Theory of statically indeterminate pin-jointed framework the material of which does not follow Hooke's law

Autor(en): **Ylinen, Arvo / Eskola, Aulis**

Objekttyp: **Article**

Zeitschrift: **IABSE congress report = Rapport du congrès AIPC = IVBH Kongressbericht**

Band (Jahr): **6 (1960)**

PDF erstellt am: **11.08.2024**

Persistenter Link: https://doi.org/10.5169/seals-6954

Ib8

Theory of a Statically Indeterminate Pin-Jointed Framework the Material of Which Does Not Follow Hooke's Law

Sur la théorie d'un treillis hyperstatique, dont le matériau ne suit pas la loi de Hooke

Über die Theorie eines statisch unbestimmten Fachwerkes bei beliebigem Formänderungsgesetz

ARVO YLINEN
Dr. Sc. Techn., Prof., Institute of Technology, Helsinki

AULIS ESKOLA
Dipl. Eng., Research Assistant

Introduction

Classical building statics is based on HOOKE's law, $\sigma = E\epsilon$, in which σ denotes the stress, E the modulus of elasticity and ϵ the unit elongation. Hence the methods of classical statics are no longer applicable if the stress in a bar of statically indeterminate framework exceeds the proportional limit of the material or if the material has no limit of proportionality whatsoever. The general theory of a statically indeterminate framework the material of which does not follow HOOKE's law may be based either on the *method of virtual displacements* or on the *principle of the minimum of the complementary energy*. Both these methods are described briefly in the following.

Method of Virtual Displacements

The axial forces produced by actual loading F in the bars of the framework are denoted by S, and the axial forces produced by fictitious loading $\bar{F}$ by $\bar{S}$. Loading F causes in the bars of the framework total elongations Δl which are assumed to be very small compared with the original length l of the bars. Furthermore, if we denote with δ the projection of the displacement of an arbitrary joint in the direction of fictitious load $\bar{F}$ acting at the joint, the principle of the virtual displacements for the real state of displacements $(\delta, \Delta l)$ and for the fictitious state of loads $(\bar{F}, \bar{S})$ can be written

$$\sum \bar{F}\delta - \sum \bar{S}\Delta l = 0. \tag{1}$$

Since no assumption was made in writing this equation regarding the elastic properties of the bars of the framework, it holds good for an arbitrary stress-strain law.

It can be proved[1]) that an expression

$$\Delta l = \frac{S l}{E A} f\left(\frac{S}{S_y}\right) \tag{2}$$

may be derived for the total bar elongation $\Delta l = \epsilon l$ from each stress-strain law which is correct in the physical respect. In the expression, A denotes the cross-sectional area of the bar, $S_y = A\,\sigma_y$ the yield point force of the bar, σ_y the yield point stress in tension and $f(S/S_y)$ a function dependent on the form of the stress-strain law. In the case of HOOKE's law $f(S/S_y) = 1$.

Because eq. (1) holds independently of what causes the changes Δl in the lengths of the bars, the elongations of bars due to a rise in temperature from some specified temperature may also be taken into consideration in the equation. If α is the coefficient of thermal expansion and t is the temperature increase, the corresponding elongation of the bar is $\alpha t l$. Sometimes the length of a bar can be changed by using some mechanical device such as a turnbuckle. Denoting such a change in length of the bar by Δ and superimposing displacements produced by various causes we can write expression (2) in a more general form,

$$\Delta l = \frac{S l}{E A} f\left(\frac{S}{S_y}\right) + \alpha t l + \Delta. \tag{3}$$

When the expression of Δl is introduced into eq. (1) the principle of virtual displacements for a framework the material of which does not follow HOOKE's law may be expressed

$$\sum \bar{F}\delta - \sum \bar{S}\left[\frac{S l}{E A} f\left(\frac{S}{S_y}\right) + \alpha t l + \Delta\right] = 0. \tag{4}$$

