Optimal design of reinforced concrete beams and frames

Optimal Design of Reinforced Concrete Beams and Frames

Dimensionnement optimal des poutres et portiques en béton armé

Optimale Bemessung der Stahlbetonbalken und -rahmen

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The following optimization problem is solved in the paper: design a structure of given geometry for maximum efficiency vs. elastic design, so that under any possible load combination certain specified minimum load factors be guaranteed against both the collapse of the structure and the first yield of its critical sections.

By linearizing the merit function and developing a method to generate all limit equilibrium constraints the problem is solved with the help of linear programming and computer techniques.

The principles involved and corresponding optimal solutions are illustrated by the examples of a reinforced concrete continuous beam and frame.

INTRODUCTION

In the last few years, limit design (as opposed to limit analysis) methods have been developed enabling limit equilibrium and serviceability conditions to be explicitly used in the mathematical formulation of the problem.

Such methods, called "serviceability" (as opposed to "compatibility") methods, have been extensively applied to reinforced concrete continuous beams [1], [2], [3] and [4]*. More recently the validity of these methods has been

* Figures within brackets refer to the list of references at the end of the paper.

investigated for a variety of basic assumptions and design criteria [5] and their extension was developed for frame design [6], [7]. A comprehensive discussion of the features of serviceability vs. compatibility methods has been presented by reference to their use in building structures [8].

The possibility of considering a variety of design criteria was earlier recognized [1], [5] and suggestions for optimizing relevant merit functions have subsequently been given [6], [7].

The problem of optimal design for reinforced concrete frames has been formulated in [6], but two major difficulties have been recognized in the actual solution of the problem: 1) the explicit expression of suitable merit functions and 2) the formulation of the limit equilibrium constraints for all possible modes of structural collapse.

Some views on the first problem are offered in papers by MASSONNET and SAVE [9] and by ANDERHEGGEN and THÜRLIMANN [10], in which the total cost or the steel volume are suggested as merit functions, respectively. The authors tested, with favorable results, the use of an "efficiency index" (defined as the ratio of the steel consumption by limit vs. elastic designs, $v = V_0/V_E$) as merit function [7].

The second problem has been amply investigated [11] and a systematic procedure has been developed for generating all possible modes of collapse and predicting the most critical combinations [12].

As a result, by using mathematical programming techniques, it is now possible, and this paper illustrates how, to produce direct designs of reinforced concrete beams and frames such that 1) optimum, 2) limit equilibrium and 3) serviceability criteria be satisfied simultaneously.

It should be noted that an optimal solution verifying the three conditions above would still have to be checked for compatibility. While it is possible to add the compatibility conditions to those already considered and to attempt the solution of the more general programming problem of optimization with compatibility constraints, this will be left for a separate investigation.

THE OPTIMAL DESIGN PROBLEM

Consider a reinforced concrete beam or frame with given geometry and moments of inertia, to resist a system of known loads varying between prescribed limits.

It will be assumed that:

1) Reinforced concrete can be idealized as an elasto-plastic material with limited ductility;

2) All possible loading conditions are considered;

3) Limit equilibrium and serviceability are basic design criteria, with compatibility to be separately investigated;

4) The optimum criterion is to accomplish the minimum volume of longitudinal reinforcement.

It will also be assumed that bending action prevails, that shear and axial forces are negligible and that inelastic rotations are concentrated at critical sections as in the simple plastic theory.

Let $j = 1, 2, \ldots s$ be the critical sections of the structure and M_{pj}^{+}, M_{pj}^{-} its corresponding positive and negative plastic moments, as in general, reinforced concrete sections may have unequal flexural resistance in the two bending directions.

A design solution is found when a set of M_{pj}^{+} and M_{pj}^{-} is derived for all critical sections of the structure such that the following conditions are satisfied:

1) Optimum: the design will provide the minimum volume of flexural reinforcing steel.

2) Limit Equilibrium: the structure will resist any loading combination of an intensity less than the prescribed ultimate load W_u and may collapse plastically for any load $W^{+} \geq W_u$.

3) Serviceability: the critical sections of the structure will remain well within the elastic range for any combination of working loads and hence will have a safety against yield not less than a prescribed minimum value, λ_1. It can be shown that provision of adequate yield safety will ensure satisfactory serviceability, i.e. acceptable cracking, stresses and deflections.

