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The Relation of Data to Calculated Failure Probabilities

Rapport entre les différents facteurs dans le calcul de la probabilité de rupture

Die Beziehung der Daten zur berechneten Bruchwahrscheinlichkeit

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By the methods of rational mechanics and the calculus of probability, we can now process the probability distributions for loads and material strengths relating to a proposed structure and calculate the 'probability of failure' to any desired number of decimal places, regardless of how scanty the data is or how poorly the curves fit the data. Clearly, the meaning of this calculated probability needs to be studied critically before it can be used with confidence in the design process. In particular, we must find ways to assess whether or not the data is really sufficient to warrant the probability statements used in the design.

The nature of probability has been studied extensively [1, 2]. In relation to the structural design problem the notion is fairly well defined; in most studies of the structural safety problem, 'probability' is usually taken in the sense of "probability-1" defined at length by CARNAP [2] (loosely called 'subjective probability'), or it is left as an undefined notion; "probability-2" ('objective probability') cannot properly be assigned any meaning in this context.

One way to employ probability(-1) in problems of structural safety is to adopt the viewpoint that it is merely a subjective measure of 'degree of belief,' or 'strength of belief'. The relation of data to the probability of failure is then very simple; data may rationally be assimilated into the input probabilities by Bayesian methods [3]. The question of what constitutes a sufficient amount of data to make a particular statement about the probability of failure, does not arise. Therefore, this paper is not relevant to 'Bayesian design.'

Alternatively, we may consider the probabilities associated with loads and strengths to be inherently unknown, auxiliary quantities. Objective statements about the probability of failure can then be made in the usual terms of statistical inference, and the subjective element in the justification of the design is greatly reduced. The viewpoint in the following, then, is that probability is not an absolute notion; rather, it has meaning only in relation to a specified body of evidence which, in this context, means: Actual results of load measurements, materials tests, model tests, prototype tests, etc., called the data. The advantage of this approach (when it is feasible) over the Bayesian approach is that it leads to propositions about the probability of failure that can be subjected to scientific inquiry.

Under normal conditions of practical design the data is, unfortunately, insufficient to make objective statements about the probability of failure of a proposed structure; for example, future loads must be guessed from measurements taken in the past. Nevertheless, it is instructive to study the rational inferences about the probability of failure that are possible under certain idealized conditions as models of reality, permitting us to estimate the amount of data required under less ideal conditions. In the following we will derive such a relationship (equation 12) between the necessary amount of data and various constants related to the design value of the probability of failure.

Consider a structure drawn at random from an infinite population of like structures and submitted to a single scalar load S drawn at random from an infinite population of loads. Let R denote the resistance of the structure, defined in such a way that failure is the event $R < S$. Resistance R and load S are assumed to be intrinsically positive, independent, continuous stochastic variables with unknown probability densities $p_R(R)$ and $p_S(S)$; information about these functions is assumed obtainable by random sampling. The data D is therefore a set of n_R resistance values and n_S load values:

$$D = \{R_i,\ i = 1,\dots,\ n_R;\ S_j,\ j = 1,\dots,\ n_S\}. \qquad (1)$$

The probability of failure is

$$p_F = \int_{R<S} p_R(R)\ p_S(S)\,dS\,dR; \qquad (2)$$

since p_R and p_S are unknown, p_F cannot be determined. The problem is instead to compute a suitable estimator C_F called the calculated probability of failure.

