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The Behavior of Viscoelastic Thin Shells of Revolution Under Constant Normal Pressure[1]

Le comportement visco-élastique des voiles minces de révolution soumis à une pression normale uniformément répartie

Das visco-elastische Verhalten von dünnen Rotations-Schalen unter konstantem normalem Druck

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Introduction

The desire and, in some cases, the necessity to utilize materials to the limit of their capacity necessitate more accurate analysis of structural problems. The consideration of time effects, which are greatly influenced by temperature (if temperature effects are experienced by the structure) require methods of analysis beyond the limits of classical theories. The attempt to rational design of structures subject to rate influences is being made through "viscoelasticity theory", which takes into account viscous (time-rate) effects.

The present paper considers the analysis of thin shells of revolution under constant normal pressure. It is assumed that the material behaves as a linear viscoelastic material which may be exposed to temperature changes.

Part I. General Theory

1. Concept of Viscoelasticity

Viscoelasticity is concerned with the analysis of materials exhibiting time effects which include delayed elasticity and viscous flow. These material res-

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ponses are associated with such phenomena as creep under constant stress and relaxation of stress at constant strains. In general, the phenomenon of viscoelastic behavior is extremely complicated. For the purpose of this analysis, the viscoelastic response is assumed to be linear, i. e. at any time instant the strain is approximately proportional to the stress.

The physical behavior of a viscoelastic material may be represented by fig. 1. First, consider the response of the general linear viscoelastic material

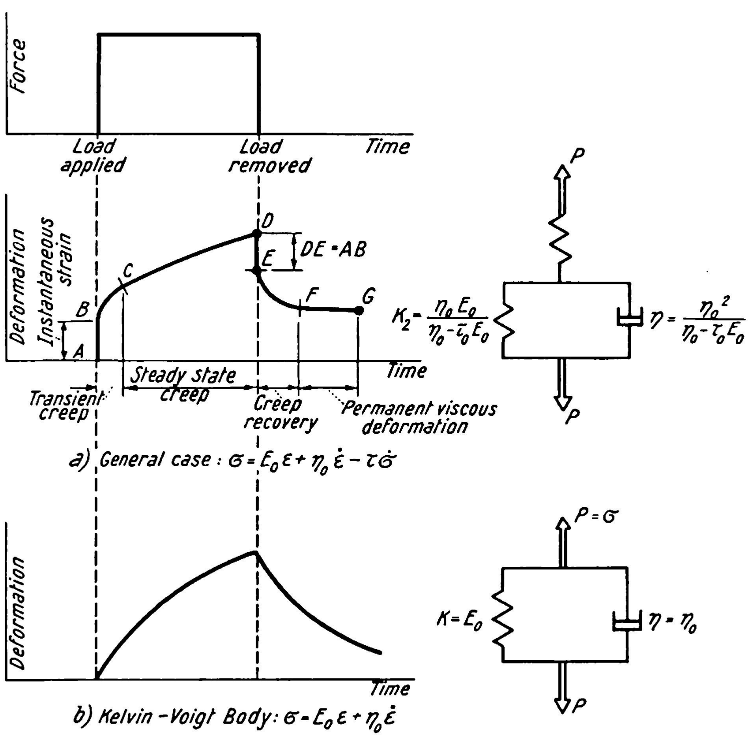

Fig. 1. Physical Behaviors and Mechanical Models.

to an applied force, represented by fig. 1a. It consists of an instantaneous elastic response AB due to spring K_1, followed by a delayed elastic deformation with viscous flow along BC and continued viscous flow along CD. When the force is suddenly removed, the instantaneous elastic deformation is immediately recovered by the spring K_1 as shown in DE. The delayed elastic deformation recovers along EF, leaving the permanent viscous deformation FG due to the dashpot η. Therefore, the general viscoelastic linear solid exhibits all three types of stress responses: instantaneous elasticity, delayed elasticity and viscous flow.

For the KELVIN-VOIGT solid, the response to force is not instantaneous but as may be seen in fig. 1b, the deformation gradually approaches an asymtotic valve. Conversely, when the force is suddenly removed, the solid does not undergo instantaneous recovery, but the deformation gradually disappears.

