A generalised method of analysis of elastic plane frames

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A Generalised Method of Analysis of Elastic Plane Frames

Une méthode généralisée d'étude des charpentes élastiques planes

Verallgemeinerte Berechnung ebener, elastischer Stabwerke

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Notation

P	Axial load
M_A, M_B	End bending moments
E	Modulus of Elasticity
I	Moment of Inertia of cross-section
A	Cross-sectional area
l	Chord length of member
l'	Length along deflected profile
$P_E = \dfrac{\pi^2 E I}{l^2}$	Euler load
$k = \dfrac{E I}{l}$	Bending stiffness
ξ	End shortening due to bowing
Δ	Total axial displacement
s, c	Stability functions
θ_A, θ_B	End rotations
$(A E)'$	Modified axial stiffness
x	Distance along the chord
s	Distance along deflection profile
y	Deflection, x along the chord

Member Behaviour

The linear theory of frameworks is a first order small deflection theory. There are some aspects of elastic frame behaviour which require second order theory to elucidate. Thus the calculation of the deflection of a tight string due to a lateral disturbing force is a non-linear problem even though the string may itself behave in a linear manner. This is an instance where change of geometry effects are significant. Another example is the determination of the distribution of loads in hyperstatic trusses which requires a consideration of the second order effects in the members themselves including the change of length due to bowing.

Consider a prismatic member under axial load and end bending moments as shown in Fig. 1.
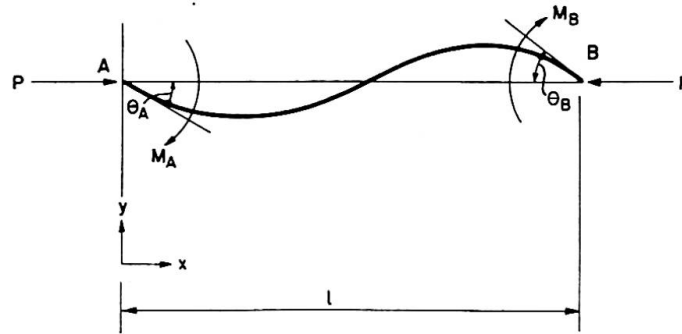

Fig. 1.

The bending moment M at section (x, y) is given by:

$$M = M_A - \frac{x}{l}(M_A + M_B) + Py = -EI\frac{d^2y}{dx^2}.$$

Using the end conditions $y = 0$ when $x = 0$ and $x = l$, the solution is:

$$y = \frac{1}{P}\left[\frac{x}{l}(M_A + M_B) - M_A\right] + A\sin\pi\sqrt{\frac{P}{P_E}}\frac{x}{l} + B\cos\pi\sqrt{\frac{P}{P_E}}\frac{x}{l},$$

where

$$A = -\frac{M_B + M_A\cos\pi\sqrt{\frac{P}{P_E}}}{P\sin\pi\sqrt{\frac{P}{P_E}}}$$

and

$$B = \frac{M_A}{P},$$

i.e.

$$\frac{y}{l} = \frac{M_B}{k}f_1\left(\frac{M_A}{M_B}, \frac{P}{P_E}, \frac{x}{l}\right)$$

and

$$\frac{dy}{dx} = \frac{M_B}{k}f_2\left(\frac{M_A}{M_B}, \frac{P}{P_E}, \frac{x}{l}\right).$$

Now length along deflected profile $\quad l' = \int_0^l ds = \int_0^l\left[1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2\right]dx,$

$\therefore$ End shortening due to bowing $\quad \xi = l' - l = \frac{1}{2}\int_0^l\left(\frac{dy}{dx}\right)^2 dx$

and

$$\therefore \quad \frac{\xi}{l} = \left(\frac{M_B}{k}\right)^2 f_3\left(\frac{M_A}{M_B}, \frac{P}{P_E}\right). \tag{1}$$

In framework analysis by the stiffness method the member end rotations are first calculated, it is, therefore, more convenient to change the variables in equation (1).

