Finite deflections of a clamped circular plate on an elastic foundation

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Finite Deflections of a Clamped Circular Plate on an Elastic Foundation[1])

Calcul des flèches finies d'une plaque circulaire encastrée sur fondation élastique

Endliche Durchbiegungen einer eingespannten Kreisplatte auf einer elastischen Fundation

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Notation

r, θ	Polar coordinates.
u, v, w	Displacements in radial, tangential, and the direction of normal vector of the undeformed middle surface of the plate, respectively.
$\epsilon_r, \epsilon_\theta$	Radial and tangential strain components.
K_r, K_θ	Changes of curvature.
$M_r, M_\theta, M_{r\theta}$	Components of bending moments per unit length of middle surface of the plate.
$N_r, N_\theta, N_{r\theta}$	Components of membrane forces per unit length of middle surface of the plate.
E	Modulus of elasticity in tension and compression.
ν	Poisson's ratio.
h	Plate thickness.
$D = \frac{E h^3}{12(1-\nu^2)}$	Flexural rigidity of the plate.

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q Intensity of uniform load in direction normal to the plate (lb. per sq. in.).

k Modulus of the elastic foundation (lb. per cu. in.).

Introduction

Although the solution to many problems involving the infinitesimal deflections of thin elastic plates on elastic foundations has been given by various techniques [1, 2], no analysis is available for the case when the maximum deflection is of the order of magnitude of the plate thickness. In the present study we consider the axisymmetric finite deflection of a thin elastic circular plate resting on an elastic foundation. The edges of the plate are clamped and the face of the plate is loaded by uniform normal pressure.

Let us denote by r the distance of a point, in the middle surface of the plate, from the geometric axis. Also, let u and w, respectively, denote radial and normal components of displacement of this point. The intensity of normal load is designated by q and the foundation modulus by k (force per unit volume). It is assumed that the direction of the reaction of the foundation upon the plate is normal to the plate and the magnitude of this reaction varies linearly with the normal deflection w. We denote the modulus of elasticity of the plate by E, Poisson's ratio by ν, the plate thickness by h, the radius by a, and the flexural rigidity by $D = E h^3/12(1-\nu^2)$. Also, we denote by N_r and N_θ the membrane forces per unit length of the middle surface of the plate in the radial and tangential directions, respectively.

Governing Equations

For deflections of the order of magnitude of the plate thickness we take the strain-displacement relations to be [3]

$$\epsilon_r = \frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2. \tag{1}$$

$$\epsilon_\theta = \frac{u}{r}, \tag{2}$$

where ϵ_r denotes the radial strain, and ϵ_θ the tangential strain. The curvature we take to be [3]

$$K_r = \frac{d^2 w}{d r^2}, \tag{3}$$

$$K_\theta = \frac{1}{r}\frac{dw}{dr}. \tag{4}$$

The finite deflections of the plate are described by the VON KARMAN equations [3]. If the foundation reaction is included these equations may be written in the form

$$D\frac{1}{r}\frac{d}{dr}\left[r\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right\}\right]-\frac{1}{r}\frac{d}{dr}\left[rN_r\frac{dw}{dr}\right]=q-kw \tag{5}$$

and

$$r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r^2N_r)\right]+\frac{Eh}{2}\left(\frac{dw}{dr}\right)^2=0. \tag{6}$$

Also, we have the equation expressing equilibrium in the radial direction [3]

$$\frac{d}{dr}(rN_r)-N_\theta=0. \tag{7}$$

The radial displacement is found from (2) together with Hooke's Law to be

$$u=\frac{r}{Eh}\left[\frac{d}{dr}(rN_r)-\nu N_r\right]. \tag{8}$$

The boundary conditions at $r=a$ are

$$\begin{aligned} w &= \frac{dw}{dr}=0,\\ \frac{u}{a} &= \frac{1}{Eh}(N_\theta-\nu N_r)=0 \\ \text{or,}\quad &\frac{d}{dr}(rN_r)-\nu N_r=0; \end{aligned} \tag{9}$$