Deflection of the Joint of a Framework. To determine the deflection δ_k of any joint k of a statically determinate framework in an arbitrary direction under the action of external loads F acting at the joints of the framework, we imagine the fictitious force $\bar{F} = F_k = 1$ acting at the joint in the direction of the displacement sought. The fictitious load system consists then of force $F_k = 1$ and the corresponding reactions. They do not produce any work since the supports are either immovable or move perpendicularly to the reactions. The first sum in eq. (4) is thus reduced to $1 \cdot \delta_k$. If we denote with $\bar{S} = S_k$ the

---

[1]) Cf. the author's investigation «*Die Knickfestigkeit eines zentrisch gedrückten geraden Stabes im elastischen und unelastischen Bereich*». Doctoral thesis. Finland's Institute of Technology. Helsinki, 1939, p. 94. — Cf. also with formula (19) of the present investigation.

forces in the bars caused by the fictitious force $F_k = 1$, and with S the axial forces caused by the given loading F, we obtain from eq. (4) the formula

$$\delta_k = \sum S_k \left[\frac{S\,l}{E\,A} f\left(\frac{S}{S_y}\right) + \alpha\, t\, l + \Delta\right] \tag{5}$$

for the deflection of joint k of the statically determinate framework. The summation must be extended over every bar of the framework.

Statically indeterminate framework with n redundants. If the framework is statically indeterminate internally it must be transformed into a statically determinate one by means of fictitious sections through each of the n bars. Replacing the unknown axial forces in the cut bars by the forces $X_1, X_2, \ldots, X_n$, we obtain a statically determinate primary system on which, in addition to the given external loading F, the n redundant forces X are acting. The forces produced in the bars of this statically determinate system by the given loading F we denote by S_0. The forces produced in any bar of the same system by the unit redundant forces $X_1 = 1$, $X_2 = 1, \ldots, X_n = 1$ we denote, respectively, by $S_1, S_2, \ldots, S_n$. Then the total axial force in a bar is

$$S = S_0 + S_1 X_1 + S_2 X_2 + \ldots + S_n X_n. \tag{6}$$

The magnitude of the redundant forces X can now be found from the conditions that the relative displacements of the two sides of each of the n fictitious sections must vanish. For these displacements, we use expression (5) which leads to the following n simultaneous equations

$$\begin{aligned}
&\sum S_1 \left[\frac{S\,l}{E\,A} f\left(\frac{S}{S_y}\right) + \alpha\, t\, l + \Delta\right] = 0,\\
&\sum S_2 \left[\frac{S\,l}{E\,A} f\left(\frac{S}{S_y}\right) + \alpha\, t\, l + \Delta\right] = 0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\sum S_n \left[\frac{S\,l}{E\,A} f\left(\frac{S}{S_y}\right) + \alpha\, t\, l + \Delta\right] = 0.
\end{aligned} \tag{7}$$

Here S denotes the expression (6). The summation in every equation includes all the bars of the framework. Supposing that equation system (7) has unique, finite solutions it is possible to determine from it the values of the redundant forces $X_1, X_2, \ldots, X_n$. When they are known, the axial force of each bar is obtained from eq. (6).

If the framework is statically indeterminate externally it must be transformed into a statically determinate one by removing the redundant supports. Replacing the unknown reactions by the forces $X_1, X_2, \ldots, X_n$ we obtain a statically determinate primary system on which, in addition to the given loading F, the n redundant forces X are acting. They can be determined in exactly the manner described above, in which case we again obtain equation system (7) in which S denotes the expression (6).

If the material of the framework follows HOOKE's law, $f(S/S_y) = 1$ and equation system (7) is reduced to the system of elasticity equations, known in the classical theory of the statically indeterminate framework.

Method of Complementary Energy

If a bar is subjected to a tensile or compressive force S acting on the end of the bar, the quantity

$$\int_0^S \Delta l \, dS \tag{8}$$

is denoted as the *complementary energy* stored in the bar. It is seen in fig. 1 as the area $0ab$. Summing up expressions (8) for all the bars of the framework and denoting the complementary energy of the framework by W^*, we obtain

$$W^* = \sum \int_0^S \Delta l \, dS. \tag{9}$$

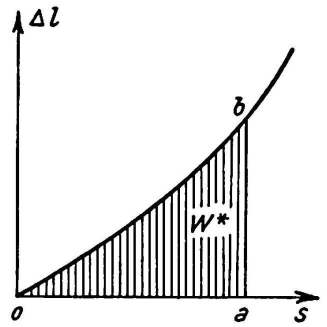

Fig. 1.