2. Fundamental Equations of Viscoelasticity

The mathematical formulation of the linear viscoelastic response of an isotropic, homogeneous body to combined thermal and mechanical load can be approximately represented by the following set of differential equations:

$\sigma_{ij,j} + F_i = \rho \ddot{\xi}_i = 0$	Equations of Motion,	(1)
$\epsilon_{ij,kl} + \epsilon_{kl,ij} - \epsilon_{ik,jl} - \epsilon_{jl,ik} = 0$	Compatibility Equations,	(2)
$P\{S_{ij}\} = 2\,Q\{E_{ij}\}, \quad \epsilon = \alpha\, T$	Equations of State,	(3)
$T_i = \sigma_{ij}\, x_i$ [2])	Boundary Conditions,	(4)

where P and Q are linear operators representing different viscoelastic materials and

$$E_{ij} = \epsilon_{ij} - \epsilon\,\delta_{ij}, \qquad S_{ij} = \sigma_{ij} - \sigma\,\delta_{ij}, \qquad \epsilon_{ij} = \tfrac{1}{2}(\xi_{i,j} + \xi_{j,i}).$$

Subject to the boundary conditions, the system of nine equations on the set of nine unknown field variables σ_{ij}, ξ_i is complete in the sense that solution of the system is unique, if the solution exists.

3. Application to Shells of Revolution

Consider a shell of revolution subjected to a constant normal axial-symmetric pressure and a temperature gradient across its shell thickness (fig. 2).

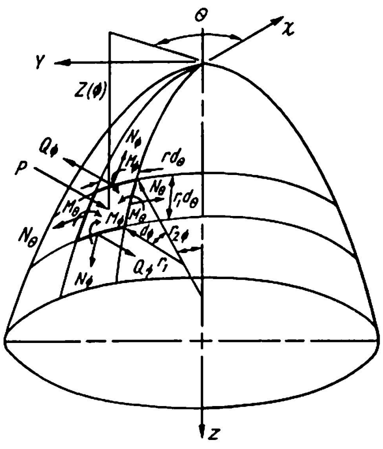

Fig. 2. Coordinate System and Shell Stress Couples and Resultants.

If it is assumed that its material behaves as a KELVIN-VOIGT body with constant viscosity, η, modulus of shear, G, and thermal expansion coefficient, α, the complete set of governing differential equations can be shown to reduce

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[2]) When surface tractions are prescribed at the surface.

to differential equations in unknowns χ and Q_ϕ, the angle of rotation of a tangent to the meridian and the meridian shear stress respectively. They are:

$$\frac{\partial^3\chi}{\partial t\,\partial r}+\frac{1}{\tau}\frac{\partial^2\chi}{\partial r^2}+\left[\frac{1}{r}+\frac{\beta}{r_1}-\frac{3}{\delta}\frac{d\delta}{dr}\right]\frac{\partial^2\chi}{\partial t\,\partial r}+\frac{1}{\tau}\left[\frac{1}{r}+\frac{\beta}{r_1}-\frac{3}{\delta}\frac{d\delta}{dr}\right]\frac{\partial\chi}{\partial r}$$
$$-\frac{1}{2r}\left[\frac{1}{r}+\frac{1}{r_2}\frac{dr_2}{dr}+\frac{f''(r)}{f'(r)}+\frac{3}{\delta}\frac{d\delta}{dr}\right]\frac{\partial\chi}{\partial t}-\frac{1}{2\tau r}\left[\frac{1}{r}+\frac{1}{r_2}\frac{dr_2}{dr}+\frac{f''(r)}{f'(r)}+\frac{3}{\delta}\frac{d\delta}{dr}\right]\chi=$$
$$=\frac{3}{\delta^3}\frac{\beta(Q_\phi r_2)}{\eta r}+\frac{3\alpha\lambda}{2\delta}\left[\frac{\partial^2\Theta}{\partial t\,\partial r}+\frac{1}{\tau}\frac{\partial\Theta}{\partial r}-\frac{3}{\delta}\frac{d\delta}{dr}\left\{\frac{\partial\Theta}{\partial t}+\frac{\Theta}{\tau}\right\}\right], \tag{5}$$