and at $r=0$ they are

$$\frac{1}{r}\frac{dw}{dr}=\text{Finite},\qquad N_r=\text{Finite}. \tag{10}$$

It is convenient to render the above equations non-dimensional. Accordingly, we introduce the following relations:

$$\begin{aligned} W &= \frac{w}{h},\\ N_r &= \frac{Eh^3}{a^2}S_r,\\ N_\theta &= \frac{Eh^3}{a^2}S_\theta,\\ P &= \frac{a^4q}{h^4E}(1-\nu^2),\\ K &= \frac{3(1-\nu^2)}{4}\frac{ka^4}{Eh^3}. \end{aligned} \tag{11}$$

Also, a dimensionless variable η is so chosen such that

$$\eta=1-\frac{r^2}{a^2}. \tag{12}$$

Thus, Eqs. (5) to (9) can be written in the following non-dimensional forms:

$$\frac{d}{d\eta}\left[(1-\eta)\frac{d^2}{d\eta^2}\left\{(1-\eta)\frac{dW}{d\eta}\right\}\right]-3(1-\nu^2)\frac{d}{d\eta}\left[S_r(1-\eta)\frac{dW}{d\eta}\right]=\tfrac{3}{4}P-KW, \quad (13)$$

$$\frac{d^2}{d\eta^2}[(1-\eta)S_r]+\frac{1}{2}\left(\frac{dW}{d\eta}\right)^2=0, \quad (14)$$

$$S_r-2(1-\eta)\frac{dS_r}{d\eta}=S_\theta. \quad (15)$$

The boundary conditions become

At $\eta=0$,

$$w=\frac{dW}{d\eta}=0,$$

$$S_\theta-\nu S_r=0, \quad (16)$$

$$2(1-\eta)\frac{d}{d\eta}(S_r)-(1-\nu)S_r=0.$$

At $\eta=1$,

$$S_r, \frac{dW}{d\eta}=\text{Finite}. \quad (17)$$

Perturbation Procedure

Let us consider a perturbation procedure based upon the smallness of the dimensionless central deflection of the plate. This technique has been used successfully by CHIEN [4] in the analysis of finite deflections of a clamped edge circular plate having no elastic foundation. We begin by denoting the dimensionless central deflection by $W_0=(w/h)_{r=0}$, and then expanding in ascending powers of W_0 each of the quantities P, W, S_r, and S_θ, viz:

$$\tfrac{3}{4}P=\alpha_1 W_0+\alpha_3 W_0^3+\alpha_5 W_0^5+\cdots \quad (18)$$

$$W=\Omega_1(\eta)W_0+\Omega_3(\eta)W_0^3+\Omega_5(\eta)W_0^5+\cdots \quad (19)$$

$$S_r=f_2(\eta)W_0^2+f_4(\eta)W_0^4+f_6(\eta)W_0^6+\cdots \quad (20)$$

$$S_\theta=g_2(\eta)W_0^2+g_4(\eta)W_0^4+g_6(\eta)W_0^6+\cdots \quad (21)$$

The choice of even and odd powers is based upon obvious physical considerations. The series (18) through (21) are next substituted into Eqs. (13) to (15), and also into the boundary conditions (16) and (17). Thus, all equations will be in the form of power series in W_0. If we equate coefficients of like powers of W_0 we then obtain a set of linearized equations. These equations may be solved successively to determine any desired number of coefficients in (18) through (21).

