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Combination of the Theories of Elasticity, Plasticity and Viscosity in Studying the Safety of Structures

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The assumption of the "failure mechanism" underlying the safety analysis of a structure must reflect the failure criterion in conjunction with the deformational response of the structure, which is determined by its geometry and by the mechanical response of the structural material.

1. Failure Criteria

Criteria of *functional* failure or "unserviceability" of a structure are derived either from the condition of conservation, during every load application beyond a few initial "proof loadings", of a stable structural form satisfying the functional requirements, or from the specification of the maximum change of this form at the end of the anticipated operational life still compatible with those requirements, and expressed in terms of an acceptable rate of change of the structural geometry. Criteria of *structural* failure have the character of instability criteria: deformation grows or separation surfaces propagate at rapidly increasing rates, but under constant or decreasing load intensities. As a result of the difference in character of the criteria of functional and structural failure the associated failure mechanisms are necessarily related to different regions of deformational response of the structure; neither criterion excludes irrecoverable deformations.

Selection of relevant design conditions presupposes the performance of two independent analyses, one for unserviceability, the other for structural failure. In both analyses the inelastic response of the structural material must be considered: in the former by limiting its magnitude either explicitly, as in the case of analysis for creep deformation of concrete structures under sustained high compressive stresses (long-span arches, prestressed girders) or of metal structures at elevated temperatures, or implicitly, as in the case of analysis for elastically constrained plastic deformation under a single load application (plastic relief of elastic stress concentrations), or for the formation of a system of stabilizing residual stresses under cyclic loading ("shakedown"); in the latter by considering the effect of the inelastic response on the development of deformational instability (elastic-plastic and creep buckling, plastic collapse mechanisms, tension instability) or on the mechanics of fracture (brittle fracture, fatigue-and creep fracture).

Inelastic deformational instability and fracture are alternative structural failure mechanisms. This can be shown under the assumption of quasistatic deformation when the applied strain work W is converted in free (elastic) energy W_F, into bound (dissipated) energy W_D and into the energy of production of new surfaces W_s or [1]

$$\frac{dW}{dt} = \frac{dW_F}{dt} + \frac{dW_D}{dt} + \frac{dW_s}{dt} \tag{1.1}$$

The rate of increase of free energy therefore

$$\frac{dW_F}{dt} = \frac{dW}{dt} - \frac{dW_D}{dt} - \frac{dW_s}{dt} \tag{1.2}$$

Formulating the failure condition of the structure by a stationary value of elastic energy $dW_F/dt = 0$

$$\frac{dW}{dt} - \frac{dW_D}{dt} = \frac{dW_s}{dt} \tag{1.3}$$

which indicates that fracture will not propagate in the presence of an effective energy dissipation mechanism absorbing the applied strain work.

Lack of recognition of the dual aspect of structural analysis has been the cause of irrelevant controversy. The proponents of the indiscriminate application of plastic "collapse analysis", comparing the results of their analysis with those of conventional elastic analysis, claim that collapse analysis is more logical and represents reality better than elastic analysis [2]. Within the framework of the well-known limitations of plastic collapse analysis (proportional loading to failure, absence of local instability and of excessive rotation in the plastic hinges) this claim is valid only with respect to structural failure, while it is the elastic analysis which represents the "reality" of functional failure.

Which of the two analyses produces the relevant design conditions therefore depends on the selected "acceptable" values of $F_N(n_s)$ and $F_N(n_F)$, where n_s and n_F denote, respectively, the number of applications of the operational and of the failure load pattern during the anticipated service life of the structure, as well as on the parameters and form of the distributions of the resistance R_s at the limit of serviceability and of R_F at structural failure respectively. The median values $\check{R}_s < \check{R}_F$ and the scatter of R_s, which in most cases depends on elastic properties, is much narrower than that of R_F, while $F_N(n_s) \doteq n_s p_{Fs}$ can always be larger than $F_N(n_F) \doteq n_F p_F$ since the consequences of functional failure are always much less severe than those of structural failure. Under the assumption of a single load spectrum containing both operational and failure loads so that $n_s = n_F$ and therefore $p_{Fs} > p_F$, the schematic representation in Fig. 1 illustrates the relation between safety analysis for functional and for structural failure.

