

The equilibrium equation in the direction of the normal to the surface at the point takes the following well-known form:

$$\frac{N_{\lambda_0}}{R_I} + \frac{N_{\mu_0}}{R_{II}} = P_z \tag{5}$$

These 3 equations (4) and (5) allow to evaluate the state of stress of a thin shell whatever, under a given field of forces, as they contain but three unknowns, N_λ, N_μ and $N_{\lambda\mu}$ (tensile forces).

2 – *Funicular surfaces*

It is easy to demonstrate that any structural surface can be in membrane equilibrium under a given field of forces but, as a rule this equilibrium is not funicular, i. e. the shell will be subject to compressive and tensile normal forces that will vary from point to point so as to achieve a static equilibrium between the internal shell forces and the external acting forces. In other words, this means that middle surfaces can be chosen for the shell such that under the acting field of forces, all the internal stresses will be of the same sign-all compressive or all tensile stresses. Such surfaces, called funiculars of the field of forces given, have important structural properties and are of great interest for Building Engineering.

Indeed if the middle surface of a very thin plain or slightly reinforced concrete shell is shaped as an anti-funicular surface of the acting forces a structure is obtained taking the maximum advantage of the strength of this building material. If, on the contrary, the material is steel the structural surface should receive the form of the funicular surface of the acting forces, if steel properties are to be used to the fullest.

In truth, all the builders have had this purpose intuitively in view with a more or less perfect degree of theoretical approximation,

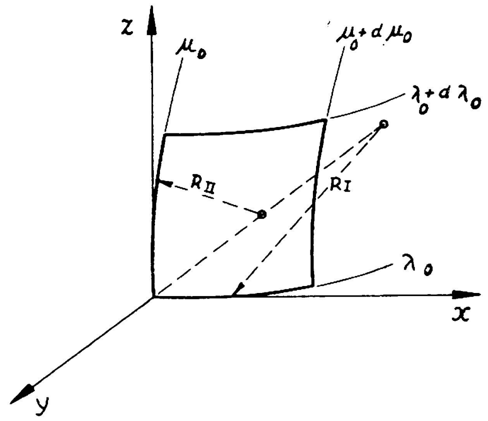

FIG. 2. Lines of curvature of a surface

what is seen in the predominant structural forms in use in all times.

Choosing at a point P a co-ordinate system λ_0, μ_0, z_0, whose z_0 — axis is normal to the shell surface and assuming the membrane to be uniformly stretched in all directions and $N_{\lambda_0} = N_{\mu_0} = \text{const.}$, the differential equation of the funicular surface may be written:

$$\begin{cases} \dfrac{1}{R_I} + \dfrac{1}{R_{II}} = -\dfrac{p z_0}{\sigma d} & \text{(6a)} \\[2ex] \sigma \dfrac{\partial d}{\partial \lambda} + \alpha p_{\lambda_0} = 0 & \text{(6b)} \\[2ex] \sigma \dfrac{\partial d}{\partial \mu} + \beta p_{\mu_0} = 0 & \text{(6c)} \end{cases}$$

in which $p_{\lambda_0}, p_{\mu_0}, p_{z_0}$ are the components of the uniformly distributed load along the λ_0, μ_0 and z_0 axes, d the shell thickness at the point in reference and σ the constant stress in the funicular surface (tensile stresses).

$$\sigma \times d = N = \text{const.}$$

The load being uniform and hydrostatic, equations 6b and 6c disappear and taking as approximate values of the membrane curvature the second order derivatives of the deflections ξ_0 of the funicular surface in relation to the OXY plane, the following equations are obtained:

$$\begin{cases} \dfrac{1}{R_I} = -\dfrac{\partial^2 \xi}{\partial x^2} \\[2ex] \dfrac{1}{R_{II}} = -\dfrac{\partial^2 \xi}{\partial y^2} \end{cases}$$

which substituted in (6a) yield:

$$\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} = -\frac{p}{N} \tag{7}$$

It is thus seen that even in this particular case the determination of a funicular surface calls for the integration of Poisson's equation

$$\nabla^2 \xi = F(x, y) \tag{8}$$

Funicular surfaces have properties in respect to plates similar to those of the funicular curves in relation to beams.

Thus expressions like $\dfrac{d^4 \xi}{dx^4} = -\dfrac{p}{EI}$; $\dfrac{d^2 \xi}{dx^2} = -\dfrac{M}{EI}$ and $\dfrac{d^2 M}{dx^2} = -p$

are equivalent to $\nabla^4 \xi = -\dfrac{p}{E I_p}$; $\nabla^2 \xi = -\dfrac{M}{E I_p}$ and $\nabla^2 M = p$ for plates,

provided that in the latter $I_p = \dfrac{e^3}{12(1-\nu^2)}$ and $M = \dfrac{m_x + m_y}{1+\nu}$.

Thus, it is seen, that as in the case of beams and arches, laws could be obtained for plates and shells connecting the shape of the funicular surface with the bending and twisting moments acting on the shell.

3 – *Search for a funicular surface by experimental means*

Scholars have always found that the search for the most adequate constant strength structural forms for certain types of acting forces is an exciting subject. Among other valuable works, mention should be made of a thesis presented at the Yugoslav Academy of Sciences in 1908 by Milankovic «Über Schalen gleicher Festigkeit». In this paper — which originated many others by other authors — Pöschl, Flugge, Forchheimer (3), etc. — it was sought to determine by analytical means the shape of some constant strength shells under acting forces with radial symmetry.

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(3) Pöschl, Th.: Bauing. 8 (1927) S. 624.
Flügge, W.: Statik und Dynamik der Schalen. Berlin 1934, S. 32.
Forchheimer, Ph.: Die Berechnung ebener und gekrümmter Behälterböden. 3 Auflage. Berlin 1931, S. 23.

The differential equation of the constant strength funicular surface can be integrated in this case and the constant strength forms of radially symmetric shells and reservoirs can be obtained by analytical or graphical means.

When no radial symmetry exists for the loads or their distribution is irregular, the mathematical tool has so far been unable to solve the

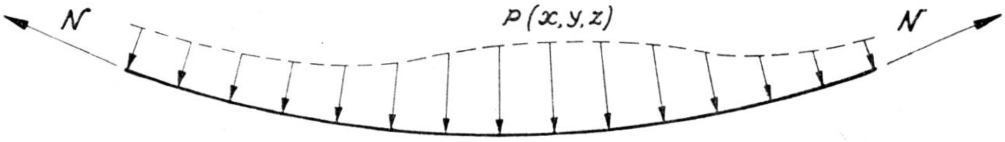

FIG. 3. Funicular equilibrium for p (x,y,z)

problem. The method we have developed half-experimental, half-analytic allows to make good this deficiency of the mathematical tools.

Consider a membrane thin and flexible enough, in tensile equilibrium under a given field of forces, fig. 3.

Obviously the shape it takes, assumed without wrinkles or folds, represents a materialization of one of the funicular surfaces of the field of forces in reference. We say one of the funicular surfaces since there is a double infinity of surfaces enjoying this property, each of them corresponding to a well-defined state of membrane equilibrium.

By a suitable variation of the membrane thickness a funicular surface of constant strength can be obtained. It suffices that the stress field in the membrane be hydrostatic and constant.