Collecting coefficients of the W_0 terms in Eq. (13) the following equation is obtained

$$\frac{d}{d\eta}\left[(1-\eta)\frac{d^2}{d\eta^2}\left\{(1-\eta)\frac{d\Omega_1}{d\eta}\right\}\right]=\alpha_1-K\Omega_1. \quad (22)$$

The corresponding boundary conditions are

$$\Omega_1(0) = \Omega_1'(0) = 0 \tag{23}$$

$$\Omega_1'(1) = \text{Finite.} \tag{24}$$

It is also necessary that

$$\Omega_1(1) = 1, \tag{25}$$

and $$\Omega_3(1) = \Omega_4(1) = \cdots = 0.$$

This first approximation is obviously the linear problem of small deflection theory. A solution of (22) may be assumed in the form of the truncated series

$$\begin{aligned} \Omega_1(\eta) &= \eta^2[1 + a_1(1-\eta) + a_2(1-\eta)^2 + a_3(1-\eta)^3] \\ &= A\,\eta^2 + B\,\eta^3 + C\,\eta^4 + D\,\eta^5, \end{aligned} \tag{26}$$

where

$$\begin{aligned} A &= 1 + a_1 + a_2 + a_3, \\ B &= -(a_1 + 2a_2 + 3a_3), \\ C &= a_2 + 3a_3, \\ D &= -a_3. \end{aligned} \tag{27}$$

The values of a_1, a_2, a_3 and α_1 are found from the following system of linear simultaneous algebraic equations

$$\begin{aligned} &28a_1 + 76a_2 + 148a_3 - \alpha_1 = -4, \\ &3a_1 + 18a_2 + 55a_3 + 0 = 0, \\ &K a_1 + (144 + K)a_2 + (912 + K)a_3 + 0 = -K, \\ &K a_1 + 2K a_2 + (3K + 400)a_3 + 0 = 0. \end{aligned} \tag{28}$$

In the numerical example to be presented later it is shown that this first approximation (26) yields results almost identical with those found by Schleicher [1]. The present technique, however, involves considerably less computational effort than does application of the Schleicher method.

Collecting coefficients of the W_0^2 terms in Eqs. (14) and (15) yields the relations

$$\frac{d^2}{d\eta^2}[(1-\eta)f_2(\eta)] + \frac{1}{2}\left(\frac{d\Omega_1}{d\eta}\right)^2 = 0, \tag{29}$$

$$g_2(\eta) = f_2(\eta) - 2(1-\eta)\frac{df_2}{d\eta}. \tag{30}$$

The corresponding boundary conditions are

$$g_2(0) - \nu f_2(0) = 0, \qquad f_2(1) = \text{Finite}. \tag{31}$$

If the value of $\Omega_1(\eta)$ determined in the first approximation is introduced in (29) and (30) the solution to these equations may again be taken in the form of the truncated series:

$$f_2(\eta) = b_0 + b_1\eta + b_2\eta^2 + b_3\eta^3 + b_4\eta^4 + b_5\eta^5 \tag{32}$$

$$g_2(\eta) = c_0 + c_1\eta + c_2\eta^2 + c_3\eta^3 + c_4\eta^4 + c_5\eta^5 \tag{33}$$

where
$$b_5 = \frac{3}{20} B^2 + \frac{4}{15} A C,$$
$$b_4 = \frac{3}{10} A B + b_5,$$
$$b_3 = \frac{1}{6} A + b_4, \tag{34}$$
$$b_2 = b_3,$$
$$b_1 = b_2 = b_3,$$
$$b_0 = \frac{2 b_1}{(1-\nu)}$$

and
$$c_0 = \frac{2 b_1 \nu}{(1-\nu)} = b_0 \nu,$$
$$c_1 = 3 b_1 - 4 b_2,$$
$$c_2 = 5 b_2 - 6 b_3, \tag{35}$$
$$c_3 = 7 b_3 - 8 b_4,$$
$$c_4 = 9 b_4 - 10 b_5,$$
$$c_5 = 11 b_5,$$

in which $A, B, C, \ldots$ are given by (27).