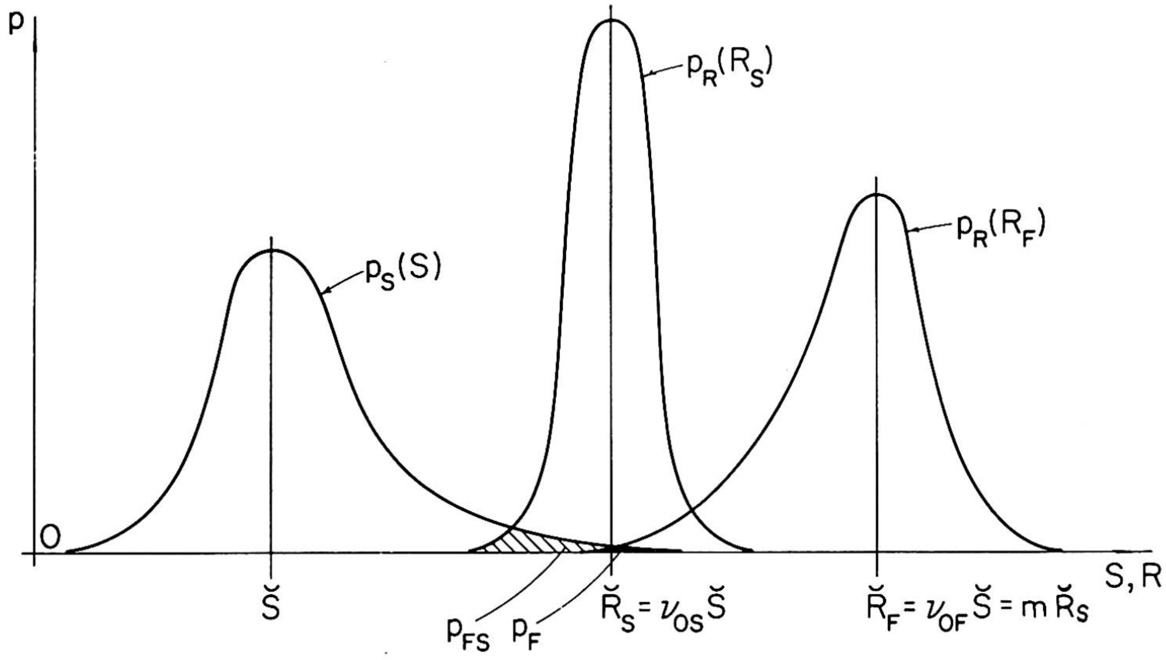

Fig. 1

It also illustrates the difference between the central safety factor for structural failure ν_{0F} and the "overload factor" $m = \check{R}_F/\check{R}_s = \nu_{0F}/\nu_{0s}$, which, in the theory of plasticity, is erroneously considered as a safety factor [3]; ν_{0s} denotes the central safety factor for functional failure. Fig. 1 shows the interrelation between the "overload factor" m of the structure, which is required to ensure the specified probabilities of functional and structural failure p_{Fs} and p_F associated respectively with the central safety factors ν_{0s} and ν_{0F}; it is this factor by which the mechanisms of elastic (functional) and plastic collapse (structural) failure must be related, and which therefore establishes the correlation between probability of failure, safety and plastic collapse analysis.

Effective safety analysis depends on the possibility of a clear separation of independent criteria of functional and of structural failure. This implies that the two criteria are associated with significantly different ranges of material response, as in the case of a functional failure mechanism based on a limited (elastic) strain and a structural failure mechanism associated with plastic or visco-elastic instability. When failure is produced by crack propagation within the range of small deformations, no such separation is possible since the two failure mechanisms are not independent and overlap: progressive damage produced by load intensities within the operational range influences the mechanism of structural failure, as in the case of catastrophic fatigue failure following "shakedown" of the cyclically loaded elastic-plastic structure.

Safety analysis in the fatigue range must be based on the concept of failure, under a single rare load of high intensity, of the structure damaged by fatigue under operational loads. The structural resistance R thus decreases with increasing damage $D(n)$ produced by n repetitions of such loads. Hence, the safety factor $\nu_F = [R(n)/S]$ decreases gradually as the density function $p_R[R(n)]$ moves towards lower values of $R(n)$ thereby increasing the associated probability of failure $p_F = p_F(n)$. This increase can be evaluated by use of the diagrams of Fig. 1 Report Ia for values of ν_0 or $\bar{\nu}$ decreasing with n according to a suitably selected function $R(n)$ [4]. Since $p_F(n)$ is not a constant, the reliability function $L(n)$ is no longer exponential but can be obtained from Eq. (2.7) in Report Ia under the approximate assumption $p_F(n) = h_N(n)$ where $h_N(n)$ is approximated by an increasing function of n of the simple form $h_N(n) = c\alpha n^{\alpha-1}$:

$$L(n) = exp\left[-\left(\frac{n}{v}\right)^{\alpha}\right] \tag{1.4}$$

with $c = v^{-1}$, where v is considered as a "return period" of fatigue failure denoting the value of n at the quantile e^{-1}. Eq. (1.4) is the well-known Third