From this expression we can establish a very simple method for calculating the deflections of joints of a statically determinate framework and for determining the redundant quantities of a statically indeterminate framework [2]).

Deflection of a joint of a framework. The deflection δ_k of an arbitrary joint k of a statically determinate framework is to be determined in the direction of force F_k acting on it. For this purpose the axial forces S_k produced by the force $F_k = 1$ are determined first. Force F_k itself then produces the axial forces $F_k S_k$. If, with the exception of force F_k, the axial forces produced by all other forces F are denoted by S_0, the axial forces

$$S = S_0 + S_k F_k. \tag{10}$$

As expression (9) of complementary energy is, according to equation (10),

[2]) The idea of using complementary energy for analysing structures was introduced by F. ENGESSER: *Zeitschr. d. Architekten- u. Ing.-Vereins zu Hannover*, Vol. 35 (1889), p. 733. Several applications are shown in a paper by H. M. WESTERGAARD: *Proc. A. S. C. E.*, Vol. 67 (1941), February, p. 199. See also HENRY L. LANGHAAR: *Journal Franklin Institute*, Vol. 256 (1953), No. 3, p. 255, and N. J. HOFF: *The Analysis of Structures*, p. 332. John Wiley & Sons, Inc., New York 1956.

a function of force F_k, W^* may be derived partially with respect to this force, in which case

$$\frac{\partial W^*}{\partial F_k} = \frac{\partial W^*}{\partial S}\frac{\partial S}{\partial F_k} = \sum \Delta l \frac{\partial S}{\partial F_k}.$$

By introducing the expression of Δl from eq. (3) and taking into account that according to eq. (10) $\partial S/\partial F_k = S_k$, we obtain

$$\frac{\partial W^*}{\partial F_k} = \sum S_k \left[\frac{S l}{E A} f\left(\frac{S}{S_y}\right) + \alpha t l + \Delta\right]. \tag{11}$$

Comparing this with eq. (5), we find that

$$\frac{\partial W^*}{\partial F_k} = \delta_k. \tag{12}$$

We have thus proved the following important theorem: *The partial derivative of the complementary energy of a framework with respect to one of the external forces acting in a joint gives the deflection of this joint in the direction of the force.*

If the deflection is desired at a joint where no force is applied, force F_k must be assumed in the direction of the desired deflection. Then

$$\delta_k = \lim_{F_k \to 0} \frac{\partial W^*}{\partial F_k}. \tag{13}$$

Statically indeterminate framework with n redundants. The framework must first be transformed into a statically determinate one either by means of fictitious sections through each of n bars or by removing the redundant supports. Replacing the unknown redundants by the forces $X_1, X_2, \ldots, X_n$ we obtain a statically determinate primary system on which, in addition to the given external loading F, the n redundant forces X act. The forces produced in the bars of this statically determinate system by the given loading F we denote by S_0. The forces produced in any bar of the same system by the unit redundant forces $X_1 = 1, X_2 = 1, \ldots X_n = 1$ we denote, respectively, by S_1, $S_2, \ldots, S_n$. Then the total axial force in a bar is

$$S = S_0 + S_1 X_1 + S_2 X_2 + \ldots + S_n X_n. \tag{14}$$

The magnitude of the redundant forces X can now be found from the conditions that the displacement in the acting points of the redundants must vanish. For these displacements we use expression (11) which leads to the following n simultaneous equations

$$\begin{aligned}
\frac{\partial W^*}{\partial X_1} &= \sum S_1 \left[\frac{S l}{E A} f\left(\frac{S}{S_y}\right) + \alpha t l + \Delta\right] = 0, \\
\frac{\partial W^*}{\partial X_2} &= \sum S_2 \left[\frac{S l}{E A} f\left(\frac{S}{S_y}\right) + \alpha t l + \Delta\right] = 0, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{\partial W^*}{\partial X_n} &= \sum S_n \left[\frac{S l}{E A} f\left(\frac{S}{S_y}\right) + \alpha t l + \Delta\right] = 0.
\end{aligned} \tag{15}$$

Here S denotes the expression (14). The summation in every equation must be extended over all the bars of the framework. A comparison of equation systems (15) and (7) shows them to be identical.