Collecting coefficients of the W_0^3 terms in Eq. (13) yields

$$\frac{d}{d\eta}\left[(1-\eta)\frac{d^2}{d\eta^2}\left\{(1-\eta)\frac{d\Omega_3}{d\eta}\right\}\right] - 3(1-\nu^2)\frac{d}{d\eta}\left[f_2(1-\eta)\frac{d\Omega_1}{d\eta}\right] = \alpha_3 - K\Omega_3. \tag{36}$$

The corresponding boundary conditions are

$$\Omega_3(0) = \left[\frac{d\Omega_3}{d\eta}\right]_{n=0} = 0, \qquad \left[\frac{d\Omega_3}{d\eta}\right]_{n=1} = \text{Finite}. \tag{37}$$

Further, from (25) we have

$$\Omega_3(1) = 0.$$

Using the results obtained in (26), (32) and (33), the boundary value problem described by (36) and (37) can be solved by means of the truncated series:

$$\begin{aligned} \Omega_3(\eta) &= \eta^2(1-\eta)[d_0 + d_1\eta + d_2\eta^2 + d_3\eta^3] \\ &= D_2\eta^2 + D_3\eta^3 + D_4\eta^4 + D_5\eta^5 + D_6\eta^6, \end{aligned} \tag{38}$$

where $D_i\,(i = 2, 3, \ldots, 6)$ satisfy the linear simultaneous algebraic equations:

$$\begin{aligned} &4D_2 - 24D_3 + 24D_3 + 24D_4 + 0 - \alpha_3 = 6(1-\nu^2)\,b_0 A, \\ &0 + 36D_3 - 144D_4 + 120D_5 + 0 = 6(1-\nu^2)[3b_0 B - 2(b_0 - b_1)A], \\ &-(360-K)D_2 - 360D_3 - 216D_4 - 840D_5 + 0 \\ &\qquad = 9(1-\nu^2)[4b_0 C - 3(b_0 - b_1)B - 2(b_1 - b_2)A], \end{aligned} \tag{39}$$

$$\begin{aligned}
&1200\,D_2+(1200+K)\,D_3+1200\,D_4+1600\,D_5+0\\
&\qquad = 12\,(1-\nu^2)\,[5\,b_0\,D-4\,(b_0-b_1)\,C-3\,(b_1-b_2)\,B-2\,(b_2-b_3)\,A],\\
&-900\,D_2-900\,D_3-(900-K)\,D_4-900\,D_5+0\\
&\qquad = -15\,(1-\nu^2)\,[5\,(b_0-b_1)\,D+4\,(b_1-b_2)\,C+3\,(b_2-b_3)\,B+2\,(b_3-b_4)\,A],\\
&D_2+D_3+D_4+D_5+D_6=0.
\end{aligned} \tag{39}$$

Collecting coefficients of the W_0^4 terms in Eqs. (14) and (15) we get the following equations:

$$\frac{d^2}{d\,\eta^2}[(1-\eta)\,f_4(\eta)]+\left(\frac{d\,\Omega_1}{d\,\eta}\right)\left(\frac{d\,\Omega_3}{d\,\eta}\right)=0, \tag{40}$$

$$f_4(\eta)-2\,(1-\eta)\frac{d\,f_4}{d\,\eta}=g_4(\eta), \tag{41}$$

as well as the boundary conditions

$$g_4(0)-\nu\,f_4(0)=0, \qquad f_4(1)=\text{Finite}. \tag{42}$$

Again for the purpose of solving this linear boundary value problem, truncated series type solutions for $f_4(\eta)$ and $g_4(\eta)$ can be employed. We take

$$f_4(\eta)=\sum_{i=0}^{10} h_i\,\eta^i, \tag{43}$$

$$g_4(\eta)=\sum_{i=0}^{10} p_i\,\eta^i, \tag{44}$$

where

$$\begin{aligned}
h_{10}&=\frac{3}{11}D\,D_6,\\
h_9&=\frac{1}{90}[24\,C\,D_6+25\,D\,D_5]+h_{10},\\
h_8&=\frac{1}{72}[18\,D\,D_6+20\,(C\,D_5+D\,D_4)]+h_9,\\
h_7&=\frac{1}{56}[12\,A\,D_6+15\,(B\,D_5+D\,D_3)+16\,C\,D_4]+h_8,\\
h_6&=\frac{1}{42}[10\,(A\,D_5+D\,D_2)+12\,(B\,D_4+C\,D_3)]+h_7,\\
h_5&=\frac{1}{30}[8\,A\,D_4+9\,B\,D_3+8\,C\,D_2]+h_6,\\
h_4&=\frac{3}{10}[A\,D_3+B\,D_2]+h_5,\\
h_3&=\tfrac{1}{3}A\,D_2+h_4,\\
h_2&=h_3,\\
h_1&=h_2=h_3
\end{aligned} \tag{45}$$