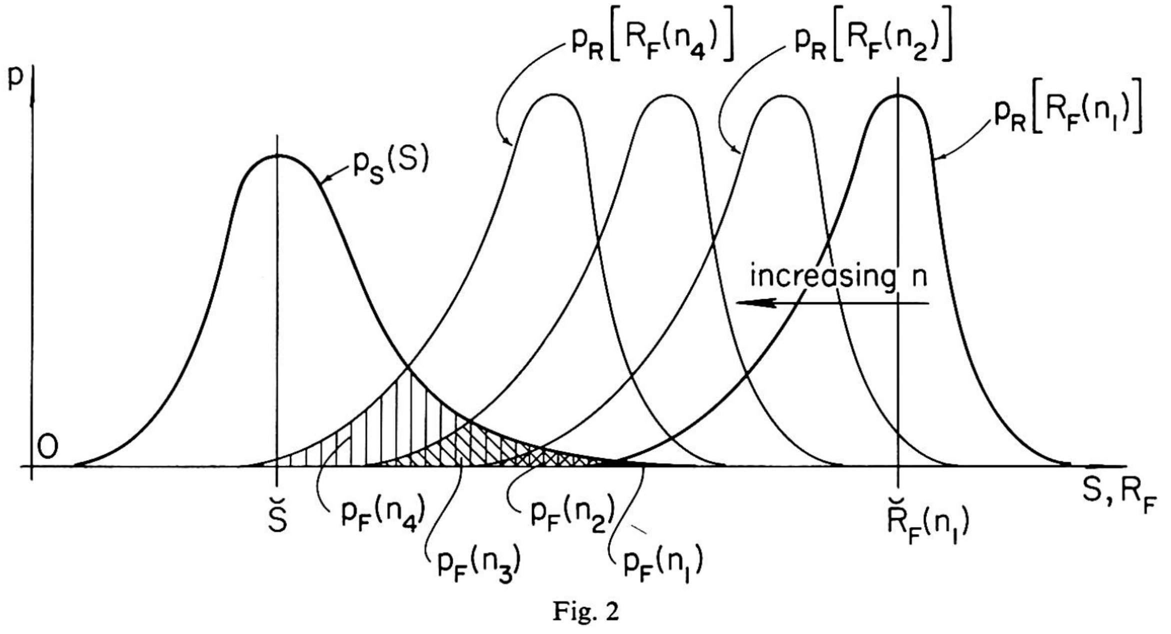

Fig. 2

Asymptotic probability function of extreme (smallest) values [5] which is widely used in the reliability analysis of fatigue sensitive structures [6]; for $a = 1$ and $v = p_F^{-1} = T_F$ Eq. (1.4) degenerates into the exponential reliability function of chance failures.

2. Probability Distribution of Failure Mechanisms

The evaluation of the statistical dispersion of the resistance R, characterizing the critical failure mechanism of a structure, becomes increasingly difficult with increasing complexity of the response of the structural material. In the case of linear material response such evaluation requires the replacement, in the relevant equations of structural mechanics, of the constant physical parameters (moduli of elasticity, coefficients of viscosity) by parameters defined in the form of probability density functions by which the material response acquires the (stochastic) character of a "random medium", and the differential equations of structural mechanics are transformed into equations with stochastic coefficients. The statistical expectation of the solution of such an equation is nearly enough the solution of the associated classical equation with the "expected" values of the parameters, which can therefore be directly introduced as an approximation. In the case of non-linear material response the direct introduction of the dispersion of the classic response equations, such as the moment-curvature relation in bending is more convenient [7] since rigorous analysis, even of linear solid stochastic media, has, so far, only been attempted in very simple cases [8]. In the simplest non-linear case of an elastic-plastic medium a few attempts have been made to introduce a stochastic distribution of the local yield limit in the formulation of a constitutive equation of such a medium and its statistical dispersion [9]. However, none of these attempts have produced results that would be useful on an engineering level.