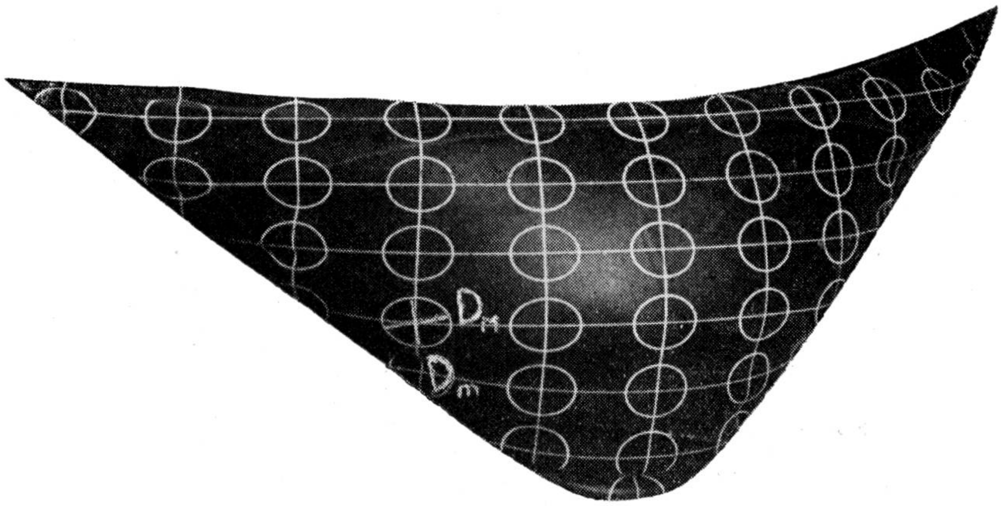

FIG. 4.

From the dimension analysis it is possible to deduce the similitude relationships between the state of stress in the elastic membrane and the state of stress in a shell whose middle surface is homologous to the model.

It is thus seen that the present method besides allowing to determine the constant strength shape of the shell surface for a given force, immediately yields the state of stress of the corresponding membrane.

4 – Experimental technique

A rubber membrane is satisfactory from this standpoint, allowing one funicular surface for a given system of forces to be very quickly obtained.

Let us suppose that on the rubber membrane that is going to be used, circumferences are drawn with diameters D_o and center at different points. Assuming the membrane to be continuous and isotropic the circumferences of small diameter D_o will change into elipses, the diameters of which D_m and D_M, fig. 4 and 5, will be variable from point to point both in magnitude and direction, allowing the state of stress of the membrane to be calculated. In fact let E and ν be the longitudinal modulus of elasticity and the Poisson's ratio of the membrane. Instead of defining the mean finite strains in the usual way $\varepsilon = \frac{D_m - D_o}{D_o}$, the summation of the infinitesimal strains may be used according to Hencky and Chilton (4).

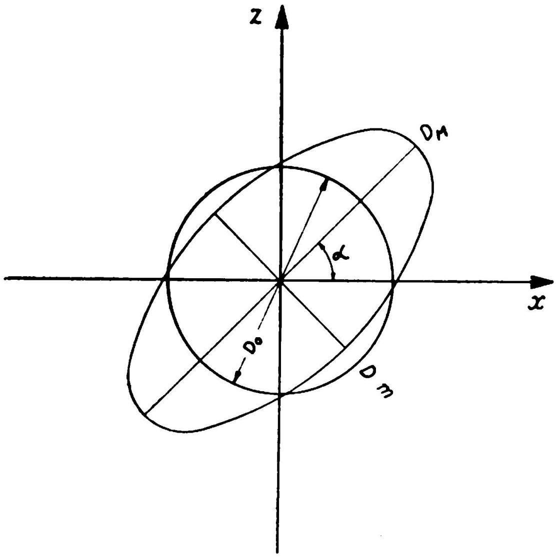

FIG. 5. Homogeneous deformation of a circle

$$\varepsilon_M = \int_{D_o}^{D_M} d\varepsilon = \int_{D_o}^{D_M} \frac{dx}{x} = \mathrm{Log}\frac{D_M}{D_o}$$
$$\varepsilon_m = \int_{D_o}^{D_m} d\varepsilon = \int_{D_o}^{D_m} \frac{dx}{x} = \mathrm{Log}\frac{D_m}{D_o} \qquad (9)$$

For strains of this magnitude and taking the deformed cross-sections of the rubber as the effective sections, Hooke's law is approximately obeyed, provided that $\varepsilon_m \leq 0.1$.

The corresponding principal stresses may be expressed by means of Hooke's law:

$$\begin{cases} \sigma_M = \dfrac{E}{1-\nu^2}(\varepsilon_M + \varepsilon_m) = \dfrac{E}{1-\nu^2}\left(\mathrm{Log}\dfrac{D_M}{D_o} + \nu\,\mathrm{Log}\dfrac{D_m}{D_o}\right) \\ \sigma_m = \dfrac{E}{1-\nu^2}(\varepsilon_m + \varepsilon_M) = \dfrac{E}{1-\nu^2}\left(\mathrm{Log}\dfrac{D_m}{D_o} + \nu\,\mathrm{Log}\dfrac{D_M}{D_o}\right) \end{cases} \qquad (10)$$

(4) E. G. Chilton — Graduation Thesis at Stanford University.

These expressions hold for rubber as a first approximation, provided a certain value ε is not exceeded. However the deformation which rubber and similar substances undergo is much too large to be covered by the classical theory of small strains. An entirely new approach is required for any adequate theory of elasticity of rubber (5).

The two families of isostatics on the membrane form two orthogonal funicular nets on the surface. The stress tensor N on the membrane has at each point of the surface a maximum component given by $N_M = \sigma_M$ d, in which d represents the thickness of the deformed membrane at that point.

Once obtained the tensorial field of a funicular surface as defined by the values of N_M and N_m at any point and the two isostatics families, the state of stress of a shell with that shape can be calculated in a first approximation. Indeed, the shell having a thickness d_b and its middle surface coinciding with the funicular surface for the applied forces to a given scale $\frac{1}{\rho}$, the following expression is obtained for each point and direction:

$$\sigma = \frac{N \rho^2}{d_b} \tag{11}$$

N being the membrane stresses in the same point and direction.

It is clear that this is but an approximate value, as deformations are set up in the shell which alter its state of stress, when the field of

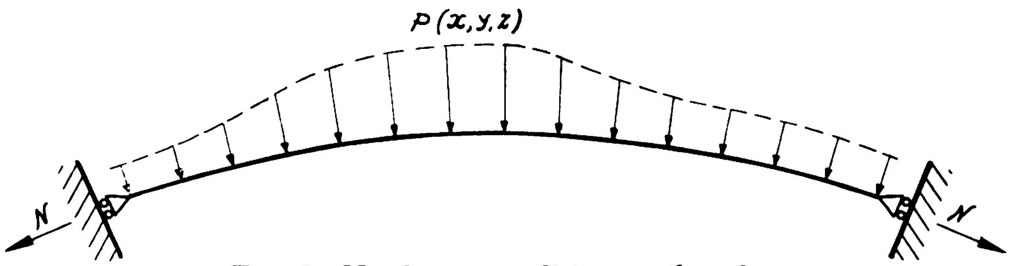

FIG. 6. Membrane condition at the edge

forces is applied. This increase of stress may be safeguarded against, by designing the shell in such a way that the membrane stresses are not much altered. The major factor disturbing the state of stress of the membrane comes from the edge conditions. For this disturbing factor to be lessened, a joint should be built allowing frictionless displacements of the edge of the shell only along a normal to the middle surface, i. e., reproducing the boundary conditions of the membrane, fig. 6. The execution of such a joint presents no technical difficulty whatsoever, improving moreover the structural behaviour of the shell and supplying more space to volumetric changes (temperature, shrinkage, etc).

(5) M. Mooney — A Theory of large elastic deformation — Journal of applied physics — Vol. 11, No. 9.