Let W_u, W and λ_o be the ultimate load, the service load and the overall load factor, respectively, when proportional loading is assumed, i.e. $W_u = \lambda_o W$. Let $M_{pj}^{\pm}$ be the maximum (minimum) elastic envelope moment at section j, and denote λ_{1j} as the yield load factor of section j, i.e. a plastic hinge will occur at this section under some particular scheme of loading at a load level $\lambda_{1j} W$.

It can be shown [4] that the design plastic moment for section j is proportional to the corresponding elastic envelope moment for the ultimate load:

$$M_{pj} = x_j \lambda_o M_j \tag{1}$$

The scale factor x_j is called the yield safety parameter [3] and is defined by the ratio of yield to ultimate load factors:

$$x_j = \lambda_{1j}/\lambda_o \tag{2}$$

It is noted that high x_j values correspond to superior serviceability and low x_j values correspond to superior economy. An optimal solution will determine x_j values that accomplish the largest overall steel savings consistent with adequate serviceability. This is taken to mean that for any section, under any loading condition, the yield load factor λ_{1j} is not smaller than a specified lower bound $\lambda_1 \geq 1$ (the case $\lambda_1 = 1$ corresponds to the formation of a plastic hinge at service loads). For a section to become a plastic hinge in some possible mechanism it is necessary that $\lambda_{1j} \leq \lambda_o$. Therefore the serviceability constraints become:

$$\lambda_1/\lambda_o \leq x_j \leq 1 \qquad (j = 1, 2, \ldots s) \tag{3}$$

A limit equilibrium condition is associated with each possible mode of plastic collapse (i) of the structure. This expresses the relationship between the energy dissipated in the plastic hinge rotations U_i and the external work E_i of the ultimate load in displacements corresponding to the mode of collapse i:

$$U_i \geq E_i \tag{4}$$

Relation (4) indicates that the structure collapses in the i^{th} mechanism at either the prescribed load factor λ_o if $U_i = E_i$ or at a larger load factor $\lambda^+ > \lambda_o$ if $U_i > E_i$.

Assuming p possible mechanisms exist and expressing U_i in terms of the plastic moments (1) and E_i in terms of the specified ultimate load, the equilibrium constraints become after simplifications:

$$\sum_{j=1}^{s} a_{ij}\, x_j \geq c_i \qquad (i = 1, 2, \ldots p) \tag{5}$$

where a_{ij} and c_i are some non-negative constants.

As the optimal solution corresponds by definition to the minimum steel volume V, the merit function is:

$$V = \int \frac{M_{pj}\, ds}{K_j} = \int \frac{\lambda_o}{K_j} M_j\, x_j\, ds = \min ! \tag{6}$$

where the integral is extended over the entire length of the structure and K_j is a constant depending on the section geometry and materials properties.

In summary the optimal design problem consists of determining the x_j values for s critical sections of the structure so that the optimum criterion (6), the limit equilibrium constraints (5) and the serviceability constraints (3) be satisfied.

This is easily identified as a typical programming problem to which the standard algorithms of mathematical programming are applicable.

MERIT FUNCTION, CONSTRAINTS AND OPTIMAL SOLUTION

Merit Function. The concrete sizes being fixed, the amount of steel reinforcement on which the design plastic moments depend is the only variable in the feasible design solutions. In current detailing practice the reinforcement provided at critical sections is maintained constant over certain lengths of the members before it is bent or terminated to conform with bond, anchorage and/or shear requirements. As a result the resisting moments provided follow a stepwise diagram, which can be idealized by a set of rectangles of depth M_{pj} and of length ℓ_j, the distance over which M_{pj} is maintained constant [8], [10].

With these assumptions and notations the steel volume associated with each plastic moment can be expressed as:

$$V_j = A_{sj}\, \ell_j = M_{pj}\, \ell_j / K_j \tag{7}$$

Therefore the merit function (6) expressing the total steel volume is linearized in the form:

$$V = \sum_j M_{pj}\, \ell_j / \sqrt[3]{\gamma_j} = \sum_j x_j\, \lambda_o\, M_j\, \ell_j / \sqrt[3]{\gamma_j} \tag{8}$$

where γ_j are the known ratios of moments of inertia for sections j to a reference moment of inertia for the structure.

Constraints. A major difficulty of the plastic analysis and design of highly indeterminate frames consists of writing explicitly the equilibrium conditions (5) representing the limit equilibrium constraints of the optimization problem. In the paper this is done by an extension of the theory of "combination of mechanisms" due to Neal and Symonds [13], which is fully developed elsewhere [11], [12].