The distribution of the failure resistance of an elastic-brittle medium based on the simplified assumption of known statistical variation of local strength has been extensively studied [10]. The resulting distribution of the resistance to brittle fracture under uniform tension in the form of the Third Asymptotic distribution of extreme (smallest) values (Weibull distribution [11])

$$P(R) = 1 - e^{-V\left(\frac{R - R_0}{R^*}\right)^{a}} \qquad (2.1)$$

where V denotes the volume, R_0 is the minimum strength, R^* a measure of the central tendency known as the "characteristic strength" and $a > 0$ a scale parameter which decreases with increasing dispension, has been found to reproduce the experimentally observed effects of size, geometry and stress distribution in fracture of brittle materials [12], such as glass, ceramics and refractory metals with values of $3 < \alpha < 8$ equivalent to coefficients of variation of 0.35 to 0.15 with respect to the mean. On the basis of this theory and at the same level of the probability of failure the resistance in pure bending R_B is related to the resistance in tension R_T by the expression [13]

$$R_B = \eta R_T \left[2(\alpha + 1) \frac{V_T}{V_B} \right]^{1/\alpha} \tag{2.2}$$

where V_T and V_B are the volumes of the tension specimen and of the beam specimen respectively, and $\eta = S_B/A_T$ is the ratio between the section modulus S_B of the beam section and the area A_T of the section stressed in tension.

Instability in the elastic range is governed by elastic and geometric parameters and the eccentricity of the compressive forces. In the simple illustrative case of a uniform elastic strut of length L with freely rotating ends, an initial eccentricity in the form of a lateral deflection is amplified by the compressive force P roughly in the ratio $(1 - c)^{-1}$, where $c = P/P_c$ is the ratio of P to the critical (buckling) force $P_c = \pi^2 EI/L^2$, the distribution of which depends on the dispersion of E and I alone, if the length can be considered as a nonstatistical parameter. Thus a relatively narrow dispersion of P_c converts c and the resulting amplification factor $(1 - c)^{-1}$ into a statistical variable of wider dispersion which, by amplifying an initial dispersion of the eccentricity, produces a still wider dispersion of the failure load of the strut due to eccentric compression. The associated probability density $p_R(P) = p_R(R_F)$ must be strongly skewed towards small values of P since the distribution of the excentricity is limited at zero and the dispersion at this limit is only affected by the dispersion of E and I.

The dispersion of the failure load of a linear visco-elastic strut is wider because of its considerable dependence on the coefficient of viscosity [14] which shows considerable scatter. Since the effect of the dispersion of the viscosity on the failure load increases with time, the dispersion of the relatively low, long time "creep-buckling" loads is necessarily wider than that of the high, short time loads. The present knowledge of the form of the distributions of the resistance to compressive failure does not yet justify the assumption of any specific probability function in the safety analysis.

The form of the dispersion of the structural resistance associated with a specific plastic collapse mechanism in bending is related to the fact that it can be expressed by a linear combination of the plastic hinge moments producing this mechanism. As a result of the central limit theorem the distribution function of the resistance at plastic collapse will tend towards a normal distribution with increasing degree of indeterminancy of the structure, independently of the form of the distributions of the individual hinge moments which are similar and depend mainly on the distribution of the yield stress. Since numerous observations have shown this distribution to be nearly enough Logarithmic

Normal [15], with a coefficient of variation with respect to the median of 0.05 to 0.15, depending on the level of control of the production process, it can be assumed that the form of the distribution function of the structural resistance at collapse varies between a Logarithmic Normal for low redundancy to a Normal for high redundancy, with coefficients of variation decreasing with increasing redundancy as a result of the central limit theorem.