It may still be feared that this method leads to shells so thin as to make probable phenomena of buckling. The theory of the buckling of shells is still in the beginning but approximate formulas are available which give satisfatory results and connect σ_{I} and σ_{II}, the principal stresses, with the thickness of the shell, the modulus of elasticity and the Poisson's ratio of the material, and the two principal radii of curvature at each point.

5 – *Application of the method to the design of dams*

The method described in 3 was developed by the author for the study of the shape of a dam to be built in a given valley.

Indeed, dams being built in plain concrete for economic reasons, the middle surface of the structural shape adopted, should coincide with

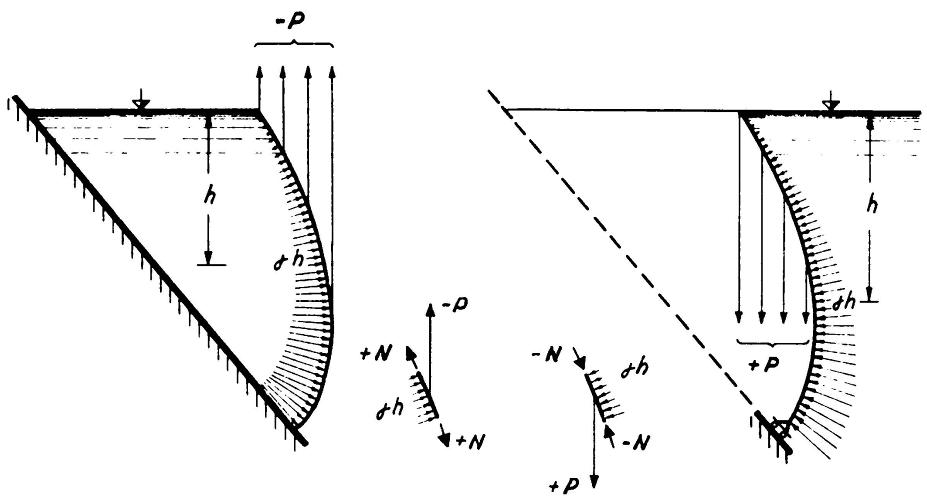

FIG. 7. Membrane equilibrium of a dam

one of the antifunicular surfaces for the forces applied. For dams, however, there are almost exclusively two importants static loads: deadweight and water pressure.

Let us consider then a rubber membrane inserted in a boundary geometrically similar to the line of contour of the valley in which the dam is to be placed. This line must be selected according to the shape and geology of the site. Be $\frac{1}{\rho}$ the scale to which this contour is reproduced. Let us take this line as the contour of a vessel for a liquid of specific gravity γ, held by a rubber membrane, fig. 7. Let us assume, additionally, that the upper contour of the membrane has no rigid connections, remaining on the free level of the fluid. Now we apply at each point of the rubber membrane a vertical load P, directed upwards, P being equal to the weight, to the scale $\frac{1}{\rho}$, of each volume element of a shell having the same middle surface but built in concrete, that is:

$$P = -2{,}4\ \gamma \frac{1}{\rho} S \tag{12}$$

S being the area of the surface of the membrane in the center of gravity of which P acts, and d the estimated average thickness of the shell in that point.

If the shape of equilibrium displays no folds or wrinkles at any point of its surface, we may say it materializes a funicular surface for the hydrostatic pressure caused by the liquid of specific gravity γ and for the deadweight of a shell, having a middle surface with that shape, so that in each point we have:

$$\frac{N_I}{R_I} + \frac{N_{II}}{R_{II}} = \gamma z + p_n \tag{13}$$

N_I and N_{II} being the principal membrane forces, R_I and R_{II} the two principal radii of curvature, and p_n the normal component of the deadweight. The field of membrane forces N can be evaluated by the method described in the foregoing paragraph, so that the state of stress of a dam having as middle surface the shape of the membrane under the action of the deadweight and hydrostatic pressure would be:

$$\sigma = \frac{N \rho^2}{d\gamma} \tag{14}$$

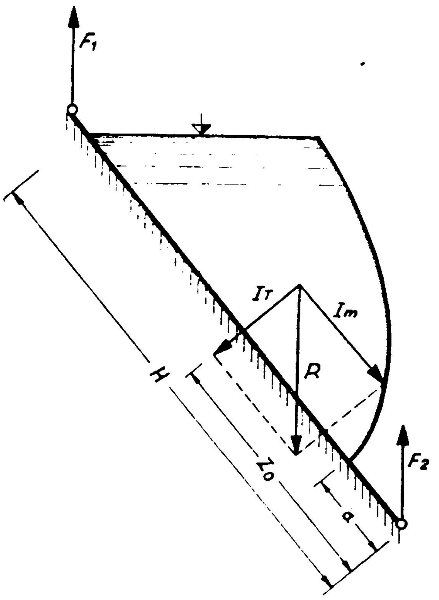

FIG. 8. Resulting vectorial equilibrium

As said above this expression holds only as a first approximation.

The following procedure, for instance, may be followed to design the dam so that the maximum stresses, as calculated by means of expression (14), do not exceed 50 kg. cm⁻², the increase of these stresses being checked by means of a more accurate method, e. g. three dimensional models [8] [9].

Likewise, the total hydrostatic pressure on the dam, can be evaluated by experimental means, the analytical determination being a difficult problem, owing to the shape of the upstream face.

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[8] G. Oberti — *La Ricerche Sperimentale su modelli como contributo al progretto delle grandi costruzioni* — Técnica Italiana — N.° 2, 1951.

[9] M. Rocha — *General review of the present status of the experimental method of structural design*. Sep. da Publicação Preliminar do 3.° Congresso da Association Internationale des Ponts et Charpentes, Cambridge e Londres, 1952.

The hydrostatic force on the membrane is given by the following vector difference, fig. 8:

$$\vec{I_m} = \vec{R} - \vec{I_t} \tag{15}$$

R being the total weight of the container held by the membrane, which is easy to measure.

$$\vec{R} = \vec{F_1} + \vec{F_2} \tag{16}$$

The value I_t, resultant of the hydrostatic pressure on the plate is also easy to compute, provided the contour of the membrane has an easy analytical expression.

$$I_t = \int_a^H \gamma z (H - z) dz \tag{17}$$

The resultant I_t acts in a direction normal to the plate, at a point at a distance Z_o from the center of application of F_2,

$$Z_o = \frac{\int_a^H (H - z) z dz}{\int_a^H (H - z) dz} \tag{18}$$

Thus by means of a simple graphical construction I_m can be determined, hence the total hydrostatic pressure on the dam is:

$$\vec{I} = - I_m \rho^3 \tag{19}$$

6 – *Illustrative example*

In order to design a dam by the foregoing method the author devised an adequate apparatus, fig. 9. This, essentially, is made up of a plate of insertion for the membrane (1), a graduated plane to which the rubber membranes are connected by means of a hoop that materializes the insertion line. All the setup hangs from a dynamometric system (3).

A tridimensional co-ordinometer was used to measure the deformed membrane and to determine its co-ordinates referred to the system of axes OXYZ. It consists of three bars representing the three axes of the co-ordinate system, its movement being controlled by one button only (7). Before starting the test, the apparatus is set up. The plate is given the desired inclination and the position of the axes relatively to the plate is recorded by means of various adjusting screws of the co-ordinometer.