The procedure is based on the fact that in the derivation of the optimal solution only a limited number, m, of equilibrium conditions (collapse modes) need be considered, which is called the critical or active set of equilibrium constraints.

The active set of constraints is identified by the fact that 1) it contains $m = s - n$ collapse mechanisms, (n is the degree of statical indeterminacy of the structure) 2) which are linearly independent, 3) have only one degree of freedom, and 4) correspond to the lowest load factors of any possible mechanism. A computer programme which generates automatically all relevant combinations of mechanisms (COMECH) and which identifies the critical set of constraints based on the above mentioned criteria has been developed [11], [12]. A flow-diagram of the COMECH programme is shown in Fig. 1.

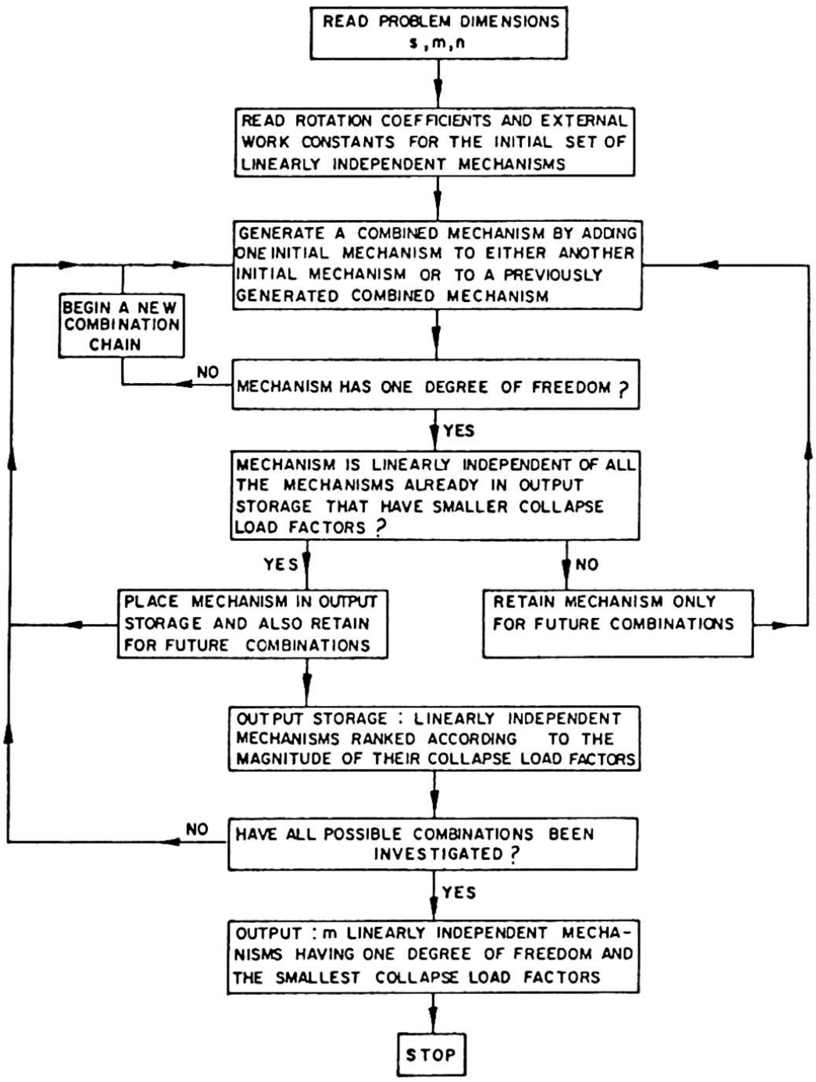


FIG. 1

Optimal Solution. With the linear merit function (8), the serviceability constraints (5) and the equilibrium constraints determined by COMECH, the simplex algorithm can be used to solve the resulting linear programming problem of optimal design [14]. A standard computer programme for the simplex algorithm is used to solve this problem.

The coefficients of the variables, the external work constants and the bounds on the variables are given as input and the following data are obtained as output:

1) The optimal design plastic moments for all critical sections;
2) The efficiency index of the optimal design, v, and
3) The effective ultimate safety of the active collapse modes.