and

$$\begin{aligned} p_0 &= h_0 - 2h_1, \\ p_1 &= 3h_1 - 4h_2, \\ p_2 &= 5h_2 - 6h_3, \\ p_3 &= 7h_3 - 8h_4, \\ p_4 &= 9h_4 - 10h_5, \\ p_5 &= 11h_5 - 12h_6, \\ p_6 &= 13h_6 - 14h_7, \\ p_7 &= 15h_7 - 16h_8, \\ p_8 &= 17h_8 - 18h_9, \\ p_9 &= 19h_9 - 20h_{10}, \\ p_{10} &= 21h_{10}. \end{aligned} \tag{46}$$

From the boundary condition (42) we have

$$h_0 = 2h_1/(1-\nu), \qquad p_0 = 2\nu h_1/(1-\nu) = \nu h_0. \tag{47}$$

In the next section it is demonstrated that no more terms in the series (18) through (21) are required for a satisfactory analysis of the problem under consideration.

Experimental Verification

For the purpose of investigating the validity of the above solution, an aluminum alloy plate was supported on coil springs and tested under normal pressure with clamped edge conditions. The elastic and geometric parameters of this system were

$$\begin{aligned} h &= 0.13 \text{ in.}, \\ E &= 10\times 10^6 \text{ lb. per in.}^2, \\ a &= 7.5 \text{ in.}, \\ \nu &= 0.3, \\ k &= 39 \text{ lb. per in.}^3. \end{aligned}$$

The solution of Eqs. (28), (34), (35), (39) and (45) to (47) leads to the following relations

$$\tfrac{3}{4} P = 6.53\, W_0 + 4.64\, W_0^3, \tag{48}$$

$$S_r = 0.95\, W_0^2 - 0.03\, W_0^4. \tag{49}$$

As a preliminary verification of the experimental procedure several tests were conducted with no elastic foundation present to stabilize the plate. Measurements of central deflections as well as outer fiber strains at the surface of the plate were found to be in excellent agreement with the predictions of CHIEN's theory [4]. Then, the springs were placed under the plate so as to give the elastic foundation effect. The experimental results for the central deflection, shown in Fig. 1, are seen to be in very satisfactory agreement with the predications of Eq. (48) based upon the present nonlinear analysis.

The outer fiber strain at the center of the plate, on the face subject to normal pressure, was determined from the membrane strain corresponding to (49) together with the bending strain as given by the usual thin plate moment-curvature relations. The normal deflections in the latter are given by (48). The strain thus predicted on the basis of the present nonlinear theory is shown in Fig. 2. Also shown on that figure is the strain at this same point as deter-