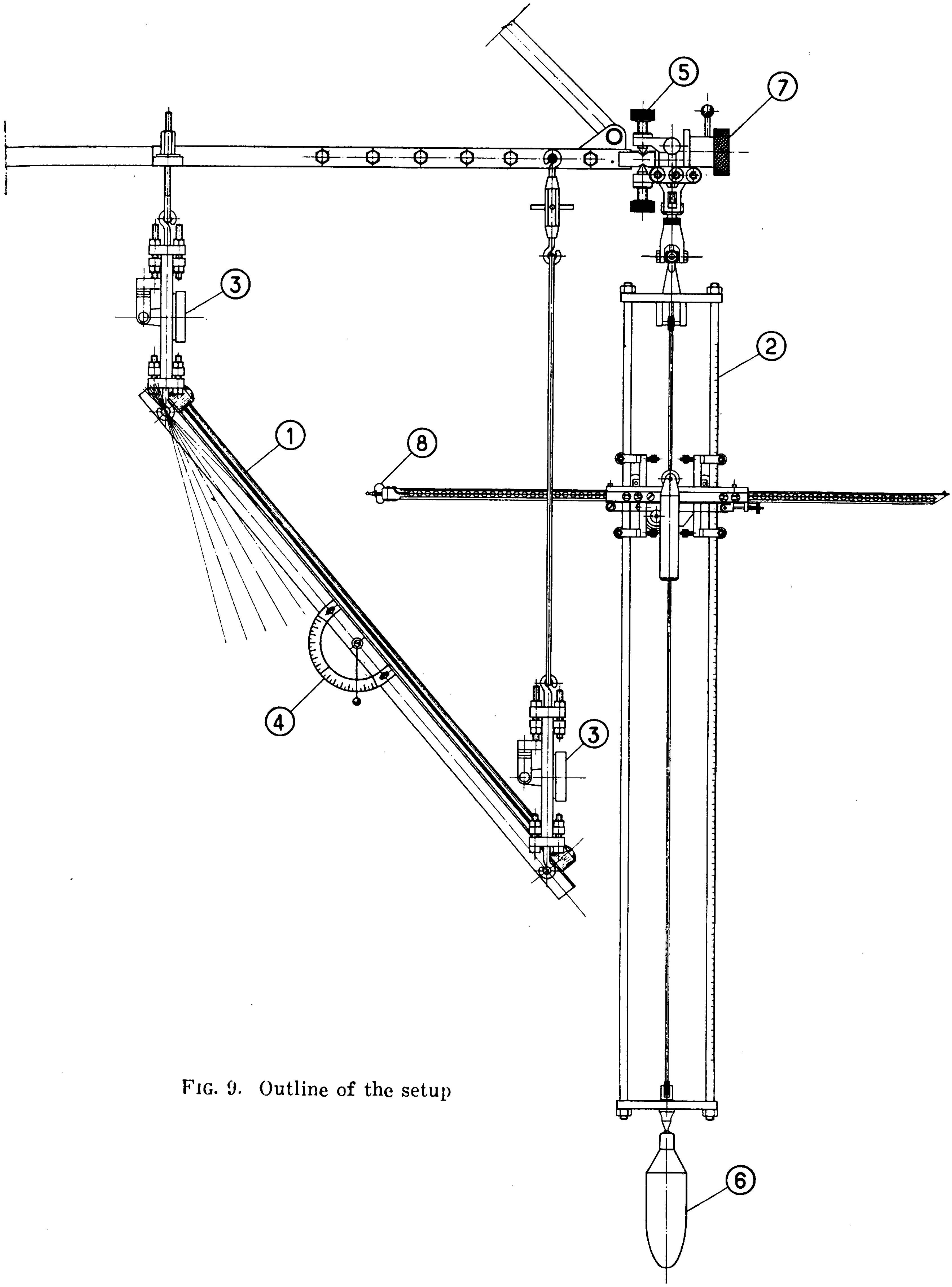


FIG. 9. Outline of the setup

The method of design, based on this experimental technique was applied to an actual case in order to check the shape and volume of a dam planned by this means ([10]).

The valley selected possessed an exceptional symmetry on the site of the dam, so that a symmetrical contour of simple analytical expression could be adopted.

Fig. 10 shows in full line the insertion contour adopted and in dotted line the topographic section obtained by cutting the valley by a plane.

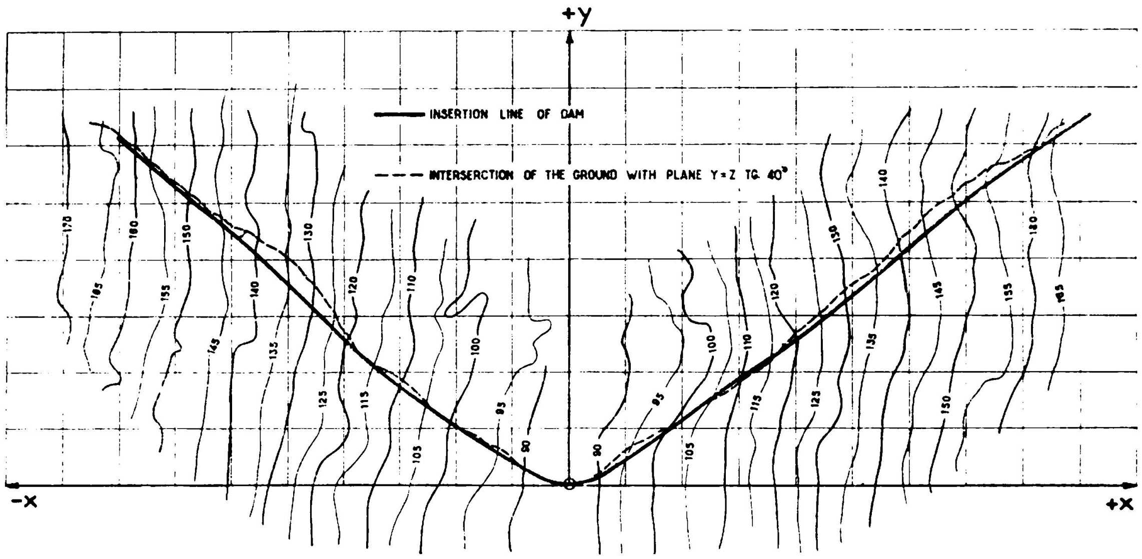


FIG. 10. Insertion lines

Relatively to the co-ordinate system OXYZ shown in the figure, in which OZ is vertical, the chosen contour is represented by the equation:

$$\begin{cases} x = \pm\sqrt{\dfrac{y\ (y + 22{,}886)}{0{,}595}} \\ y = z \operatorname{tg} 40^{\circ} \end{cases} \tag{20}$$

This is a branch of hyperbola resting on a plane making a 50° angle with the horizontal plane.

Pure vulcanized rubber was used to manufacture the membrane; it was calendered in both directions, in order to ensure a more perfect isotropy. The elastic properties were evaluated up to values $\varepsilon \cong 0{,}1$, fig. 11.

Furthermore it was sought to give such thicknesses to the membrane as to obtain surfaces of equilibrium not very different from the most recent shapes of arch dams in order to prevent constructional objections.

([10]) The study of this dam was carried out in collaboration with Mr. Peres Rodrigues.

After some trials the shape of equilibrium shown in fig. 12 was selected. It corresponds to a distribution of thickness allowing the stress in the membrane not to exceed 50 kg. cm^{-2} as we can see in table I and fig. 13 in which the magnitudes and directions of the principal stresses were evaluated.

Afterwards the membrane was measured by means of the pendulum, and the co-ordinates in various points of the same vertical profile were determined, table II. In fig. 14 we show the horizontal curves of the deflected membrane.