The fundamental member equations introduced in ref. [1] are

$$M_A = sk\theta_A + sck\theta_B,$$
$$M_B = sck\theta_A + sk\theta_B$$

and these combine with equation (1) to give:

$$\frac{\xi}{l} = \theta_B^2\, f_4\!\left(\frac{\theta_A}{\theta_B},\ \frac{P}{P_E}\right) = \beta\,\theta_B^2.$$

Graphs of β (plotted logarithmically for convenience) against $\dfrac{\theta_A}{\theta_B}$ for various values of $\dfrac{P}{P_E}$ are given in Fig. 2.
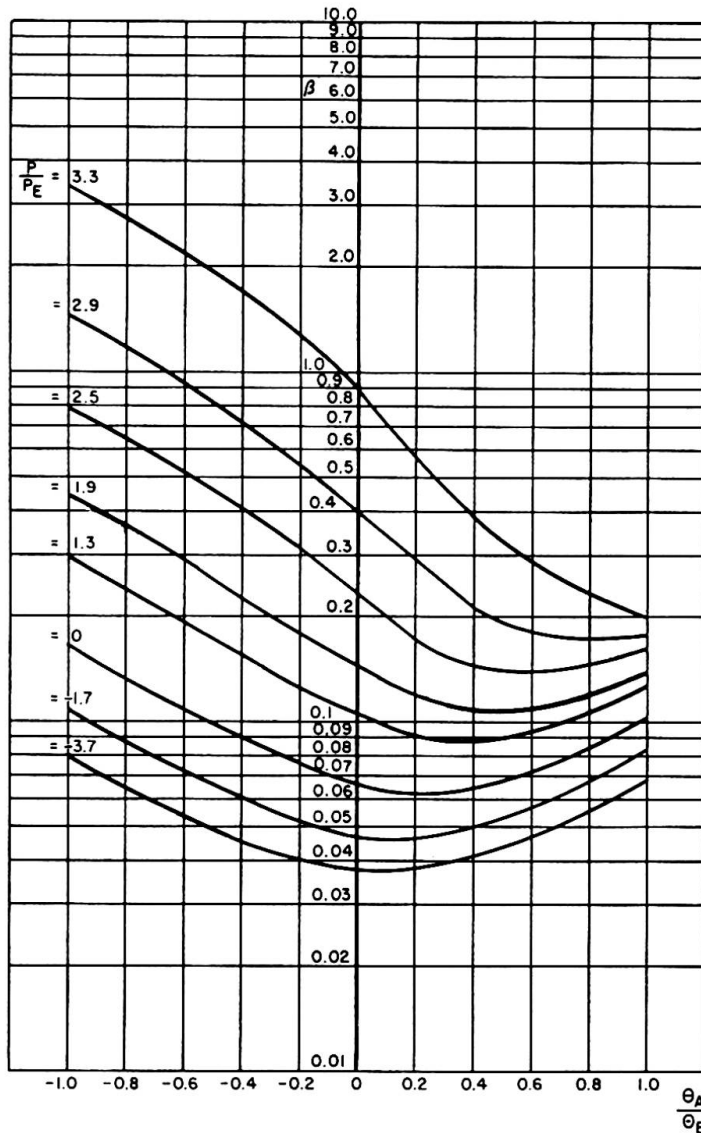

Fig. 2.

The effect of bowing can be incorporated in frame analysis by the introduction of a modified axial stiffness.

Thus the total axial displacement $\varDelta$ is given by:

$$\varDelta = \frac{P\,l}{A\,E} + \xi = \frac{P\,l}{(A\,E)'}.$$

Where $(A\,E)'$ is the modified axial stiffness.

$$\frac{Pl}{AE} + \beta l \theta_B^2 = \frac{Pl}{(AE)'},$$

$$\frac{AE}{(AE)'} = 1 + \frac{\beta \theta_B^2}{\dfrac{P}{P_E} \dfrac{\pi^2 I}{A l^2}}.$$

Frame Analysis

An iterative method is required since the modified member stiffness coefficients are functions of β, θ_B and P; also β is a function of P since it involves