EXAMPLES

Example 1. Given is the five-span continuous beam with the geometry and loading in Fig. 2a. Live loads are 2.5 times higher than the dead loads and can be applied to any or all of the spans. Dead and live load factors of 1.5 and 1.8 respectively are assumed, conforming to the American practice. An overall load factor $\lambda_o = 1.715$ is implied. A minimum yield load factor $\lambda_1 = 1.2$ is specified and therefore the minimum x_j value permitted is $x_j = \lambda_1/\lambda_o = 1.2/1.715 = 0.7$.

With the conventional arrangement of the reinforcement as in Fig. 2b, the elastic moment envelope coefficients in Fig. 2c and assuming $K_j = K =$ const., the merit function is given by expression (9).

All possible modes of collapse are indicated in Fig. 2d and are labelled by (a), (b) and (c). The corresponding limit equilibrium conditions are given by expressions (10).

The serviceability constraints are given by expression (11).

Therefore the problem statement is:

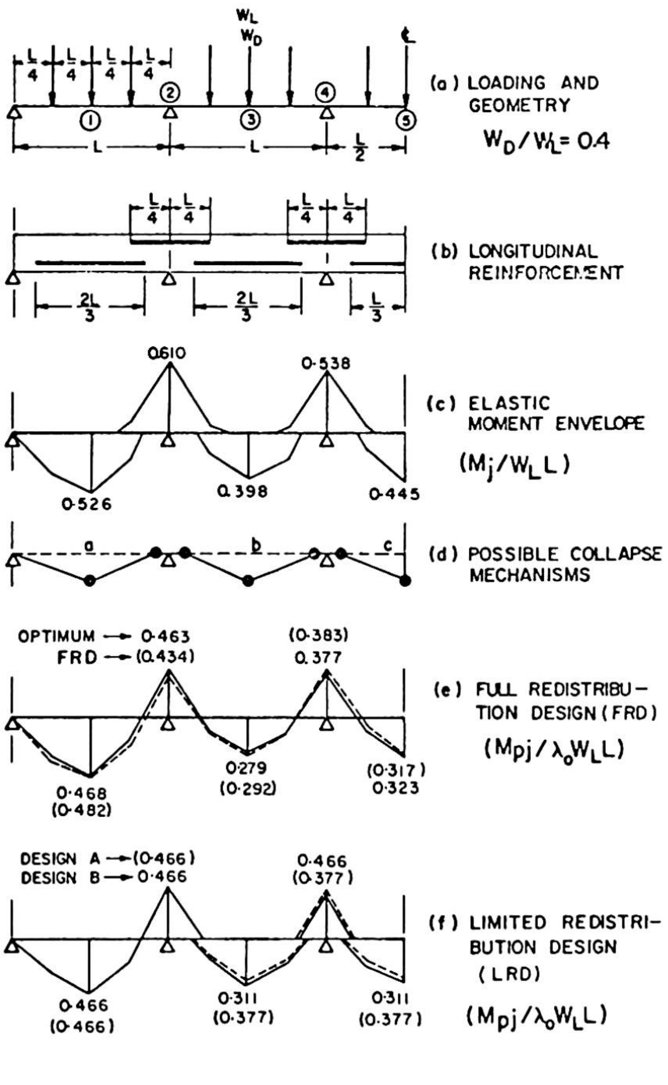


FIG. 2

minimize:

$$KV = \frac{2}{3} 0.526\, x_1 + \frac{1}{2} 0.610\, x_2 + \frac{2}{3} 0.398\, x_3 + \frac{1}{2} 0.538\, x_4 + \frac{1}{3} 0.445\, x_5 \quad (9)$$

subject to:

$$1.052\, x_1 + 0.610\, x_2 \geq 1.4 \quad (10a)$$

$$0.610\, x_2 + 0.796\, x_3 + 0.538\, x_4 \geq 1.4 \quad (10b)$$

$$1.076\, x_4 + 0.890\, x_5 \geq 1.4 \quad (10c)$$

and:

$$0.7 \leq x_j \leq 1 \quad (j = 1, 2, \dots 5) \quad (11)$$

The set of x_j values corresponding to the optimal design is given in the last column of Table 1.

It is to be noted that the optimal solution corresponds to conditions (10) becoming equalities, i.e. to mechanisms (a), (b) or (c) occuring at the prescribed load factor $\lambda_o = 1.715$. Therefore this is a full redistribution design. The corresponding bending moment diagram is illustrated by the full lines in Fig. 2e.

