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Post-Critical Behavior

Comportement post-critique

Überkritisches Verhalten

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1. INTRODUCTION

The problem of post-critical behavior of structural elements and structures is not new. The load-carrying capacity of structures in the post-critical range has been attracting attention of structural engineers for many years, and it has been successfully utilized in many practical designs. The analysis of the post-critical behavior of structures has been an interesting and challenging task of applied mechanics; even very early works in the theories of bars, plates, and shells contain investigations of buckling and post-buckling states.

Thin-walled structures represent the area in which the post-critical behavior is undoubtedly of greatest importance. Numerous significant contributions on various aspects of thin-walled structures were presented at the previous congresses of IABSE. Most recently, the 8th Congress, held in New York in 1968, had a theme dealing exclusively with thin-walled structures [1]. The reports by Prof. Winter [2], Dr. Scalzi [3], and Prof. Massonnett [4] give very extensive and enlightening accounts of the status up to 1968. The discussion in the Final Report contains many original theoretical and practical developments. The comments by Prof. Beer [5] emphasize the great potential of structures working in the post-critical range.

Consistent with the spirit of Theme I of the present Congress, this report will concentrate on the progress of the general theory, necessarily nonlinear, of the post-critical behavior of structures. The formulation of the problem, methods of solution, and recent results for various types of structures will be reviewed. An intensive effort of many researchers in the field of mechanics of solids and structural mechanics generated, especially in the past two decades, an immense amount of original and important contributions in the field of the post-critical behavior and the related nonlinear analysis of structures. This makes the writing of a report in this field an extremely difficult task, forcing certain selection of the presented topics. Here, the effects of large deformations, or geometrical nonlinearities, will be emphasized at the expense of the effects of the nonlinear material properties.

The scope of the problem of the post-critical behavior of structures can be defined by examining typical load deflection relations.

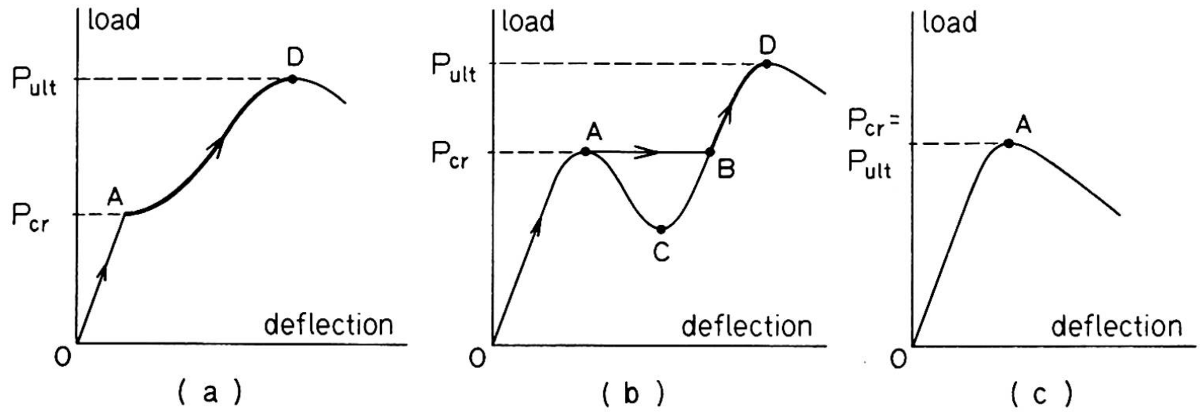

Fig. 1.

If a properly defined load parameter is plotted against a deflection component of the structure, a diagram of one of the types shown in Fig. 1 is usually observed. From the origin 0 to the point A, the structure is in the fundamental path of equilibrium. At the point A, which is referred to as the first critical point, the path changes either by bifurcation buckling to A - D, Fig. 1a, or by snap buckling to B - D, Fig. 1b. Occasionally, no state of equilibrium exists for loadings above the critical point A (Fig. 1c). The point D in Figs. 1a and 1b represents the ultimate state, at which the structure fails by fracture, total buckling, or plastic flow. It is the path between the point A and the point D in Figs. 1a and 1b which corresponds to the post-critical state of the structure, and which is the main subject of this report.

An engineer's interest in the post-critical state of a structure is based on the fact that the ultimate load exceeds sometimes considerably the critical load, and the structure can be perfectly serviceable in the post-critical range. On the other hand, his caution in utilizing the post-critical loading capacity of the structure is also well founded, since frequently the deflections increase quite rapidly, and the deformations may become irreversible. The above factors justify a thorough investigation of the problem of the post-critical behavior.