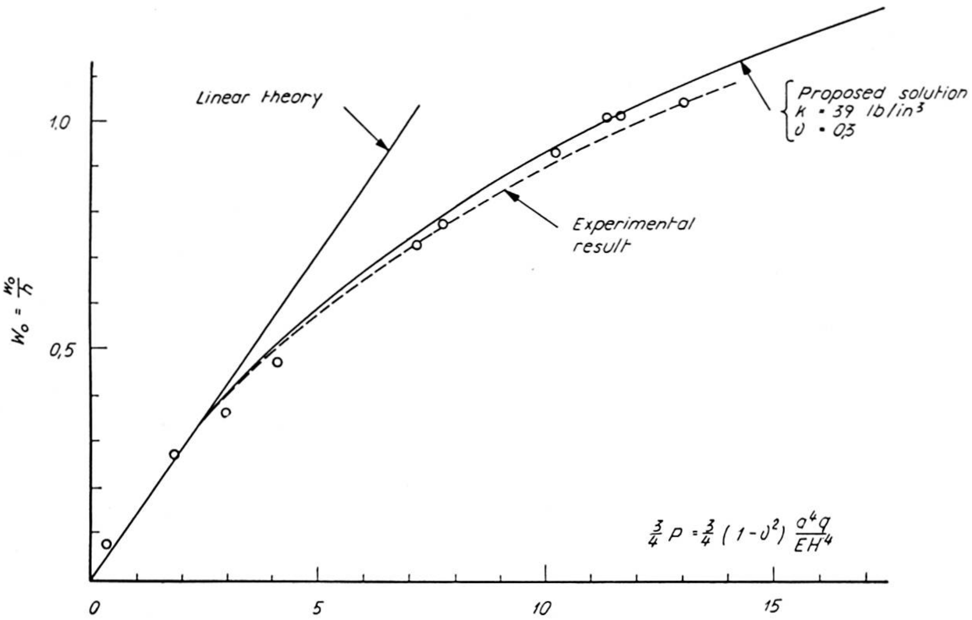

Fig. 1. Variation of central deflection with load (with elastic foundation).

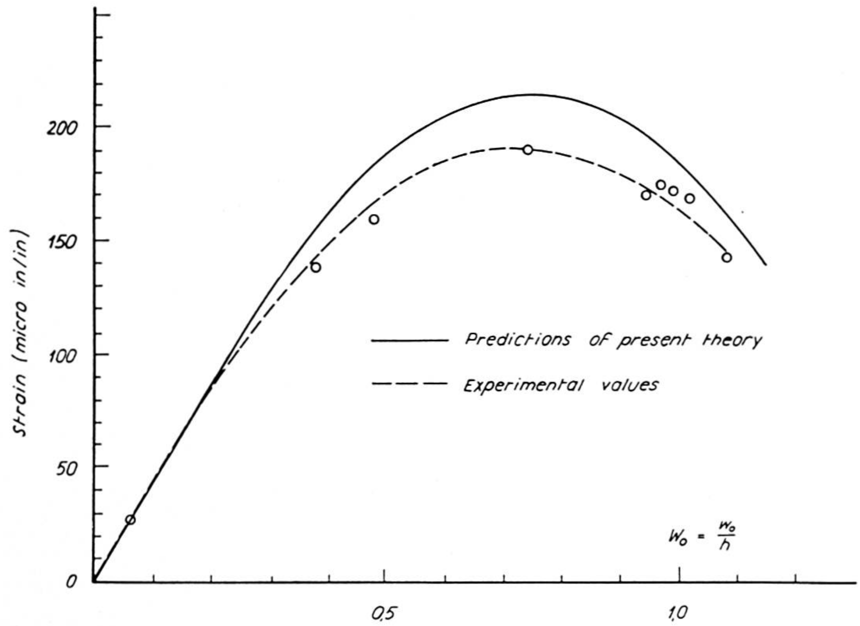

Fig. 2. Comparison of theoretical and experimental outer fiber radial strains at center of plate (with elastic foundation).

mined experimentally by use of electric strain gages. The agreement, although satisfactory is not, of course, as good as was the agreement of deflections.

Fig. 3 indicates a comparison of various significant stresses in the plate; a) when the elastic foundation is present, and b) when it is absent. These relations are all based upon values given by the present nonlinear analysis. From these curves it is evident that the elastic foundation is more effective in reducing the central bending stress than in reducing the central membrane stress.