The effect of material response on the dispersion of tension instability loads can be easily illustrated for the case of extension of a uniform strut of cross section A and length L of incompressible material, of stress-strain relation

$$\sigma = M \varepsilon_L{}^n \tag{2.3}$$

where $\varepsilon_L = \ln(L/L_0)$ and $0 < n < 1$. Since $P = \sigma A$, the instability conditions $dP = \sigma dA + A d\sigma = 0$ in conjunction with the incompressibility condition $dV = d(AL) = A dL + L dA = 0$ produces the expression $(d\sigma/\sigma) = d\varepsilon_L$ or $(d\sigma/d\varepsilon_L) = \sigma$ and therefore, from Eq. (2.3), the "instability strain" $\varepsilon_L = n$. The associated instability force therefore

$$P = A_0 M (n/e)^n \tag{2.4}$$

since $A = A_0 e^{-\varepsilon_L}$. Introducing the "strain-hardening coefficient" n as a statistical variable to reproduce the dispersion of the observed stress-strain relations, the probability function $P_R(R_F)$, where $R_F = P$, can be obtained from Eq. (2.4). Thus, for instance, for the rather wide range of variations of the instability strain $0.25 > \varepsilon_L = n > 0.125$ the range of P is enclosed between $1.83 A_0 M > P > 1.48 A_0 M$. Since the workhardening relations for one and the same material are usually reproducible within a much narrower range of scatter of n, the dispersion of the resistance under conditions of tension instability is so narrow as to be practically non-statistical.

3. Limitations of the Probabilistic Approach to Safety Analysis

The most obvious limitations of the probabilistic approach to safety analysis are the existence of non-random effects in structural reliability such as the existence of non-random loads, and of effects of accuracy of load and stress analysis, quality of workmanship and level of local inspection during construction. Some of these effects are, however, reflected in the selection of the distribution functions and parameters of the probabilistic analysis.

Thus the level of material inspection influences both the width of the dispersion of the critical material parameters as well as the form of their distribution. According to existing observations a coefficient of variation with respect to the median of $v = 0.05$ represents an exceptionally high level of control of both strength and yield stress of structural metals as well as of structural concrete. A coefficient of variation $v = 0.10$ to 0.12 represents an average level, while values $v > 0.15$ are an indication of inadequate quality control. These latter conditions are, moreover, characterized by "extreme value" distributions of the material parameters [16], while adequate control levels are reflected by Logarithmic Normal distributions.

Non-random loads, such as dead load, can be added to the mean or median operational load intensity, thereby reducing the coefficient of variation of the load which determines the central safety factor ν_0 of the structure. Thus a coefficient of variation of the operational load $v = 0.20$ is reduced to $v = 0.10$ for a ratio of 1:1 of dead load to operational load and to $v = 0.05$ for a ratio of 3:1, with resultant reduction of the central safety factor with increasing dead load.

Effects of accuracy of analysis and quality of workmanship require consideration outside of the framework of probabilistic analysis, which therefore must be considered to produce only a minimum value of the safety factor, to be corrected for the nonstatistical effects by a suitable rating procedure [17]. Numerical values proposed for such "rating factors" by which an objective safety factor is to be multiplied have, however, no rational basis and can, therefore, not be related to an objective probability of failure.

The fact that the form of the distribution functions of the relevant parameters cannot be determined from actual observations within a range significant for safety analysis has caused the most serious objections to the use of such analysis. It has also given rise to proposals to introduce non-parametic methods [18] in preference to specific distribution functions. Since these proposals are impractical in view of the impossibility to obtain acceptably low values p_F on the basis of a non-parametric approach, the problem of the selection of the form of the distribution functions of S and of R on the basis of existing or obtainable data appears to be the principal limitations to the general acceptance of the probabilistic interpretation of structural safety.

No rational solution of this problem is possible, however, without the realization that the problem is one of selecting probabilistic models that generate relevant distribution functions and not of selecting functions to fit observations, simply because the number of observations, particularly of the significant material parameters, can never be large enough to reach the probability range significant for structural reliability analysis. On the other hand, within the practical range of observations statistical fitting of data cannot lead to such discrimination between probability functions that would justify extrapolation into the significant probability range.

The simplest and most effective probability models are based on the concepts of "rare" and of "extremal" phenomena generating respectively Poisson and related discrete distributions and Extremal distributions [19]. These distributions are germane to structural safety analysis which is concerned with rare or extreme high load intensities in conjunction with rare or extreme low values of structural resistance. Once a relevant form of the distribution has been selected only a limited number of observations is required for estimation of its parameters, and extrapolation can be justified not on the basis of curve fitting in the central range, but on the much firmer bases of physical relevance. Hence, small numbers of observations of data that can be classified as "rare" or "extremal" are much more useful than a large series of data of indeterminable character. For instance, observations of highest yearly flood levels can be reliably extrapolated on the basis of extremal distributions [20], while the entity of daily water level records would be useless for this purpose.