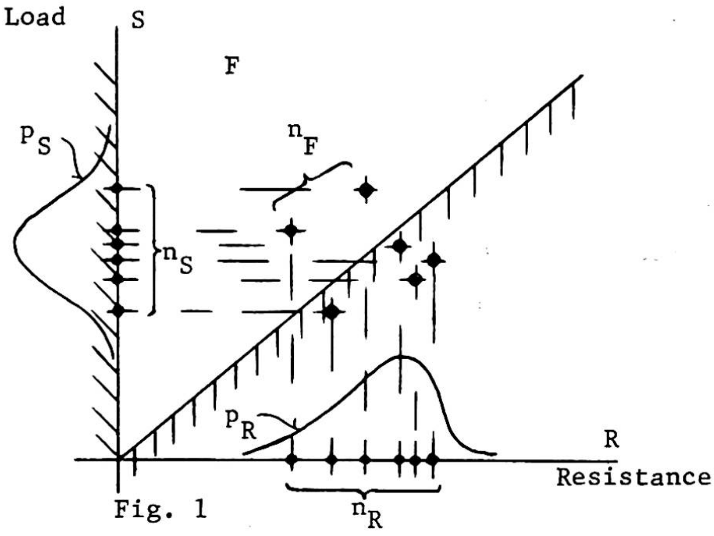

Fig. 1

To make inferences about the probability of failure p_F, it is necessary to derive suitable statistics of the stochastic variable C_F.

The simplest way to obtain such an estimator is to draw from the data D a sample W of n ($\leq n_R$, $\leq n_S$) pairs (R,S) of resistance and load values, at random and without replacement, see Fig. 1. Then, W is a random sample of the parent population $\{(R,S)\}$, and the elements of W are stochastically independent. Let n_F denote the number of outcomes of the failure event $R < S$ in the sample W. Evidently, n_F is the total number of "successes" in n independent Bernoulli trials with probability p_F of "success". Therefore, n_F is distributed according to the binomial distribution

$$b(1,n,p_F) = np_F(1-p_F)^{n-1} \tag{3}$$

with mean np_F and variance $np_F(1-p_F)$. It follows that the estimator $f_F = n_F/n$ is similarly distributed with mean $m = p_F$, variance $\sigma^2 = p_F(1-p_F)/n$, and coefficient of variation $v = \sigma/m = 1/\sqrt{np_F/(1-p_F)}$. The relative failure frequency f_F is therefore an unbiased estimator of p_F. It is discrete valued ($f_F \in \{0, 1/n, 2/n, \ldots, 1\}$), so that in order to get sufficient resolution it is required that n_F be large in comparison with unity. Assuming that n_F is greater than 9 and neglecting p_F in comparison with unity, it can be shown [4] that f_F is approximately normally distributed with mean p_F and coefficient of variation $1/\sqrt{np_F}$.

In this context, the most appropriate way to indicate the precision of an estimate of p_F is by means of confidence intervals [4]. First, a confidence coefficient α is selected. Taking the distribution to be normal with mean C_F and coefficient of variation $1/\sqrt{nC_F}$ gives the following approximate confidence limits for p_F computed from the calculated probability of failure:

$$L^- \approx C_F(1-N^{-1}(\alpha)/\sqrt{nC_F}), \quad L^+ \approx C_F(1+N^{-1}(\alpha)/\sqrt{nC_F}); \tag{4}$$

$N^{-1}(\ )$ denotes the inverse function of the normal probability integral. In a long sequence of repetitions the confidence interval between L^- and L^+ will contain the probability of failure p_F nearly a fraction α of the time.

To illustrate, assume that the data D consists of $n_S = 10^5$ and $n_R = 10^4$ random samples of load and resistance, respectively. The largest random sample W of independent elements that can be drawn contains $n = 10^4$ (R,S)-pairs. Assume that $n_F = nC_F = 16$ is the number of failure events in such a sample. If a confidence coefficient α = 90 per cent is considered suitable, we get from a table of the normal probability integral that $N^{-1}(0.9) = 1.645$. Equations (4) then give $L^- = (1-0.41)C_F$ and $L^+ = (1+0.41)C_F$. The following continued inequality may be written down:

$$(0.59)\left(\frac{16}{10^4}\right) < p_F < (1.41)\left(\frac{16}{10^4}\right); \tag{5}$$

it may be asserted that this inequality is satisfied with probability 0.9. In other words, chances are nine out of ten that the value of p_F lies between 0.00094 and 0.00226. Independent random pairing of load and resistance values is clearly a very inefficient way of processing the data, in the present case using only 10^4 out of a possible maximum of $n_R n_S = 10^9$ combinations of load and strength.