$$\frac{\partial^2(Q_\phi r_2)}{\partial r^2}+\left[\frac{1}{r}+\frac{\beta}{r_1}-\frac{1}{\delta}\frac{d\delta}{dr}\right]\frac{\partial(Q_\phi r_2)}{\partial r}+\frac{1}{2r}\left[\frac{3}{r}-\frac{2}{r_2}\frac{dr_2}{dr}-\frac{f''(r)}{f'(r)}-\frac{1}{\delta}\frac{d\delta}{dr}\right](Q_\phi r_2)=$$
$$=\frac{3\eta\beta\delta}{r}\left(\frac{\partial\chi}{\partial t}+\frac{\chi}{t}\right)+\frac{\rho\lambda}{2}\left[-\frac{3r_2}{2r}\left(1-\frac{r_2}{r_1}\right)+\frac{d}{dr}\left(\frac{r_2^2}{r_1}\right)-\frac{1}{\delta}\frac{r_2^2}{r_1}\frac{d\delta}{dr}\right. \tag{6}$$
$$\left.-\frac{3}{2}\frac{dr_2}{dr}+\frac{3r_2}{\delta}\frac{d\delta}{dr}\right]+3\alpha\delta\lambda\eta\left[\frac{\partial^2 T_m}{\partial t\,\partial r}+\frac{1}{\tau}\frac{\partial T_m}{\partial r}\right],$$

where $\beta=f'(r)\,[1+f'(r)^2]^{1/2}$, $\lambda=[1+f'(r)^2]^{1/2}$, $Z=f(r)$, $f'(r)$ is slope of tangent to a meridian, T_m is the temperature function of the middle surface.

Part II. Example of Practical Application

2. Conical Shell

As an example, we will restrict ourselves to thin conical shells (fig. 3). The temperature is assumed steady, axial-symmetric and constant across the shell thickness. The governing differential equation reduces to (Reference 1)

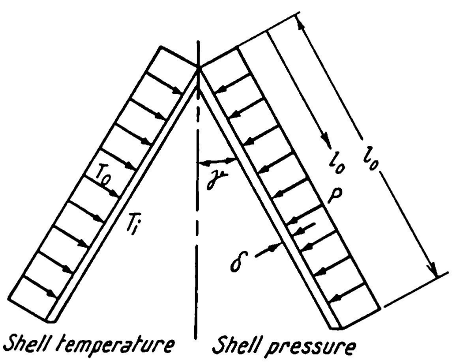

Fig. 3. Shell Temperature and Pressure Distributions.

$$l\frac{\partial^5\chi}{\partial t\,\partial l^4}+4\frac{\partial^4\chi}{\partial t\,\partial l^3}+\frac{l}{\tau}\frac{\partial^4\chi}{\partial l^4}+\frac{4}{\tau}\frac{\partial^3\chi}{\partial l^3}+\frac{9}{\delta^3}\frac{\cot^2\gamma}{l}\left(\frac{\partial\chi}{\partial t}+\frac{\chi}{\tau}\right)= \tag{7}$$

$$=+\frac{9}{2}\frac{p}{\delta^2\eta}-\frac{9\,\alpha\cot\gamma}{\delta^2}\left[\frac{\partial^2 T_m}{\partial t\,\partial l}+\frac{1}{\tau}\frac{\partial T_m}{\partial l}\right]+\frac{3\,\alpha^2 l}{2}\frac{\partial^3}{\partial l^3}\left[\frac{\partial\Theta}{\partial t}+\frac{\Theta}{\tau}\right]+\frac{9\,\alpha^2}{2}\frac{\partial^2}{\partial l^2}\left[\frac{\partial\Theta}{\partial t}+\frac{\Theta}{\tau}\right].$$

Substitution of the boundary conditions $\chi=0$ when $t=0$ for all values of l and when $l=l_0$ for all values of t yields for the solution (1)

$$\chi=\left[A_1\left(Z_1+\frac{2\,Z_2'}{y}\right)+A_2\left(Z_2-\frac{2\,Z_1'}{y}\right)+\frac{\delta^2}{9}\tan^2\gamma\, l\right]\frac{9}{2}\frac{p}{G\,\delta^3}\left(1-e^{-t/\tau}\right), \tag{8}$$

$$Q_\phi=-54\frac{p\cot^2\gamma}{\delta^2}\left[A_1\left(\frac{Z_2}{y^2}-\frac{2\,Z_1'}{y^3}\right)-A_2\left(\frac{Z_1}{y^2}+\frac{2\,Z_2'}{y^3}\right)\right], \tag{9}$$

$$N_\theta=27\frac{p\cot^2\gamma}{\delta^2}\left[A_1\left(\frac{4\,Z_1'}{y^3}-\frac{2\,Z_2}{y^2}+\frac{Z_2'}{y}\right)+A_2\left(\frac{4\,Z_2'}{y^3}+\frac{2\,Z_1}{y^2}-\frac{Z_1'}{y}\right)\right]-p\,l\tan\gamma, \tag{10}$$