When the values of the redundant forces have been calculated from equation system (15) the axial force of each bar is obtained from eq. (14).

Eqs. (15) state that *the redundant forces $X_1, X_2, \ldots, X_n$ have such magnitudes as to give the complementary energy stored in the framework a stationary value with respect to variations in stress*. It can be shown that *it is a minimum.*

Stress-Strain Function

For the numerical calculations it is necessary to have an analytical expression approximating the actual stress-strain curve of the material. For this purpose we use in the following the function

$$\epsilon = \frac{\sigma}{E} \frac{1 - c\left(\frac{\sigma}{\sigma_y}\right)^n}{1 - \left(\frac{\sigma}{\sigma_y}\right)^n}, \tag{16}$$

in which exponent n denotes a positive integer and $c\,(<1)$ a dimensionless parameter whose value depends on the shape of the stress-strain curve. If n is an odd number, the absolute value $(|\sigma|/\sigma_y)^n$ of the stress ratio must be used. Accuracy sufficient for practical purposes is obtained by selecting $n = 1$, in which case

$$\epsilon = \frac{\sigma}{E} \frac{1 - c\frac{|\sigma|}{\sigma_y}}{1 - \frac{|\sigma|}{\sigma_y}}. \tag{17}$$

This contains three free parameters E, σ_y and c the values of which should be determined so that the stress-strain function agrees suitably with the stress-strain diagram. With the value $c = 1$ function (17) is reduced to HOOKE's law.

To obtain a general idea of the form of the stress-strain diagrams represented by function (17) we write it in a more suitable form for graphical representation by multiplying both sides by the ratio E/σ_y, which gives

$$\frac{E\epsilon}{\sigma_y} = \frac{\sigma}{\sigma_y} \frac{1 - c\frac{|\sigma|}{\sigma_y}}{1 - \frac{|\sigma|}{\sigma_y}}. \tag{18}$$

The dimensionless[3]) stress-strain diagrams according to this equation may be seen from fig. 2 where σ/σ_y is plotted against $E\epsilon/\sigma_y$, with c as the parameter.

---

[3]) Cf. the author's investigation, p. 27, footnote 1.

We see that the greater the value of c, the smaller the deviation of the stress-strain diagrams from the broken line formed by HOOKE's straight line $\sigma/\sigma_y = E\,\epsilon/\sigma_y$ and the horizontal line $\sigma/\sigma_y = 1$ corresponding to the yield point stress. The stress-strain diagrams are symmetrical with respect to the origin of coordinates. The following values of parameter c should be selected for different materials: Steels St 37 and St 52, $c = 0.997$, Magnesium alloy, $c = 0.975$.

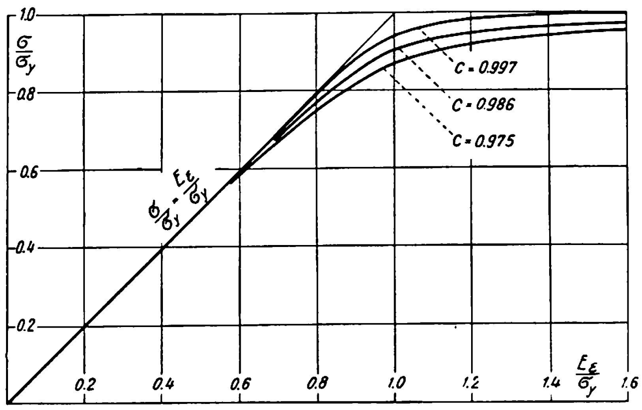

Fig. 2. Dimensionless stress-strain diagrams according to eq. (18).