The analytical definition of the dam was based on the following geometric procedure: The upstream and downstream faces were obtained

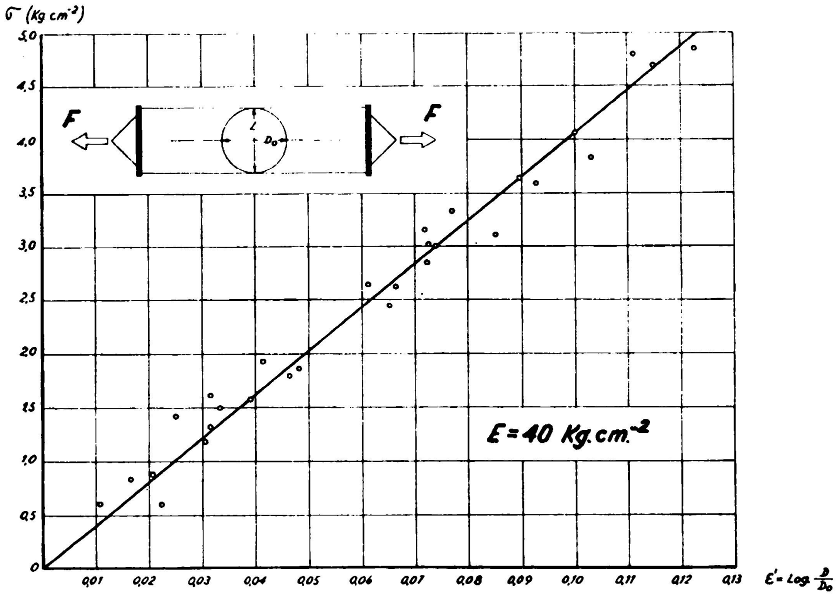


FIG. 11. Stress-strain relationship for rubber

from a middle surface, approaching as much as possible the surface of the membrane to the scale of the dam, by marking along the normal for both sides half the thickness calculated for each point.

The solid — dam — is therefore totally defined, once the equation of the middle surface and the law of the variation of thickness are settled.

The analytical definition of a middle surface as near as possible to the surface of the membrane and at the same time simple, was obtained in the following manner:

The horizontal curves are conic sections having 5 directrixes (also conics) intersecting at point (0, 0, 0). Thus the analytically defined geometric surface has no less than 17 points of contact with the surface obtained experimentally, fig. 15.

The general equation for the horizontal curves is therefore:

$$y - c - \sqrt{b - a x^2}$$

The parameters a, b, c are obtained by the condition of contact with the following directrixes:

Directrix d_e — Corresponding to the branch of hyperbola, defining the insertion line of the dam

$$\begin{cases} x_e = \pm \dfrac{y_e\,(y_e + 22{,}886)}{0{,}595} \\ y_e = z \operatorname{tg} 40^{\circ} \end{cases}$$

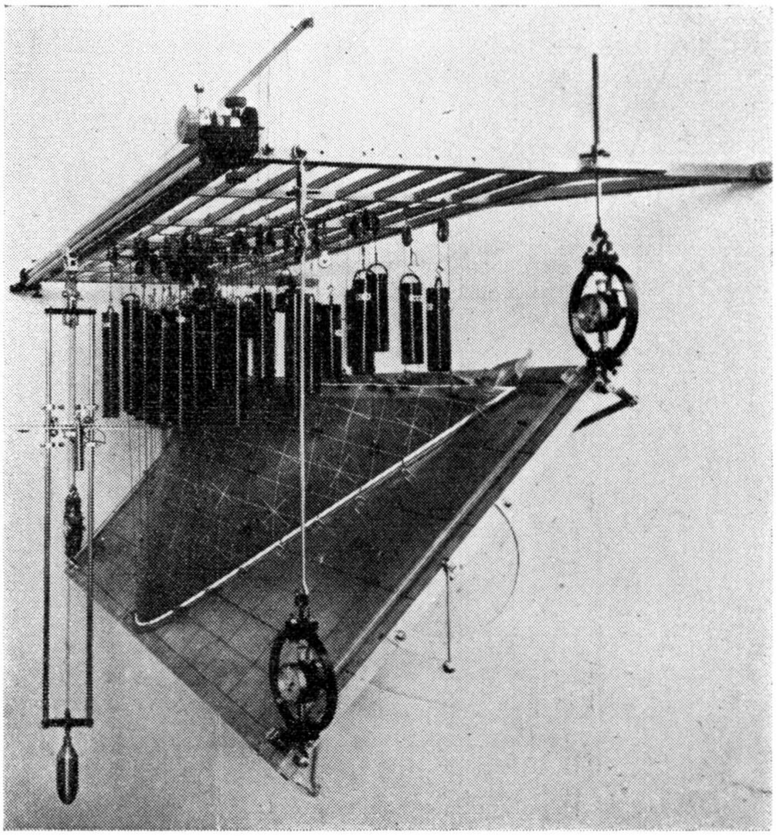


FIG. 12.

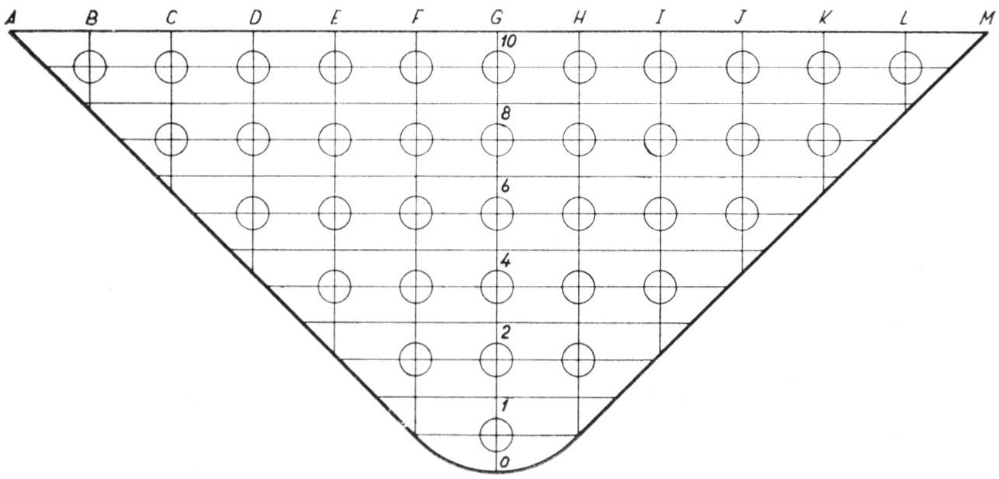


FIG. 13. Points at which the stress was measured

Directrix d_0 — Vertical profile through point (0, 0, 0)

$$\begin{cases} y_0{}^2 - (0{,}579\ Z + 24{,}240)\ y_0 + (0{,}225\ Z^2 - 5{,}7087\ Z) = 0 \\ x_0 = 0 \end{cases}$$

Directrixes d_m — Two branches of a conic section given by the expressions

$$\begin{cases} x_m{}^2 - (3{,}094\ y_m + 2{,}511)\ x_m + (5{,}587\ y_m{}^2 + 0{,}920\ y_m) = 0 \\ y_m = \dfrac{Z}{2} \end{cases}$$