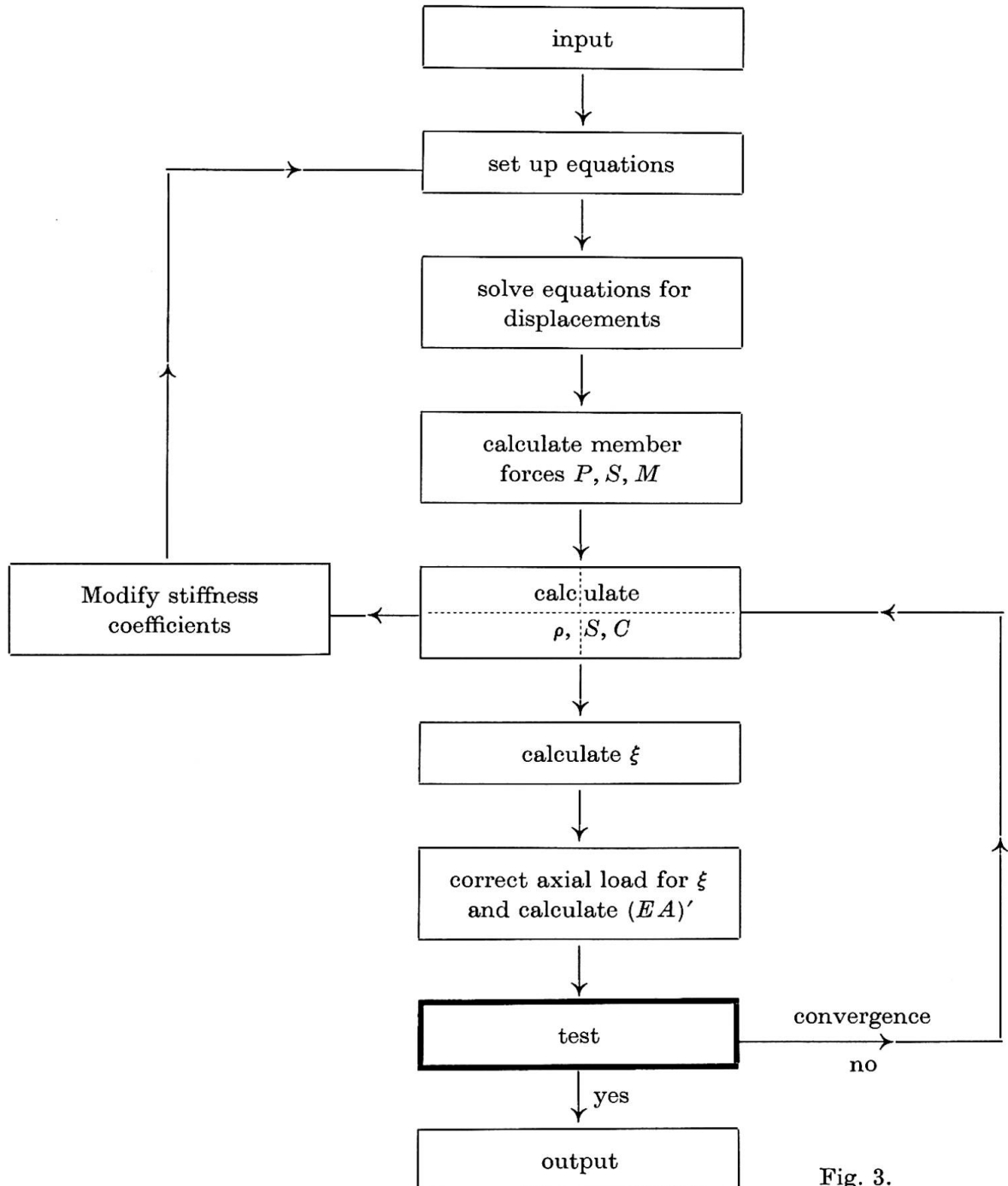Fig. 3.

the stability functions s and c. The path followed by the calculation can be clearly seen in the flow diagram in Fig. 3, it begins with a linear analysis which corresponds to the assumption that $\beta = 0$ and yields estimates of the joint rotations and the axial loads in the members. These enable β and $(A\,E)'$ to be calculated and the estimates of the axial loads in the members and the values of the stability functions to be improved. The process can be repeated until similar results are produced on successive cycles.

A computer programme which will carry out the calculation automatically has been prepared for the Ferranti Atlas computer.

Example

Consider the hyperstatic cantilever truss shown in Fig. 4. The value of E has been taken as 13 000 ton/sq. in. and the cross-sectional properties of the members are given in Table 1.