$$N_\phi=54\frac{p\cot^2\gamma}{\delta^2}\left[A_1\left(\frac{2\,Z_1'}{y^3}+\frac{Z_2}{y^2}\right)+A_2\left(\frac{2\,Z_2'}{y^3}-\frac{Z_1}{y^2}\right)\right]-\frac{p\,l\tan\gamma}{2}, \tag{11}$$

$$M_\theta=-\frac{q}{2}\frac{p\cot\gamma}{\delta}\left[A_1\left(\frac{Z_1'}{y}+\frac{2\,Z_1}{y^2}+\frac{4\,Z_2'}{y^3}\right)+A_2\left(\frac{Z_2'}{y}+\frac{2\,Z_2}{y^2}-\frac{4\,Z_1'}{y^3}\right)\right]-\frac{p\,\delta^2\tan^2\gamma}{4}+\frac{\alpha^2\delta^3\,\Theta}{2\,G}, \tag{12}$$

$$M_\phi=-\frac{q}{2}\frac{p\cot\gamma}{\delta}\left[A_1\left(\frac{Z_1'}{y}-\frac{Z_1}{y^2}-\frac{2\,Z_2'}{y^3}\right)+A_2\left(\frac{Z_2'}{y}-\frac{Z_2}{y^2}+\frac{2\,Z_1'}{y^3}\right)\right]-\frac{p\,\delta^2\tan^2\gamma}{4}+\frac{\alpha^2\delta^3\,\Theta}{2\,G}, \tag{13}$$

$$u=l\left[\frac{e^{-t/\tau}}{6\,\delta\,\eta}\int_0^t\left(2\,N_\theta-N_\phi\right)e^{t/\tau}\,dt-\alpha\,T_m\right], \tag{14}$$

where $Z=Z(y)$, $y^2=\frac{12\cot\gamma\, l}{\delta}$, $Z_1=\mathrm{ber}\,(y)$, $Z_2=-\mathrm{bei}\,(y)$, $Z'(y)$ is derivative of function with respect to y, and the constants A_1 and A_2 are determined from the shell edge conditions.

1. Numerical Example

Consider a conical shell after fig. 3, axial-symmetrically loaded by $p=10$ lb./sq. in. with a constant temperature difference across the shell thickness of

500° F. The apex angle 2γ is 90° and l_0 is 120 inches. The shell is considered fixed so that the boundary conditions are as follows:

$$l = l_0, \qquad u = 0, \qquad \frac{du}{dl} = 0. \tag{15}$$

The resulting membrane forces and bending moments are shown in fig. 4.

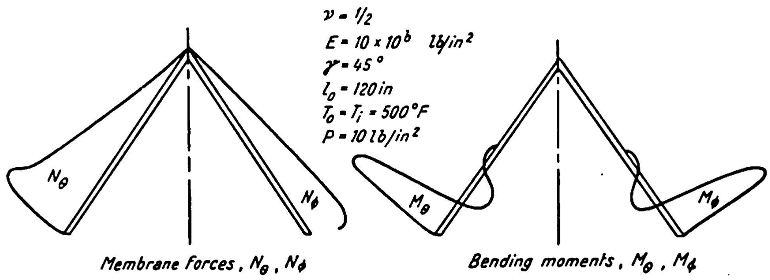

Fig. 4.

Part III. Comparison of Linear Viscoelastic and Elastic Results

Compare now, the linear viscoelastic stresses, moments, and deformation with those of a corresponding elastic analysis.

Linear Viscoelastic Analysis

$$\chi_v = \left[\frac{9}{2}\frac{p}{\delta^3 G} A_1\left(Z_1 + \frac{2Z_2'}{y}\right) + \frac{9}{2}\frac{p}{\delta^3 G} A_2\left(Z_2 - \frac{2Z_1'}{y}\right) + \frac{p\tan^3\gamma}{24\,G}\right](1 - e^{-t/\tau}),$$

$$(N_\phi)_v = 54\frac{p\cot\gamma}{\delta^2}\left[A_1\left(-\frac{2Z_1'}{y^3} + \frac{Z_2}{y^2}\right) + A_2\left(-\frac{2Z_2'}{y^3} - \frac{Z_1}{y^2}\right)\right] - \frac{p\,\delta\tan^2\gamma}{4} y^2,$$