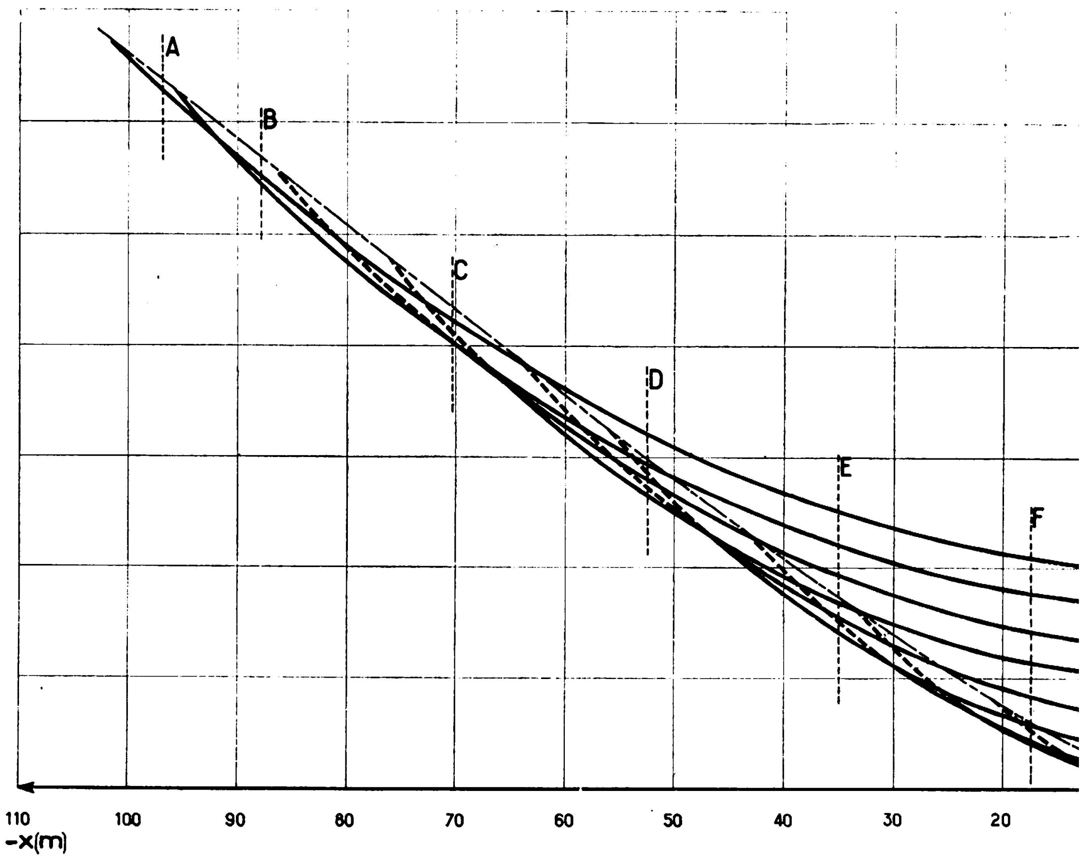


FIG. 14.

Horizontal curves of the membrane

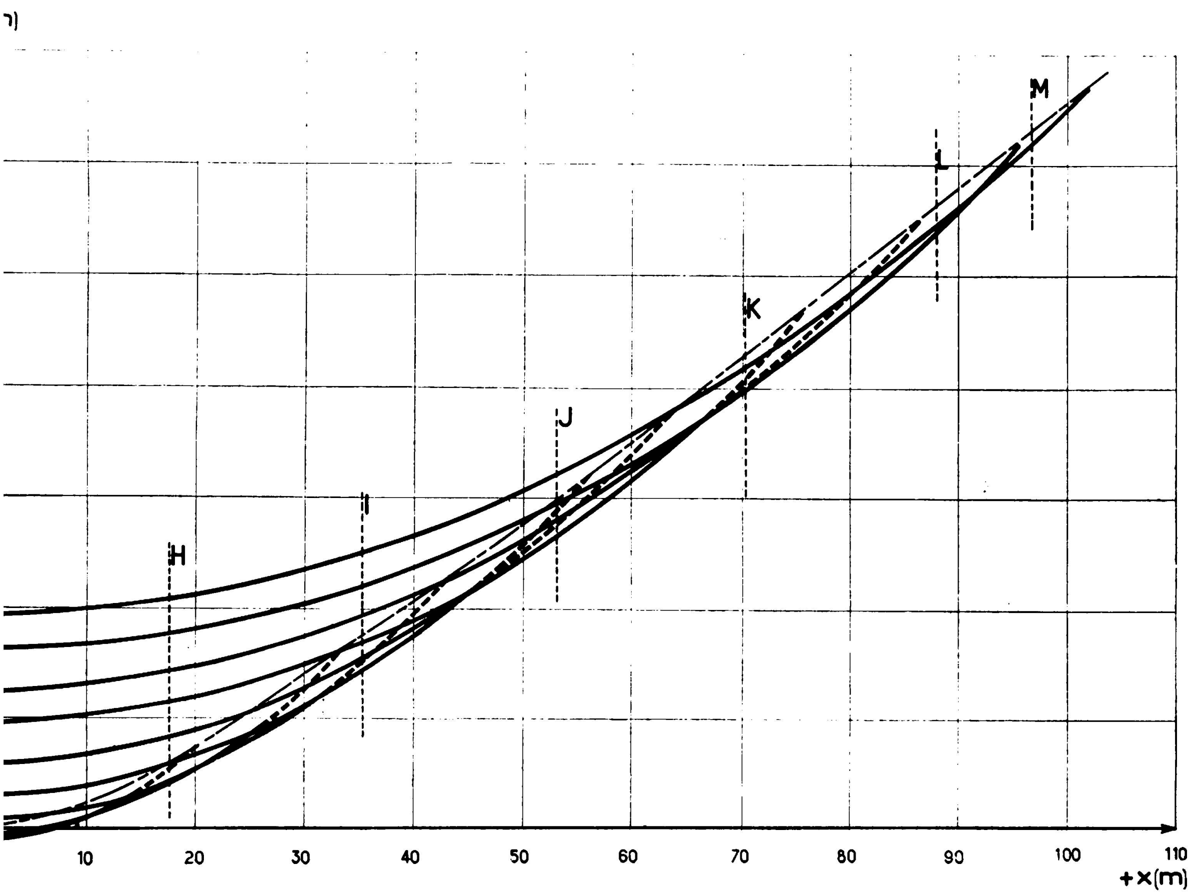
In fig. 16 are indicated the vertical profiles of the surface, as analitically defined, and the points obtained experimentally from the co-ordinate measurements on the stretched membrane. The maximum differences do not exceed 3 mm to a scale of 1/200, that is to say they are less than 60 cm in the prototype.

The variation of the thickness of the dam was chosen in such a way, as to correspond to the loads fixed for the deadweight and, within the limits imposed by this condition, to bring about the best possible distribution of stresses inside the dam.

Thus two laws of variation for the thickness were selected according to the boundary conditions, fig. 17.

1st Case — *Built-in contour* — The thickness along the normal varies only, with respect to the elevation i. e. it does not change along a horizontal curve.

2nd Case — *Shell with a perimetric contour joint* — This joint should be made in such a way as not to disturb to a large extent the membrane conditions of the contour.