For the sake of comparison the elastic solution and three additional limit design solutions are provided as follows:

- a full redistribution design (FRD) based on [3], column 5 in Table 1 and bending moment diagram in dotted lines, Fig. 2e.
- two limited redistribution designs (LRD) labelled A and B respectively, with equal design moments at a number of sections for convenience of steel placing. These solutions are given in Table 1, columns 3 and 4, and the corresponding bending moments are illustrated in Fig. 2f.

TABLE 1

EXAMPLE 1: DESIGN SOLUTIONS

$x_j = M_{pj}/\lambda_o M_j$

Section (j)	ELASTIC DESIGN	LRD:A	LRD:B	FRD	OPTIMAL DESIGN
1	1.000	0.887	0.887	0.916	0.889
2	1.000	0.763	0.763	0.712	0.760
3	1.000	0.946	0.781	0.732	0.700
4	1.000	0.700	0.866	0.712	0.700
5	1.000	0.848	0.700	0.712	0.726

Lower serviceability limit $x_j = 0.700$

In Table 2 a summary is given of the effective safety against collapse for the various designs. If λ_i' is the actual load factor for a particular design and collapse mode i the ratio λ_i'/λ_o is indicative of the relative conservatism of various solutions.

In the same table the efficiency index is given for all solutions studied. As expected, it is noted that the larger is the relative safety against collapse the less economical is the design. The optimal solution is evidently the most efficient while providing exactly the required safety in all possible modes of failure. Note that the full redistribution design (FRD) is very close to the efficiency of the optimal solution.

TABLE 2

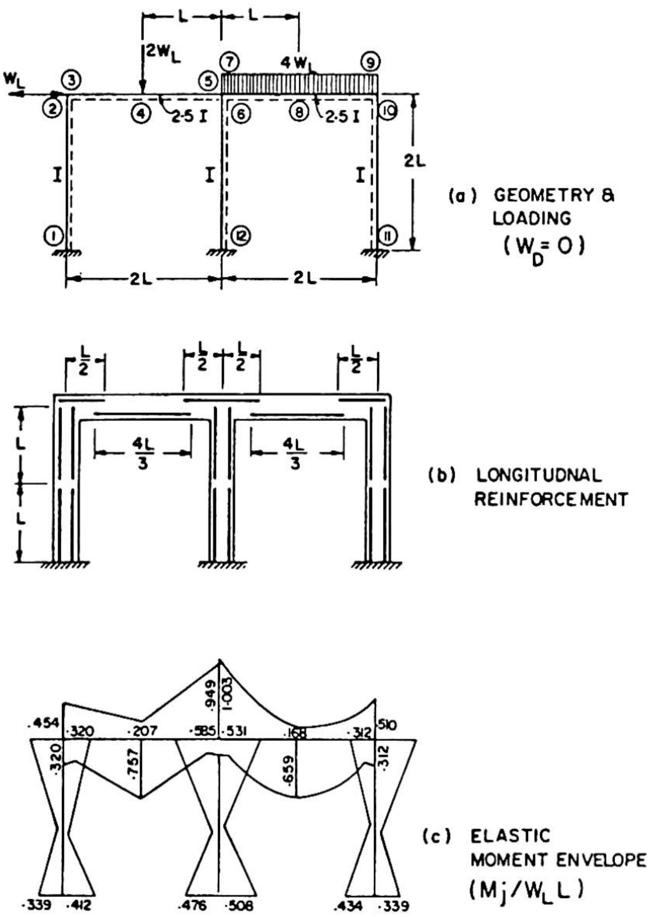
EXAMPLE 1: STRUCTURAL SAFETY AND EFFICIENCY

λ_i'/λ_o

Mechanism (i)	ELASTIC DESIGN	LRD:A	LRD:B	FRD	OPTIMAL DESIGN
a	1.189	1.000	1.000	1.000	1.000
b	1.389	1.142	1.111	1.000	1.000
c	1.405	1.075	1.111	1.000	1.000
$v = V_O/V_E$	1.000	0.828	0.813	0.767	0.766

Example 2. The frame in Fig. 3a [13] is to be designed for any possible combination of the applied live loads, assuming zero dead load. With an overall load factor $\lambda_o = 1.8$ and a minimum yield load factor $\lambda_1 = 1.2$, the minimum permissible value of $x_j = 1.2/1.8 = 0.667$.

The reinforcing details are provided in Fig. 3b and the elastic moment envelope is indicated in Fig. 3c. A merit function of the form (8) is obtained, which is not reproduced here for the sake of brevity.