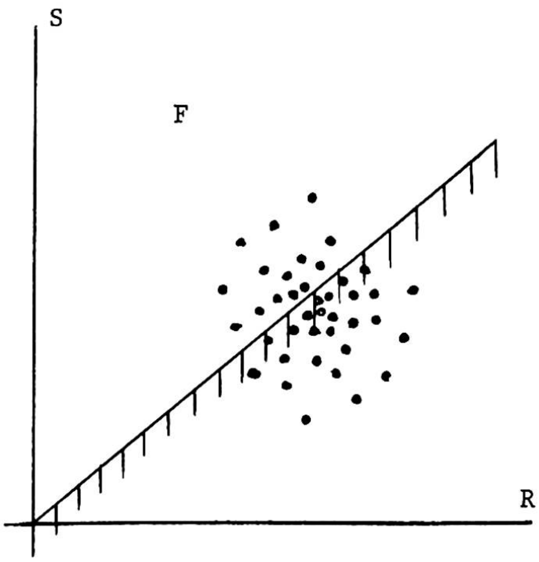

Fig. 2

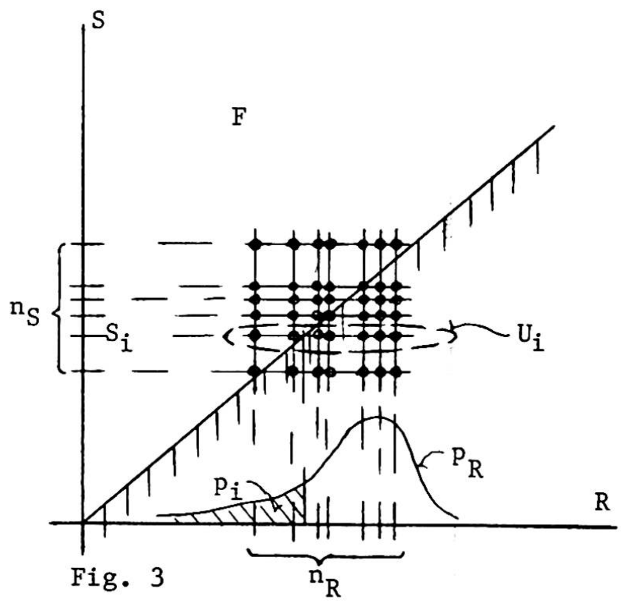

Fig. 3

Fig. 2 illustrates a sample consisting of a total of $n_R n_S$ pairs obtained by independent random sampling. Fig. 3 shows all the (R,S)- pairs that can be formed from the data. The ordering of the pairs in this figure suggests a stochastic dependence which, according to the sign of the correlation between sample elements, may either increase or decrease the variance of the estimated probability of failure in comparison with independent random sampling using the same sample size. Nevertheless, the relative failure frequency, C_F, in the sample is an unbiased estimator of the probability of failure,

$$m(C_F) = p_F, \tag{6}$$

since every sample element was obtained by random sampling. To compute the variance, consider a sub-sample U_i (Fig. 3) consisting of n_R pairs formed by one of the load values, S_i, paired with all the resistance values $R_1, \ldots, R_{n_R}$. A conditional probability of failure at this load level, p_i, may be associated with the sub-sample:

$$p_i = \int_0^{S_i} p_R(R)dR. \tag{7}$$

As before, the elements of the sub-sample constitute a sequence of n_R independent random Bernoulli trials. The number of failure events, n_i, at load level S_i is therefore binomially distributed with mean $n_R p_i$ and variance $n_R p_i(1-p_i)$. However, it is also observed that the n_S sub-samples constitute a sequence of independent random samples, for the n_R resistance values may be considered to be drawn a priori, thereby dividing the load range into $n_R + 1$ intervals establishing for each interval an associated probability that a load value will fall in the interval. As the loads are drawn independently and at random, the outcomes $n_i (i = 1, \ldots, n_S)$ are stochastically independent. Accordingly, the estimator

$$C_F = \sum_{i=1}^{n_S} \frac{1}{n_R n_S} n_i, \tag{8}$$

has the mean value

$$m(C_F) = \frac{1}{n_R n_S} \sum_{i=1}^{n_S} n_R p_i \tag{9}$$

and the variance

$$\sigma^2(C_F) = \frac{1}{(n_R n_S)^2} \sum_{i=1}^{n_S} n_R p_i (1-p_i). \tag{10}$$