The expression below for function $f(S/S_y)$ appearing in formula (2) follows from eq. (18):

$$f\left(\frac{S}{S_y}\right) = \frac{1 - c\dfrac{|S|}{S_y}}{1 - \dfrac{|S|}{S_y}}. \tag{19}$$

By introducing it into eqs. (3), (4) and (5) we arrive at the elongation of the bar

$$\Delta l = \frac{S\,l}{E A}\,\frac{1 - c\dfrac{|S|}{S_y}}{1 - \dfrac{|S|}{S_y}} + \alpha\, t\, l + \Delta, \tag{20}$$

the equation of virtual displacements

$$\sum \bar{F}\,\delta - \sum \bar{S}\left[\frac{S\,l}{E A}\,\frac{1 - c\dfrac{|S|}{S_y}}{1 - \dfrac{|S|}{S_y}} + \alpha\, t\, l + \Delta\right] = 0 \tag{21}$$

and the displacement of joint k of the framework

$$\delta_k = \sum S_k\left[\frac{S\,l}{E A}\,\frac{1 - c\dfrac{|S|}{S_y}}{1 - \dfrac{|S|}{S_y}} + \alpha\, t\, l + \Delta\right]. \tag{22}$$

When the expression $f(S/S_y)$ from the eq. (19) is introduced into equation system (7) or (15) these equations can be expressed in the form

$$\sum S_1 \left[\frac{S\,l}{EA}\,\frac{1-c\dfrac{|S|}{S_y}}{1-\dfrac{|S|}{S_y}}+\alpha t l+\Delta\right] = 0,$$

$$\sum S_2 \left[\frac{S\,l}{EA}\,\frac{1-c\dfrac{|S|}{S_y}}{1-\dfrac{|S|}{S_y}}+\alpha t l+\Delta\right] = 0, \qquad (23)$$

. . . .

$$\sum S_n \left[\frac{S\,l}{EA}\,\frac{1-c\dfrac{|S|}{S_y}}{1-\dfrac{|S|}{S_y}}+\alpha t l+\Delta\right] = 0,$$

where S denotes expression (6).

Illustrative Example

In order to illustrate the method described above we analyse the plane framework with one redundant shown in fig. 3. As a redundant force, we take the reaction X_c at the intermediate support C. Removing this support we obtain a statically determinate simple framework on two supports. The framework is made of steel with $E = 2{,}100{,}000$ kg/sq. cm., $\sigma_y = 2{,}400$ kg/sq. cm. and $c = 0.997$. The cross section of all the bars of the framework is $A = 10$ sq. cm. The temperature of the framework is assumed to be constant.

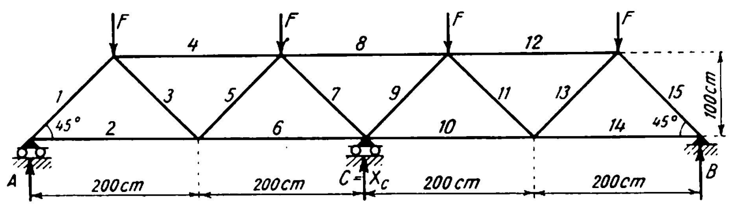

Fig. 3.

The magnitude of the redundant reaction X_c as a function of the loading F may be computed from the first eq. (23) in the form

$$\sum S_c (S_0 + S_c X_c)\, l\, \frac{S_y - c\,|S_0 + S_c X_c|}{S_y - |S_0 + S_c X_c|} = 0. \qquad (24)$$

Here S_0 denotes the axial forces in the bars of the statically determinate primary system due to loads F, and S_c the axial forces due to unit redundant force $X_c = 1$. The forces S_0, S_c and the values of X_c corresponding to the different values of loading F are given in Table 1. The final axial forces are obtained from formula (6) in the form

$$S = S_0 + S_c X_c.$$

Table 1.