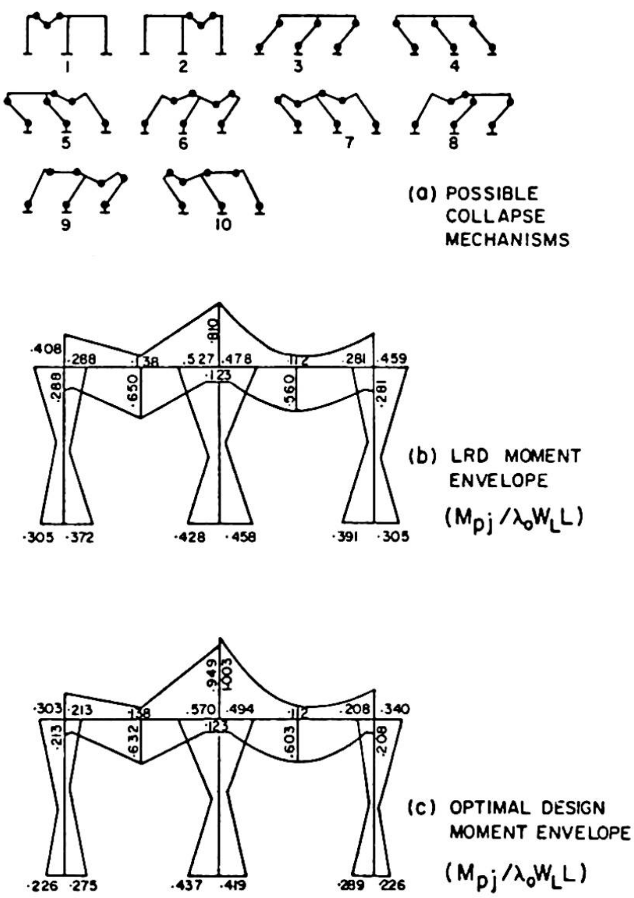


FIG. 3 FIG. 4

Of all the potential collapse modes, the COMECH programme [11] identifies the 10 mechanisms in Fig. 4a, which correspond to the critical or active set of limit equilibrium constraints.

By using the simplex computer programme the optimum solution is found and the corresponding $x_j^{\pm}$ values are listed in the last columns of Table 3. Also listed are the elastic solution ($x_j^{\pm} = 1 = \text{const.}$) and a limited redistribution design based on [7].

The B.M. diagrams associated with these designs are represented in Figs. 4b and 4c, which are obtained by scaling down the bending moments in Fig. 3c with the corresponding x_j values in Table 3.

TABLE 3

EXAMPLE 2: DESIGN SOLUTIONS

$x_j = M_{pj}/\lambda_o M_j$

Section (j)	ELASTIC DESIGN x_j^+	ELASTIC DESIGN x_j^-	LIM. RED. DESIGN LRD x_j^+	LIM. RED. DESIGN LRD x_j^-	OPTIMAL DESIGN x_j^+	OPTIMAL DESIGN x_j^-
1	1.000	1.000	0.900	0.900	0.667	0.667
2	1.000	1.000	0.900	0.900	0.667	0.667
3	1.000	1.000	0.900	0.900	0.667	0.667
4	1.000	1.000	0.858	0.667	0.835	0.667
5	1.000	1.000	0.667	0.853	0.667	1.000
6	1.000	1.000	0.900	0.900	0.931	0.975
7	1.000	1.000	0.667	0.807	0.667	1.000
8	1.000	1.000	0.850	0.667	0.915	0.667
9	1.000	1.000	0.900	0.900	0.667	0.667
10	1.000	1.000	0.900	0.900	0.667	0.667
11	1.000	1.000	0.900	0.900	0.667	0.667
12	1.000	1.000	0.900	0.900	0.824	0.919

Lower serviceability limit $x_j = 0.667$

The effective collapse safety of the elastic, LRD and optimal solutions is indicated in Table 4, along with the corresponding efficiency indices. It should be noted that the optimal design enables 8 modes of collapse at the prescribed ultimate load with an overall steel reduction of about 20% vs. the elastic solution. The limited redistribution design generates only 2 modes of failure at $\lambda_o W$ with a steel saving of 14% vs. the elastic design.