It is assumed that sufficient restraints are provided to prevent out of plane

Table 1

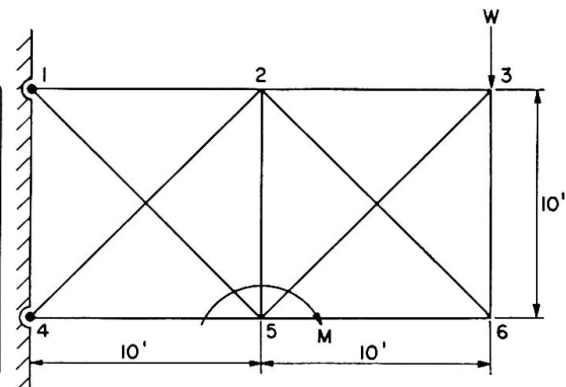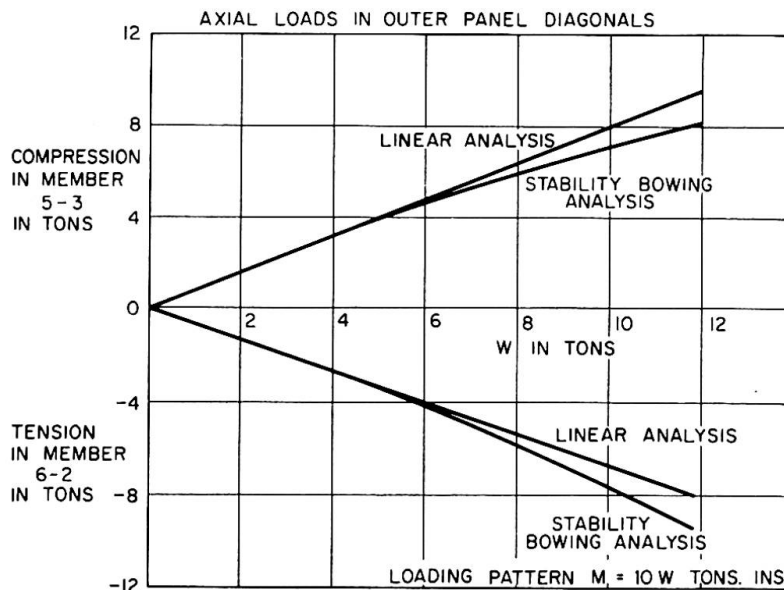| Members	Area in.²	Inertia in.⁴
1—2, 2—3	1.88	0.68
4—5, 5—6	3.56	4.28
5—2, 6—3	1.88	0.68
4—2, 5—3 5—1, 6—2	2.12	0.98

Fig. 4.

Fig. 5.

buckling. Fig. 5 shows various aspects of the non-linear behaviour of the truss as determined by the computer programme. The deviations from linear behaviour are to be noted.

This work forms part of a comprehensive investigation into the behaviour of framed structures using the Atlas Computer and the authors are indebted to Mr. G. Arnold and Mr. B. Dyer, Research Assistants in the Department, who have carried out the programming.

References

1. MERCHANT and SALEM: "The Use of Stability Functions in the Analysis of Rigid Frames." International Association for Bridge and Structural Engineering. Stockholm July 1960.

Summary

In previous work on the stability of framed structures it has been assumed that the distortions are small enough for the equations to be written in terms of the original geometry.

The present paper in addition to stability effects deals with the effects of axial changes in the length of members due to bending and it is illustrated by a consideration of the changes in the proportions of the shear carried in the diagonals of a hyperstatic cantilever truss. The calculations have been carried out using an automatic digital computer.

Résumé

On avait admis jusqu'ici dans tous les travaux sur la stabilité des charpentes, que les déformations étaient assez faibles pour permettre de formuler les équations en fonction de la géométrie initiale.

En plus de l'influence de la stabilité, la présente communication traite des effets des modifications axiales dues à la flexion sur la longueur des pièces. On a pris pour exemple l'étude de la variation de la part de l'effort tranchant supporté par les diagonales d'une ferme cantilever hyperstatique. On a utilisé pour les calculs une calculatrice automatique digitale.

Zusammenfassung

In früheren Arbeiten über die Stabilität von Stabwerken wurde angenommen, daß der Einfluß der Verformung bei den geometrischen Angaben vernachlässigt werden kann.

Die vorliegende Arbeit befaßt sich neben den Einflüssen der Stabilität auch mit den Längenänderungen der Stäbe infolge Biegung. Als Beispiel wird das Verhältnis der Diagonalkräfte infolge Querkraft in einem statisch unbestimmten Konsolfachwerk untersucht. Die Berechnungen wurden mit Hilfe eines Digital-Computers ausgeführt.