2. FORMULATION OF THE PROBLEM

The equations describing the post-critical behavior of a structure cannot be based on the assumption of small displacements and small displacement gradients. The strain-displacement relations should include at least some of the second order terms, and the analysis of stress should take into account the effect of the deformed configuration. There are many excellent books and papers on the foundations of the nonlinear mechanics of solids. The current state, as well as the historical development, can be obtained from the works of F. D. Murnaghan [6], C. Truesdell [7], V. V. Novozhilov [8], [11], A. E. Green and W. Zerna [9], T. C. Doyle and J. L. Ericksen [10], C. Truesdell and R. A. Toupin [12], A. C. Eringen [13], L. I. Sedov [14], and M. A. Biot [15].

The following comments are made in order to clarify the position of various special equations of structural mechanics within the general theory of continuous media. In the general theory, three types of coordinate systems are being used*: (a) Material (or Lagranean) coordinate system x, with x^A being the coordinates of a particle P_0 in the initial configuration of the body B_0 at the initial time t_0; (b) Spatial (Eulerian) coordinate system z, with z^k being the coordinates of the position P of the particle in the deformed, or current configuration B at time t; (c) Convected coordinate system ξ, which deforms with the body in such a way that to the coordinates ξ^α corresponding to the subsequent positions P of a particle P_0 remain constant. For small, or "infinitesimal", deformations the distinction between different types of coordinate systems disappears. The problems of finite deformation in structural mechanics are usually formulated in the fixed material system x or in the convected system ξ, although it is very seldom that the choice of the method of description is stated explicitly. (Also, in many cases, these two descriptions are formally very similar).

If $u_A(x,t)$ are the components of the displacement vector in the system x, the components of the material strain tensor e_{AB} = in this system are

$$e_{AB} = \frac{1}{2}(u_{A|B} + u_{B|A} + u_{C|A}u^C{}_{|A}) \tag{2.1}$$

The components of the same tensor in the convected system ξ are

$$e_{\alpha\beta} = \frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha} + u_{\gamma|\alpha}u^\gamma{}_{|\beta}) \tag{2.2}$$

where $u_\alpha(\xi,t)$ are the components of the displacement vector in the system ξ at P_0; the covariant differentiation in (2.2) is performed in the initial configuration, i.e. using the Christoffel symbols corresponding to the configuration of the system ξ at t_0. If at t_0 the system ξ coincides with the system x, the values

*) The subscripts and superscripts A, B, C,..., k, l, m,..., α, β, γ,... assume values 1, 2, 3; repeated indices imply summation; partial differentiation with respect to a coordinate is denoted by a comma followed by the index of the coordinate; covariant differentiation with respect to a coordinate is denoted by a bar followed by the index of the coordinate.

of the corresponding components of e_{AB} and $e_{\alpha\beta}$ are identical. Frequently, the systems x and ξ at t_0 are orthogonal cartesian; the system ξ at t, however, becomes some curvilinear system, following deformation of the body.

The description of the state of stress in continuum mechanics is most frequently accomplished in terms of: (a) The spatial, or Cauchy, stress tensor associated with the position P in the deformed state and with the components p^{kl} in the fixed system z; (b) The Kirchhoff two-point tensor t^{Al}, associated with the particle P_0 in the system x and with the position P in the system z; (c) The material, or Piola-Kirchhoff, stress tensor s^{AB}, associated with the particle P_0 in the system x. The relations between these three tensors are

$$t^{Al} = J\, p^{kl} \frac{\partial x^A}{\partial z^k}, \quad s^{AB} = J\, p^{kl} \frac{\partial x^A}{\partial z^k} \frac{\partial x^B}{\partial z^l} \tag{2.3}$$

where $J = dV/dV_0 = \rho_0/\rho$, with dV_0 and dV being the initial and the deformed volume elements, and ρ_0 and ρ, the initial and the current densities, respectively. In the convected coordinates ξ,

$$s^{\alpha\beta} = t^{\alpha\beta} = J\, p^{\alpha\beta} \tag{2.4}$$

If $\underset{\sim}{p}_{(n)}$ is the stress vector referred to unit area in the deformed state and acting on the area element whose unit normal vector in the deformed state is $\underset{\sim}{n}$, its components are

$$p^l_{(n)} = p^{kl} n_k \tag{2.5}$$

For the stress vector $\underset{\sim}{s}_{(n)}$ referred to unit area in the initial state and acting on the area element whose unit normal vector in the initial state is $\underset{\sim}{n}_0$, we have the components in the z system

$$s^l_{(n)} = t^{Al} n_{0A} = s^{AB} \frac{\partial z^l}{\partial x^B} n_{0A} \tag{2.6}$$

In the absence of body forces, the equations of equilibrium in terms of the tensors p^{kl} and s^{AB} are

$$p^{kl}{}_{|k} = 0, \quad p^{kl} = p^{lk} \tag{2.7}$$

$$[s^{AB}(\delta_B^C + u^C{}_{|B})]_{|A} = 0, \quad s^{AB} = s^{BA} \tag{2.8}$$

In the convected coordinate systems ξ, they read

$$p^{\alpha\beta}{}_{|\alpha} = 0, \quad p^{\alpha\beta} = p^{\beta\alpha} \tag{2.9}$$

$$[s^{\alpha\beta}(\zeta_\beta^\gamma + u^\gamma{}_{|\alpha})]_{|\alpha} = 0, \quad s^{\alpha\beta} = s^{\beta\alpha} \tag{2.10}$$

It should be noted that the covariant differentiation in (2.9) is performed in the deformed configuration of the system ξ, while the differentiation in (2.10) is performed in the initial configuration. (The lack of the displacement gradients in eq. (2.9) is apparent only; since the coordinate system is that of the deformed configuration, the effect of the displacement is included.)