TABLE 4

EXAMPLE 2: STRUCTURAL SAFETY AND EFFICIENCY

λ_i'/λ_o

Mechanism (i)	ELASTIC DESIGN	LRD	OPTIMAL DESIGN
1	1.46	1.26	1.26
2	1.42	1.20	1.28
3	1.31	1.18	1.00
4	1.31	1.18	1.00
5	1.16	1.00	1.00
6	1.17	1.01	1.00
7	1.17	1.00	1.00
8	1.19	1.04	1.00
9	1.21	1.06	1.00
10	1.25	1.09	1.00
$v = V_0/V_E$	1.000	0.860	0.803

CONCLUSIONS

Optimal solutions can be derived for reinforced concrete beams and frames for minimum steel consumption, with adequate safety against both the structural collapse of structures and the first yield of their critical sections.

With the assumptions adopted in the paper the optimal design becomes a linear programming problem, which can be solved by using digital computers.

Examples of optimal design presented indicate savings of 20 - 23% in steel consumption vs. the elastic solutions based on the ultimate strength design for the sections.

While the techniques described are straightforward when applied using a digital computer, they appear prohibitive for hand calculation in design offices.

However standard optimal solutions may be computed and tabulated for typical beams in the same way as in [4].

Data in Tables 2 and 4 confirm that the full redistribution design (FRD) is nearly as efficient as the optimal design, a result which has been anticipated in some previous studies [7], [8]. Because of this feature and of the relative simplicity of serviceability methods it appears that approaching full redistribution is a realistic and practical objective in the limit design of reinforced concrete frames.

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NOTATION

a_{ij}, c_i	constants defining the internal and external work in the limit equilibrium equations
A_{sj}	flexural reinforcement of section j
i	index referring to the mode of collapse (mechanism)
j	index referring to critical sections of the structure
K_j	a constant which depends on the section geometry and materials properties and defines the flexural reinforcement of section j.
ℓ_j	total distance over which M_{pj} prevails
m	number of independent mechanisms
M_j	elastic moment envelope value at section j
$M_{pj}^{\pm}$	design plastic moments at section j
n	degree of statical indeterminacy of the structure
p	total number of possible collapse modes (mechanisms)
s	total number of critical sections of the structure
V_E	steel volume required by elastic design
V_O	steel volume required by optimal design
$v = V_o/V_E$	efficiency index of the structural design

W	service (dead + live) loads
W_D, W_L	dead and live service loads, respectively
$W_u = \lambda_o W$	specified ultimate load
λ_o	specified ultimate load factor
λ_1	specified yield load factor
λ_i'	effective ultimate load factor in the i^{th} collapse mode
λ_{1j}	minimum effective yield load factor of section j
$x_j = \lambda_{1j}/\lambda_o$	yield safety parameter of section j

SUMMARY

Optimal solutions can be derived for reinforced concrete beams and frames for minimum steel consumption, with adequate safety against both the structural collapse of structures and the first yield of their critical sections. With the assumptions adopted in the paper the optimal design becomes a linear programming problem, which can be solved by using digital computers.

While the techniques described are straightforward when applied using a digital computer, they appear prohibitive for hand calculation in design offices. However standard optimal solutions may be computed and tabulated for typical beams in the same way as in [4].

RÉSUMÉ

Il est possible d'arriver à un dimensionnement optimal des poutres et portiques en béton armé pour un minimum d'armature, avec sécurité adéquate contre la ruine totale de la structure et la première rupture dans une section critique. L'économie contre la méthode élastique est de l'ordre de 20-23 %.

La technique de calcul décrite est directe pour une calculatrice électronique, main plutôt difficile pour le calcul manuel. Cependant des solutions-standard pourraient être préparées en tabelles pour des poutres typiques, comme dans [3]. D'ailleurs une méthode approchée (table 4) donne d'assez bons résultats.

ZUSAMMENFASSUNG

Die optimale Bemessung der Stahlbetonbalken und -rahmen ist bei gleicher Sicherheit gegen Traglast sowie erstem plastischen Gelenk der gefährdeten Querschnitte für ein Minimum an Bewehrung möglich. Die Ersparnis gegenüber der elastischen Verfahren beträgt 20-23 %.

Während die beschriebene Methode auf Digital-Computern einfach anzuwenden ist, erscheint sie für die Handrechnung nicht empfehlenswert. Wie auch immer, die standardisierten Optimumslösungen mögen in der gleichen Weise wie in [3] berechnet und tabelliert werden. Im übrigen ergibt ein Näherungsverfahren (Tafel 4) hinreichend genaue Ergebnisse.