It is important to note, moreover, that the knowledge of the form of the distribution of both S and of R is required only if the dispersion of both variables is roughly of the same magnitude. It is easily seen from Fig. 2 of Report Ia that a moderately large dispersion of the load intensity reduces the significance of the form of the distribution of the structural resistance: for coefficients of variation $v_S = 0.20$ the relations between p_F and ν for Logarithmic Normal and for Extremal distribution with $v_R = 0.10$ are practically identical, in spite of the fact that a dispersion of resistance characterized by $v_R = 0.10$ is not very narrow. Since a coefficient of variation of the load intensity of $v_S = 0.20$ is not exceptional (coefficients of variation of $v_S = 0.18$ for windloads [21] and of $v_S = 0.177$ for floor loading [22] have been determined) while $v_R = 0.10$ is at the upper limit of dispersion of material parameters for adequate inspection levels, the conclusion seems justified that the form of the distribution of material parameters is significant only when the load intensity is of very narrow dispersion or non-statistical.

The assessment and justification of a quantitative risk of failure designated as "acceptable" has been attempted by either of two methods: (a) by comparison of the risk of structural failure with other risks deemed "acceptable" because they are usually provided for by insurance coverage, or (b) by the introduction of a "decision rule" or course of action by which a certain measure of "effectiveness" of the structure is optimized. Current preoccupation with "decision theory" and "optimization" as important aspects of "systems design" has resulted in attempts to apply similar concepts to the determination of an acceptable risk of structural failure, selecting a suitable measure of "effectiveness" to be optimized, such as the weight of the structure or its cost, introduced as a function of the probability of failure characterizing the design, or introducing simultaneous objectives, such as minimizing the cost while maximizing the safety of the structure.

The application of these methods can, however, not remove the necessity to introduce, at some point in the analysis, a subjective value judgement, for instance the assessment of the relative importance of alternative objectives or, in the case of a single objective, such as minimum cost, the assessment of the ratio of the cost of the structure and the cost of its failure. For the latter case this can be shown by introducing the simplest possible criterion that the total cost of the structure should be minimized. This cost is made up of the first cost $A(p_F)$ and the capitalized cost of failure $C(p_F)$, which is $p_F \cdot C(p_F) \cdot Q$, where Q is the capitalization factor and p_F the probability of failure referred to one year of operation. Hence the condition

$$A(p_F) + p_F C(p_F) \cdot Q \rightarrow min \tag{3.1}$$

or

$$\frac{dA(p_F)}{dp_F} + Qp_F \frac{dC(p_F)}{dp_F} + QC(p_F) = 0 \tag{3.2}$$

will furnish the value of p_F by which the total cost of the structure is optimized provided the dependence of A and C on p_F can be established. If $A(p_F)$ is introduced as a decreasing function of p_F of the simple form [23]

$$\frac{dA}{dp_F} = -\frac{c}{p_F} \quad \text{or} \quad A = -c \ln p_F + B \tag{3.3}$$

and the cost of failure written in the form $C = C' + C''$ is assumed to consist of two parts, the cost of reconstruction $C' \sim A$ and a part C'' that is independent of the reconstruction cost but somehow expresses the general cost of the failure, Eq. (3.2) with $p_F Q << 1$ takes the form

$$p_F{}^* \doteq \frac{c}{QA}\left(1 + \frac{C''}{A}\right)^{-1} \sim \frac{c}{QC''} \tag{3.4}$$

provided $C''/A >> 1$. Hence the analysis contains the ratio between the first cost of the structure and the cost of its failure or the cost of failure itself as a prominent parameter by the selection of which the numerical value of the "acceptable" risk can be changed by several orders of magnitude. Instead of selecting an "acceptable" risk by subjective considerations, such as a comparison with other risks, the cost of failure is thus selected in terms of the first cost of the structure. While this latter procedure may be less arbitrary, it is shown that a subjective decision at some point of the procedure cannot be avoided; only the point at which it is to be made can be varied.

One of the objections to the probabilistic approach to safety that has been raised is that no real meaning can be associated with probabilities of the very small magnitude (10^{-4}–10^{-8}) used in this approach, particularly since the distributions in this range cannot be known from statistical inference. It must be recognized, however, that the distributions are not selected by statistical inference but by arguments of physical relevance, and that the actual values of the probabilities are less important than the fact that their use permits the imposition of a *uniform* reliability measure on all parts of a structure for which no other method is available.

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