where
$$A_1 = \frac{-\left(\dfrac{\alpha\,\delta^4\,\Theta\,G\tan\gamma}{18P} - \dfrac{\delta^3\tan^3\gamma}{36}\right)}{\left(\dfrac{\dfrac{2Z_{10}'}{y_0^3} + \dfrac{Z_{20}}{y_0^2}}{\dfrac{2Z_{20}'}{y_0^3} - \dfrac{Z_{10}}{y_0^2}}\right)\left[\dfrac{Z_{20}'}{y_0} - \dfrac{Z_{20}}{y_0^2} + \dfrac{2Z_{10}'}{y_0^3}\right] - \left[\dfrac{Z_{10}'}{y_0} - \dfrac{Z_{10}}{y_0^2} - \dfrac{2Z_{20}'}{y_0^3}\right]},$$

$$A_2 = \frac{\left(\dfrac{\alpha\,\delta^4\,\Theta\,G\tan\gamma}{18P} - \dfrac{\delta^3\tan^3\gamma}{36}\right)}{\left[\dfrac{Z_{20}'}{y_0} - \dfrac{Z_{20}}{y_0^2} + \dfrac{2Z_{10}'}{y_0^3}\right] - \left(\dfrac{-\dfrac{2Z_{20}'}{y_0^3} - \dfrac{Z_{10}}{y_0^2}}{-\dfrac{2Z_{10}'}{y_0^3} + \dfrac{Z_{20}}{y_0^2}}\right)\left[\dfrac{Z_{10}'}{y_0} - \dfrac{Z_{10}}{y_0^2} - \dfrac{2Z_{20}'}{y_0^3}\right]}.$$

Elastic Analysis

$$\chi_E = B_1\left(Z_1+\frac{2\,Z_2'}{y}\right)+B_2\left(Z_2-\frac{2\,Z_1'}{y}\right)+\frac{3}{8}\,\frac{p\tan^3\gamma}{E\sqrt{12\,(1-\nu^2)}}\,y^2,$$

$$(N_\phi)_E = 4\,E\,\delta\cot\gamma\left[B_1\left(-\frac{2\,Z_1'}{y^3}+\frac{Z_2}{y^2}\right)+B_2\left(-\frac{2\,Z_2'}{y^3}-\frac{Z_1}{y^2}\right)\right]-\frac{p\,\delta\tan^2\gamma}{8\sqrt{12\,(1-\nu^2)}}\,y^2,$$

where

$$B_1 = \frac{-\left(\dfrac{\alpha\,\delta\,\Theta\tan\gamma}{2}\sqrt{\dfrac{1+\nu}{12\,(1-\nu)}}-\dfrac{P\tan^2\gamma}{8\,E}\sqrt{\dfrac{3\,(1+\nu)}{1-\nu}}\right)}{\left(\dfrac{-\dfrac{2\,Z_{10}'}{y_0^3}+\dfrac{Z_{20}}{y_0^2}}{-\dfrac{2\,Z_{20}'}{y_0^3}-\dfrac{Z_{10}}{y_0^2}}\right)\left[\dfrac{Z_{20}'}{y_0}-(1-\nu)\left(\dfrac{2\,Z_{20}}{y_0^2}-\dfrac{4\,Z_{10}'}{y_0^3}\right)\right]-\left[\dfrac{Z_{10}'}{y_0}-(1-\nu)\left(\dfrac{2\,Z_{10}}{y_0^2}+\dfrac{4\,Z_{20}'}{y_0^3}\right)\right]},$$

$$B_2 = \frac{\left(\dfrac{\alpha\,\delta\,\Theta\tan\gamma}{2}\sqrt{\dfrac{1+\nu}{12\,(1-\nu)}}-\dfrac{P\tan^3\gamma}{8\,E}\sqrt{\dfrac{3\,(1+\nu)}{1-\nu}}\right)}{\left[\dfrac{Z_{20}'}{y_0}-(1-\nu)\left(\dfrac{2\,Z_{20}}{y_0^2}-\dfrac{4\,Z_{10}'}{y_0^3}\right)\right]-\left(\dfrac{-\dfrac{2\,Z_{20}'}{y_0^3}-\dfrac{Z_{10}}{y_0^2}}{-\dfrac{2\,Z_{10}'}{y_0^3}+\dfrac{Z_{20}}{y_0^2}}\right)\left[\dfrac{Z_{10}'}{y_0}-(1-\nu)\left(\dfrac{2\,Z_{10}}{y_0^2}+\dfrac{4\,Z_{20}'}{y_0^3}\right)\right]}.$$

If $\nu = \frac{1}{2}$, $E = 3\,G$, then $B_1 = \frac{9}{2}\,\frac{P}{\delta^3\,G}A_1$, $B_2 = \frac{9}{2}\,\frac{P}{\delta^3\,G}A_2$ and

$$4\,E\,\delta\cot\gamma\,B_1 = 54\,\frac{P\cot\gamma}{\delta^2}A_1 \quad\text{and}\quad 4\,E\,\delta\cot\gamma\,B_2 = \frac{P\cot\gamma}{\delta^2}A_2.$$

Thus, $\chi_v = \chi_E\,(1-e^{-t/\tau})$ and $(N_\phi)_v = (N_\phi)_E$.