Along one horizontal curve of elevation z the thickness measured along the normal, changes according to the expression:

$$e_x = e_0 - \frac{e_0 - e_n}{d^2} x^2$$

where e_0 is the thickness of the middle cross-section as defined by the law concerning the built-in case and e_n represents the thickness

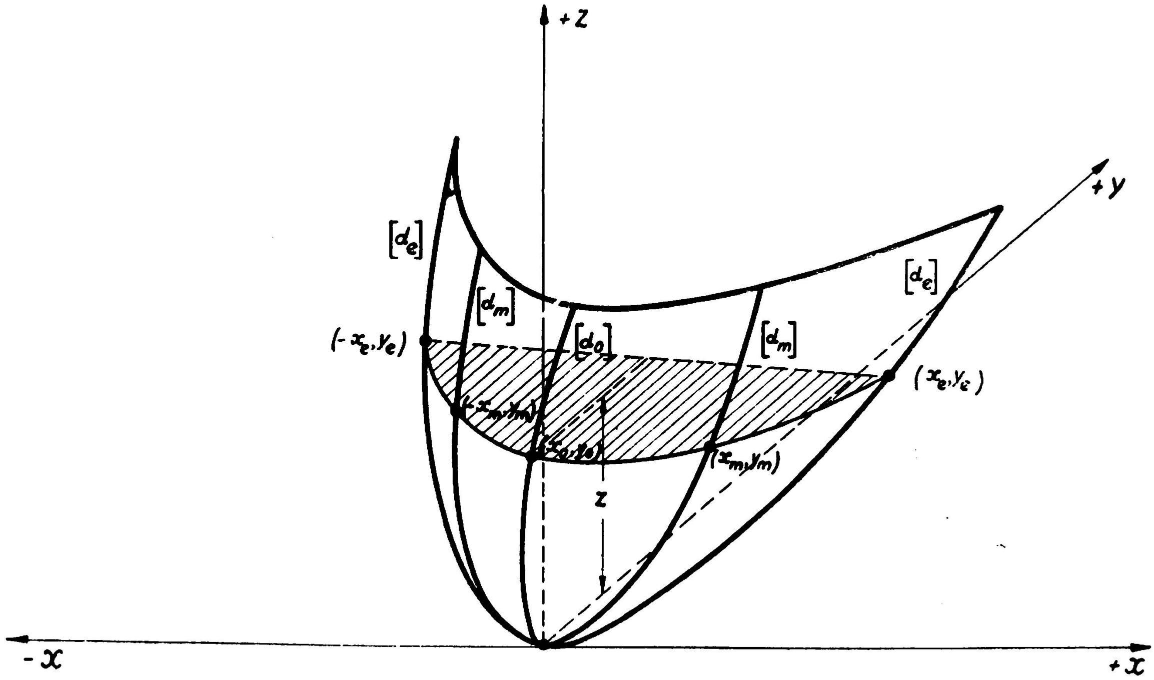


FIG. 15. Analytic expression of the middle surface

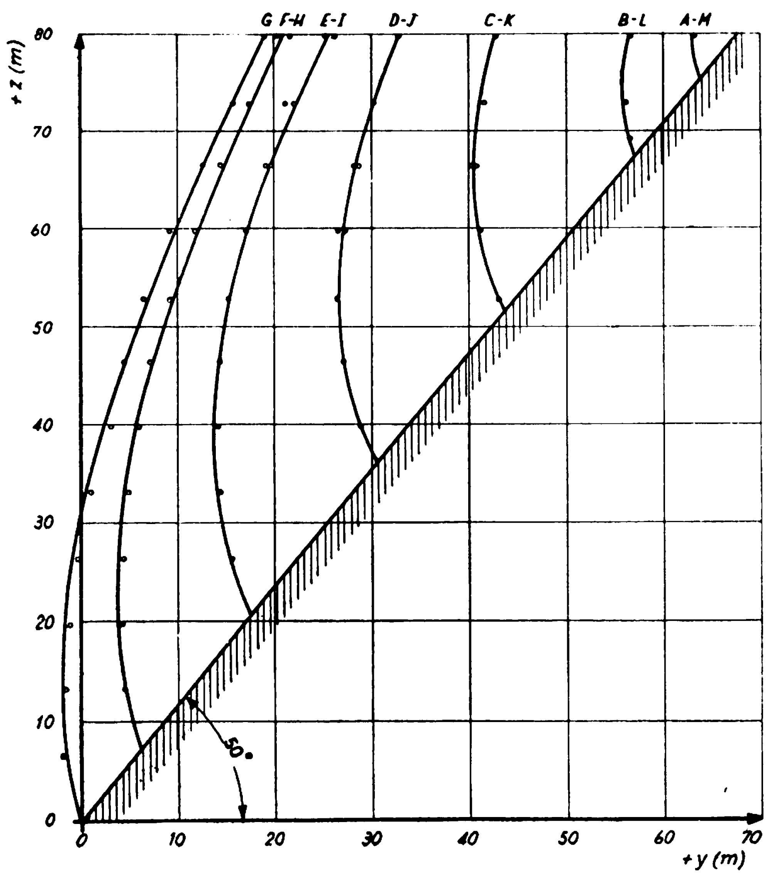


FIG. 16. Computed and experimental profiles

in the insertion contour:

$$e_n = -0{,}00146\ z^2 + 0{,}0567\ z + 6$$

d being the half-chord of the horizontal curves at that level;

$$d = \frac{z\,(z + 27{,}274)}{0{,}845}$$

That is, in this case the thickness can be somewhat smaller along the contour as the perimetral joint diminishes the moments due to the differences between the funicular surface and the middle surface, when deflected under load.

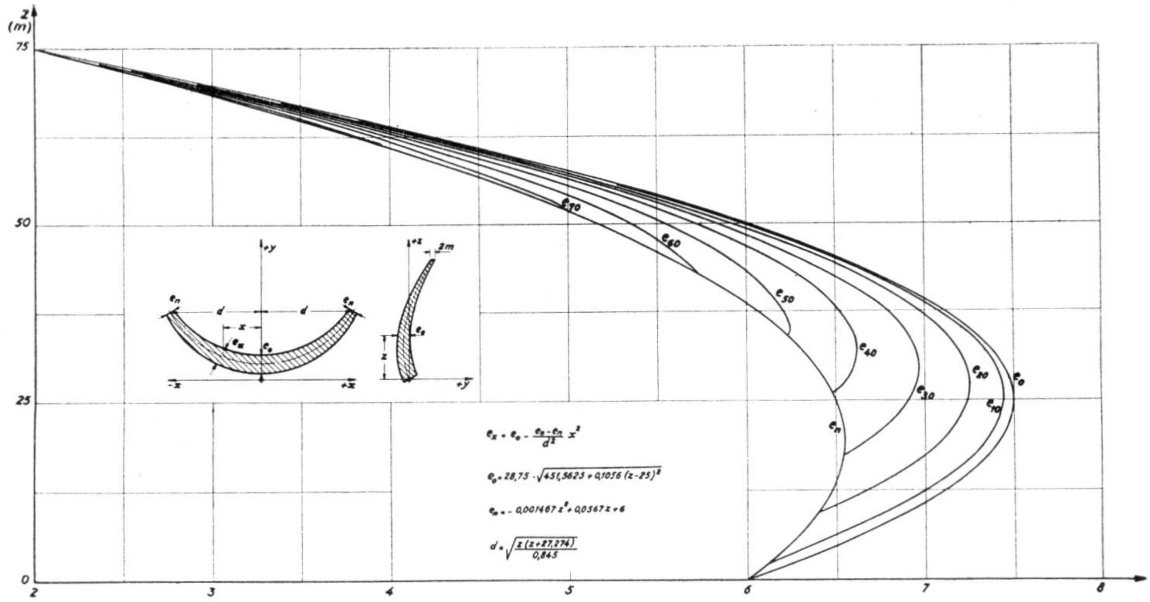


FIG. 17. Law of variation of the thickness

From these data the dam was designed with the following characteristics, fig. 18.