Bar	S_0	S_c	F (kg)	X_c (kg)
1, 15	$-2\sqrt{2}\,F$	$\frac{1}{\sqrt{2}}$	5 800	14 099
2, 14	$2\,F$	$-0{,}5$	13 000	31 535
3, 13	$\sqrt{2}\,F$	$-\frac{1}{\sqrt{2}}$	13 500	32 680
4, 12	$-3\,F$	1	13 944	33 560
5, 11	$-\sqrt{2}\,F$	$\frac{1}{\sqrt{2}}$	15 000	33 870
6, 10	$4\,F$	$-1{,}5$	16 971	33 941
7, 9	0	$-\frac{1}{\sqrt{2}}$		
8	$-4\,F$	2		

In fig. 4 X_c is plotted against loading F. The straight line 1 represents the supporting reaction X_c of a framework following HOOKE's law as a function of loading F. Curve 2 represents reaction X_c of a framework the material of which follows law (17). The maximum value of the supporting reaction is $X_{c\,max} = 33{,}941$ kg which is attained when $F = 16{,}971$ kg. The compressive stress in the bars 1, 7, 9 and 15 has then reached the yield point $\sigma_y = -2{,}400$ kg/sq. cm.

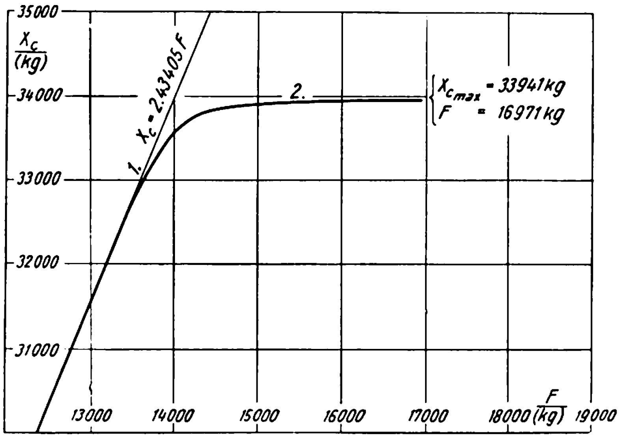

Fig. 4.

Summary

The general theory of a statically indeterminate framework the material of which does not follow HOOKE's law may be based either on *the method of virtual displacements* or on *the principle of the minimum of complementary energy.* Both methods are explained. For numerical calculation the authors present a new stress-strain function, (17), which contains three parameters: E, σ_y and c. As an example of application of the method a statically indeterminate plane framework in fig. 3, the redundant reaction X_c of which is shown in fig. 4 as a function of load F, is analyzed.

Résumé

Dans le cas d'un treillis hyperstatique, dont le matériau ne suit pas la loi de HOOKE, on pourra baser la théorie, soit sur le *principe des déplacements virtuels,* soit sur le *principe de l'énergie complémentaire minima.* Ces deux méthodes font l'objet de l'examen de la présente étude.

Pour le calcul numérique, l'auteur présente une nouvelle loi de déformation (17) qui comprend les trois paramètres E, σ_y et c. La méthode est utilisée dans la résolution du treillis hyperstatique plan de la fig. 3, dont la réaction d'appui X_c en fonction de l'effort F est donnée en fig. 4.

Zusammenfassung

Die allgemeine Theorie eines statisch unbestimmten Fachwerkes bei beliebigem Formänderungsgesetz kann entweder auf die *Methode der virtuellen Verschiebungen* oder auf das *Prinzip vom Minimum der Ergänzungsarbeit* aufgebaut werden. Die beiden Verfahren werden in der Arbeit erläutert. Für die numerischen Berechnungen wird ein neues Formänderungsgesetz (17) verwendet, das drei freie Parameter E, σ_y und c enthält. Als Anwendungsbeispiel der Methode wird das in Fig. 3 dargestellte, statisch unbestimmte ebene Fachwerk behandelt. Die statisch unbestimmte Auflagerkraft X_c als Funktion der äußeren Belastung F ist in der Fig. 4 graphisch dargestellt.