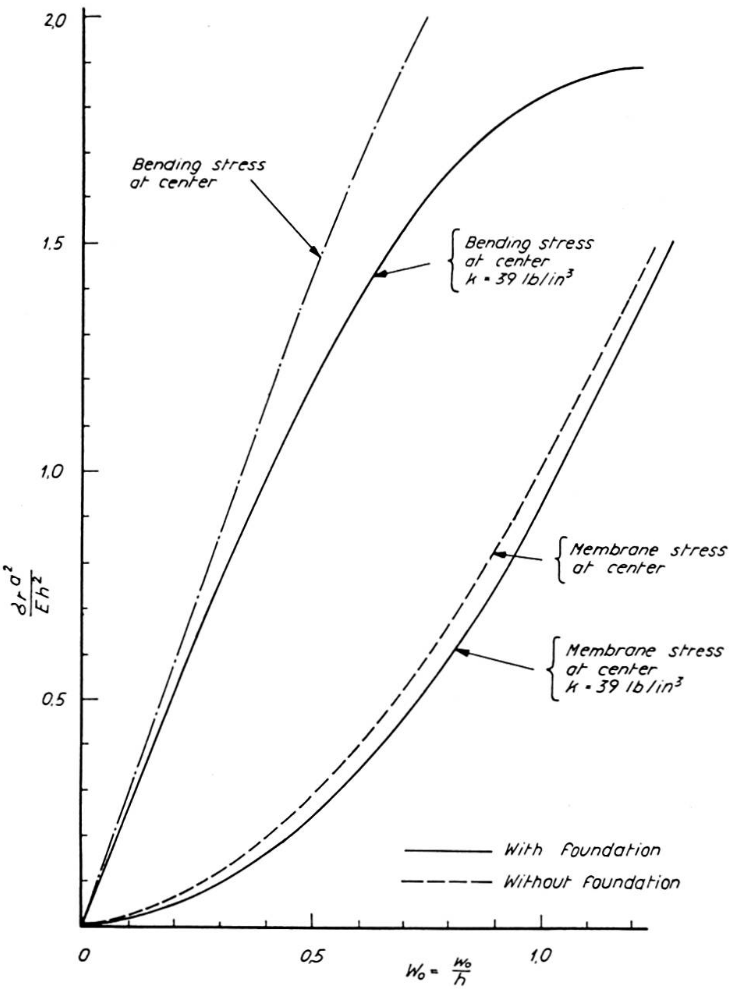

Fig. 3. Various stresses in a clamped edge circular plate ($\nu = 0.3$) with and without elastic foundation.

Conclusions

The validity of a perturbation type analysis for the nonlinear elastic behavior of a clamped edge circular plate on an elastic foundation has been established through experimental verification.

The present nonlinear analysis indicates that, for a given load intensity, the presence of the elastic foundation has little effect on the membrane stress at the center of the plate. However, the elastic foundation is extremely effective in reducing the outer fiber bending stresses at the center of the plate.

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Summary

The problem of the nonlinear large deflections of a thin circular plate supported on an elastic foundation is treated by the method of successive approximations based upon the smallness of central deflections. The edges of the plate are clamped and the face of the plate is subject to uniform normal pressure. Results of the analysis are shown to be in satisfactory agreement with experimental data obtained from tests of an aluminum plate.

Résumé

La méthode d'approximations successives, basée sur la petitesse des déformations centrales, permet d'aborder le problème des grandes déformations non-linéaires des plaques minces, circulaires, sur fondation élastique. Les bords de la plaque sont encastrés et la plaque est soumise à une charge uniformément répartie. L'auteur montre que les résultats obtenus à l'aide de cette méthode concordent de façon satisfaisante avec les résultats d'essais effectués sur une plaque en aluminium.

Zusammenfassung

Das Problem der nichtlinearen großen Durchbiegungen einer dünnen Kreisplatte auf einer elastischen Fundation wird durch die Methode der sukzessiven Näherung, basierend auf der Kleinheit der Mittendurchbiegungen, behandelt. Der Rand der Platte ist eingespannt und die Plattenfläche ist durch gleichförmigen Druck belastet. Die Ergebnisse der Untersuchung ergeben eine zufriedenstellende Übereinstimmung mit Werten, die bei Versuchen an einer Aluminiumplatte gemessen wurden.

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