Neglecting p_i in comparison with unity for all $i = 1, \ldots, n_S$, eliminating $m(C_F)$ from equations (7) and (9), and inserting the result into equation (10) gives for the estimator C_F the coefficient of variation

$$V(C_F) = \sigma(C_F)/m(C_F) \approx 1/\sqrt{n_R n_S P_F} \tag{11}$$

Thus, as a good approximation, the coefficient of variation of C_F has the same value as if all $n_R n_S$ sample pairs had been obtained by independent random sampling. We may therefore use equations (4) with $n = n_R n_S$ to determine the confidence limits for the probability of failure. To illustrate, let $n_S = n_R = 100$, yielding 10^4 (R, S)-pairs, and assume that 16 of these pairs represent failures. This data yields the same confidence interval as found above, equation (5). The calculated probability of failure, $C_F = n_F/n_R n_S$ according to Fig. 3, is believed to utilize the data in the most efficient way possible.

The amount of data required for a specified confidence coefficient α, a target "design" probability of failure P_F, and a specified maximum width βP_F of the confidence interval (symmetric about P_F) is easily computed from equation (4) to be

$$n_R n_S > [2N^{-1}(\alpha)/\beta]^2/P_F \tag{12}$$

For example, assume that we seek to design the structure so that the probability of failure "with 90 per cent confidence" ($\alpha = 0.9$) is a number between 10^{-3} and 10^{-4}. We select the target probability of failure $P_F = 5.5 \times 10^{-4}$ and choose $\beta = 9/5.5$ in order that the confidence limits $(1 \pm \beta)P_F$ coincide with the specified limits $p_F = 10^{-3}$ and $p_F = 10^{-4}$. Equation (12) gives the result that the product $n_R n_S$ must be greater than 7,500. For example, n_R must be greater than 150 is n_S equals 50. Alternatively, if we demand that the probability of failure equals $10^{-6} \pm 5\%$, with 95% confidence, the required amount of data is increased to $n_R n_S > 1.5 \times 10^9 = (50{,}000)(30{,}000)$.

While the specific case studied here is greatly idealized, it serves to give an idea of the amount of data required in probabilistic design, unless one is content with giving merely a subjective meaning to the term 'probability of failure'. The value of $n_S n_R$ according to equation 12 may be taken as a rough lower bound for the data required to make an objective statement about the probability of failure in the form of a confidence interval. The amount of data that, as a practical possibility, can be collected does not seem out of proportion to the amount required in probabilistic design, assuming that reasonable standards of precision are prescribed.

References

1. R.v Mises: Probability, Statistics and Truth. William Hodge and Co., London (1939).
2. R. Carnap: Logical Foundations of Probability. University of Chicago Press (1950).
3. J.R. Benjamin: Probabilistic Models for Seismic Force Design, Proc. ASCE, Vol. 94, No. ST5 (May 1968), pp. 1175 - 1196.
4. A. Hald: Statistical Theory with Engineering Applications. John Wiley and Sons, Inc., New York (1952).

Summary

Statistical considerations must be used to supplement purely probabilistic considerations in structural reliability studies if concepts such as the probability of failure are to have more than a mere subjective meaning. In this contribution, the amount of data required to make confidence interval statements about the probability of failure is estimated by the methods of mathematical statistics.

Résumé

Nous voulons ajouter des considérations statistiques aux considérations probabilistiques des études de sécurité dans le domaine de la construction, afin d'élever ces dernières au-dessus du niveau purement subjectif. Dans cette étude, nous proposons, à l'aide des méthodes de statistiques mathématiques, d'évaluer la quantité requise de données pour établir les intervalles de confiance autour de la probabilité de ruine.

Zusammenfassung

Ueber rein wahrscheinlichkeitstheoretische Ueberlegungen hinausgehende statistische Betrachtungen sind für die Studien der Sicherheitskriterien im Hochbau erforderlich, falls Begriffe wie "Bruchwahrscheinlichkeit" usw. mehr als mit bloss subjektiver Bedeutung belegt sein sollen. In der vorliegenden Arbeit wird aufgrund eines speziellen Modells eine Abschätzung für den Bedarf an Datenmaterial vorgenommen, um Konfidenzgrenzen für die berechnete Bruchwahrscheinlichkeit angeben zu können.