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Appendix

χ	angle of rotation of a tangent to a meridian.
l	distance along cone element, measured from vertex.
l_0	length of the cone element
t	time.
τ	relaxation time, $\tau = \eta / G$.
G	shear modulus of elasticity.
γ	cone half-angle.
η	coefficient of viscosity.
δ	shell thickness.
α	thermal expansion coefficient.
T_m	mean shell temperature above a reference temperature $= \frac{T + T_i}{2}$.
Θ	temperature differential across the shell divided by the shell thickness $= \frac{T + T_i}{\delta}$.
r, r_1, r_2	radii of shell.
p	normal pressure, positive pressure directed towards the shell.
Q_ϕ	shear stress per unit length.
N_θ, N_ϕ	membrane forces per unit length.
M_θ, M_ϕ	bending moments per unit length.
u	radial displacement on any xy-planes.
T_1	temperature outside the shell at the nose of the cone, or where $l = 0$.
T_2	temperature outside the shell at the edge of the cone, or where $l = l_0$.
T_i	temperature inside the shell.

Summary

Thin shells of revolution under constant normal pressure are considered as linear viscoelastic bodies to study time dependency of stresses and deformations of shell type structures.

It is shown that for the linear viscoelastic problem, temperature effects are considered, if the temperature dependence of material properties can be neglected, the elastic solution and results can be used with slight modification. If the material is assumed to be incompressible, i. e. Poisson's ratio is $^1/_2$, the following relationship between linear viscoelastic and elastic stresses, moments, and deformations were observed:

1. The linear viscoelastic stresses and moments are identical to the elastic stresses and moments.
2. The viscoelastic deflections differ from the elastic deflection by a time factor $(1-e^{-t/\tau})$, where τ is the relaxation time of the material.

Résumé

L'auteur considère des voiles minces de révolution, soumis à une pression normale uniformément répartie, comme des corps linéairement visco-élastiques; il étudie ainsi la variation des contraintes et des déformations de ce type de structures en fonction du temps. Il montre que pour un voile linéairement visco-élastique, lorsque l'on considère l'effet de la température, les solutions de la théorie de l'élasticité sont applicables avec quelques modifications, pour autant que les propriétés du matériau soient pratiquement indépendantes de la température. Pour un matériau admis incompressible (coefficient de Poisson 0,5), les relations entre les contraintes, moments et déformations visco-élastiques d'une part et élastiques d'autre part sont les suivantes:

1. Les moments et contraintes d'un voile linéairement visco-élastique sont identiques à ceux donnés par la théorie de l'élasticité.
2. Les déformations visco-élastiques sont égales aux déformations élastiques multipliées par le coefficient dépendant du temps $(1-e^{-t/\tau})$, où τ caractérise le temps de relaxation du matériau.

Zusammenfassung

Es werden dünne Rotations-Schalen unter gleichmäßig verteiltem Normaldruck als linear visco-elastische Körper aufgefaßt, um allgemein die Zeitabhängigkeit der Spannungen und Deformationen von solchen Gebilden zu untersuchen.

Es wird gezeigt, daß für das linear visco-elastische Problem bei Betrachtung der Temperatureinflüsse die Lösungen der Elastizitätstheorie mit kleinen Änderungen angewendet werden dürfen, falls die Temperaturabhängigkeit der Materialeigenschaften vernachlässigt werden kann. Unter der Annahme inkompressiblen Materials, d. h. daß die Poisson-Zahl $\frac{1}{2}$ ist, konnten die folgenden Beziehungen zwischen den linear visco-elastischen und den elastischen Spannungen, Momenten und Deformationen festgestellt werden.

1. Die linear visco-elastischen Momente und Spannungen sind identisch mit den Momenten und Spannungen nach der Elastizitätstheorie.
2. Die visco-elastische Durchbiegung unterscheidet sich von der elastischen Durchbiegung durch einen Zeitfaktor $(1-e^{-t/\tau})$, wobei τ die Relaxationszeit des Materials ist.