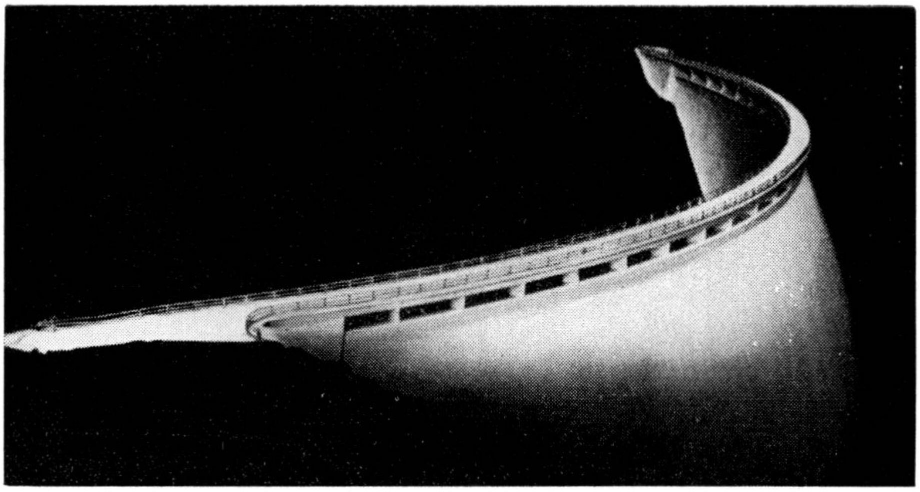


FIG. 18

Height	H = 77 m
Semi-axes of the ellipse of crest	a = 434,9 m b = 207,7 m
Angular width between the normals to the ellipse in the contour	φ = 92°
Maximum thickness of the dam	e_m = 7,5 m
Volume of concrete	V = 54 × $10^3 m^3$
Maximum length chord	C = 198 m
Maximum membrane stress (compressive)	= 46 kg. cm^{-2}

Table I

Point	Model							Prototype		
	d (mm)	ε_M (10^{-2})	ε_m (10^{-2})	σ_M (Kg cm^{-2})	σ_m (Kg cm^{-2})	N_M (Kg cm^{-1})	N_m (Kg cm^{-1})	d_b (m)	$\sigma'_b = N_M \frac{\rho^2}{d_b}$ (Kg cm^{-2})	$\sigma''_b = N_m \frac{\rho^2}{d_b}$ (Kg cm^{-2})
10B - 10L	0,65	3,1	-	1,3	0	0,08	0	2,00	20	0
10C - 10K	0,65	4,2	-	1,7	0	0,11	0	2,00	26	0
10D - 10J	0,65	5,6	-	2,3	0	0,15	0	2,00	35	0
10E - 10I	0,65	6,3	-	2,5	0	0,16	0	2,00	40	0
10F - 10H	0,65	6,3	-	2,5	0	0,16	0	2,00	40	0
10G	0,65	7,2	-	2,9	0	0,19	0	2,00	46	0
8C - 8K	1,10	6,4	1,3	2,6	0,3	0,29	0,03	4,10	34	4
8D - 8J	1,10	6,4	1,3	2,6	0,3	0,29	0,03	4,10	34	4
8E - 8I	1,10	7,8	1,2	3,3	0,5	0,36	0,05	4,10	42	6
8F - 8H	1,10	8,0	1,0	3,4	0,6	0,37	0,07	4,10	41	8
8G	1,10	8,6	1,0	3,7	0,7	0,40	0,08	4,10	46	9
6D - 6J	1,10	9,3	0	4,1	1,2	0,45	0,14	5,80	38	11
6E - 6I	1,10	9,4	1,0	4,0	0,9	0,44	0,09	5,80	37	8
6F - 6H	1,10	9,6	1,2	4,1	0,7	0,45	0,08	5,80	37	7
6G	1,10	9,9	1,4	4,2	0,6	0,46	0,07	5,80	38	6
4E - 4I	1,65	9,0	2,4	3,6	1,1	0,60	0,19	7,00	41	13
4F - 4H	1,65	9,0	2,4	3,6	1,1	0,60	0,19	7,00	41	13
4G	1,65	9,1	0	4,0	1,2	0,66	0,20	7,00	46	14
2F - 2H	1,65	8,0	2,0	3,2	1,2	0,54	0,20	7,00	37	14
1G	1,7	4,7	0.4	2,0	0,4	0,34	0,08	6,40	26	10

TABLE II

Profiles	y = f (x,z) (mm)						
Horizontal curves	A — M	B — L	C — K	D — J	E — I	F — H	G
0							40
2						62	35
4				172	106	69	55
6			172	160	110	80	70
8			225	167	128	105	97
10	334	300	238	192	159	140	134

SUMMARY

The author begins by presenting the advantages of putting to avail the three dimensionality in the structural behaviour of a shell, by giving it a double curved shape, so as to obtain an almost exclusively compressive state of stress. He then introduces the notion of funicular surface, a generalization of the funicular curve to the three dimensions.

A mathematical basis is given for an analytical approach to the problem but owing to its complexity, it is only sketched.

It is shown that the problem is very easy to solve by experimental means, the state of stress of the funicular surface being likewise calculated by a quick method, whatever the shape obtained.

The method expounded is then applied to the design of an 80 m high dam.

ZUSAMMENFASSUNG

Der Verfasser weist in erster Linie auf die Zweckmässigkeit der Ausnützung dreier Dimensionen beim Entwurf einer Schale hin. Es soll dieser eine doppelte Krümmung gegeben werden, so dass die Eigenspannungen fast auschliesslich und in allen Richtungen Druckspannungen sind Dann wird der Begriff «Seilfläche», eine Erweiterung der Seilkurven auf drei Dimensionen, dargelegt.

Der Verfasser erörtert eine mathematische Grundlage zur analytischen Betrachtung dieser Frage, die aber wegen ihrer Komplexität nur leicht skizziert wird.

Es wird die Möglichkeit gezeigt, wie man das Problem experimentell leicht lösen kann, wobei auch der Spannungszustand in der Seilfläche ungeachtet der gefundenen Form rasch festzustellen ist.

Die beschriebene Methode wird dann angewendet zur Berechnung einer 80 m hohen Staumauer.

RESUMO

O autor começa por apresentar a vantagem de se tirar partido das três dimensões no comportamento estrutural duma cúpula dando-lhe dupla curvatura por forma que o estado de tensão seja quase exclusivamente de compressão em todas as direcções. Introduz em seguida a noção de superfície funicular, generalização a 3 dimensões das linhas funiculares.

Apresenta uma base matemática para o tratamento analítico deste problema mas, dada a sua complexidade fica apenas esboçada.

Seguidamente mostra a possibilidade de resolver aquele problema com grande simplicidade por via experimental podendo também calcular-se o estado de tensão na superfície funicular por meio dum método rápido, qualquer que seja a forma obtida.

O método é aplicado a título de exemplo no dimensionamento duma barragem de 80 m de altura.

RÉSUMÉ

L'auteur présente d'abord les avantages que l'on peut avoir à tirer parti des trois dimensions dans le comportement structural d'une coupole, en lui donnant une forme à double courbure de façon à obtenir presque exclusivement des contraintes de compression. Il introduit ensuite la notion de surface funiculaire, généralisation des courbes funiculaires aux trois dimensions.

Il présente une base mathématique permettant de traiter analytiquement le problème, mais, étant donné sa complexité, cette méthode n'est qu' ébauchée.

Il montre ensuite la possibilité de résoudre ce problème très aisément par voie expérimentale, la contrainte à la surface funiculaire pouvant être également calculée au moyen d'une méthode rapide, quelle que soit la forme obtenue.

À titre d'exemple la méthode presentée est appliquée au calcul d'un barrage de 80 m de hauteur.