In structural mechanics, the components $p^{\alpha\beta}$, $t^{\alpha\beta}$, $s^{\alpha\beta}$, or s^{AB} are used. With the assumption $J \approx 1$, which appears to be justified in most practical problems; and if the systems x and ξ coincide at t_0, we have

$$s^{AB} = s^{\alpha\beta} = t^{\alpha\beta} \approx p^{\alpha\beta}, \text{ for } A = \alpha,\ B = \beta. \tag{2.11}$$

The theory presented by M. A. Biot [15] differs from the above outline. Biot's theory is geared towards problems of stability and solutions of large deformations problems in small incremental steps. The acknowledged efficiency of the incremental methods of solution, makes Biot's theory an attractive tool in structural analysis.

For most structural material in the elastic range, the relation between the stress tensors (s^{AB} or $s^{\alpha\beta}$) and the strain tensors (e_{AB} or $e_{\alpha\beta}$) can be assumed in the form of Hooke's law. An exposition of the theory of plasticity for arbitrary deformation can be found in the paper by A. E. Green and Naghdi [17]. Some problems of the theory of viscoelasticity for finite deformations are presented in the paper by Oldroyd [18] and in the book by A. E. Green and J. E. Adkins [16]. A discussion of the constitutive equations of various materials is beyond the scope of this paper. Let us only point out that the use of the material or convected coordinates offers distinct advantages also in this part of the problem. In fact, most of the constitutive relations used in the analysis of inelastic structures subjected to small deformations can be adapted to the present problem without major changes. Comprehensive reviews of these relations are given in the article by A. M. Freudenthal and H. Geiringer [19] and in the book by T. H. Lin [20].

The procedure of derivation of the fundamental equations for specific types of structures follows the general ideas of the three-dimensional mechanics of solids. The strain-displacement relations, the equations of equilibrium in the deformed configuration, and the stress-strain relations must be established. As a rule, the assumption of small strains can be made; moreover, satisfactory theories can be developed by taking into account that only some of the displacements and displacement gradients are large (e.g. normal deflections of beams and plates).

A theory of moderately large deflections of plates has been proposed by Th. von Karman in 1910 [21] (see also S. Timoshenko and S. Woinowsky-Krieger [22]). It retains all the basic assumptions of the classical (linear) theory of thin plates. The expressions, however, for the extensional strain components in the plane of the plate contain the squares of the gradients of the normal deflection, i.e.,

$$e_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial^2 w}{\partial x}\right)^2, \text{ etc.} \qquad (2.12)$$

Similarly, the deformed configuration of the plate is taken into account in the equations of equilibrium which contain terms of the type

$$s_{xx} \frac{\partial^2 w}{\partial x^2}, \; s_{yy} \frac{\partial^2 w}{\partial y^2}, \text{ etc.} \qquad (2.13)$$

The resulting system of equations may be used in its original form, or it can be reduced to three equations with the three displacement components of the middle plane, or two equations for the normal deflection and a stress function can be written. The choice of the final form of the equations depends on the method of solution.

For curved panels, K. Margeurre [23] proposed a theory whose assumptions are similar to those of Kármán's plate theory. There are several versions of nonlinear theories of shells, which differ mainly in the degree of precision in which the geometry of the shell is taken into account. This situation is parallel to the variety of linear theories of shells. A discussion of this can be found in the works of L. H. Donnell [24], J. L. Synge and W. Z. Chien [25], W. Z. Chien [26], V. S. Vlasov [27], A. S. Volmir [28], Kh. M. Mushtari and K. Z. Galimov [29], J. L. Sanders,Jr. [30], P. M. Naghdi and R. P. Nordgren [31], W. T. Koiter [32].

Among the methods of solution of the nonlinear problems of post-critical behavior, the finite element method appears to be unusually versatile and effective. The basic ideas and relations of this method, including stability and large deformations, are presented in the papers by J. H. Argyris [33], and J. H. Argyris, S. Kelsey, and H. Kamel [34], and the books by O. C. Zienkiewicz [35], and J. S. Przemieniecki [36]. The papers by J. J. Turner, E. H. Dill, H. C. Martin and R. J. Melosh [37], H. C. Martin [38], R. H. Mallett and P. V. Marcal [39] concentrate on the buckling and nonlinear problems.*

For a linear elastic structure, the finite element method results in a system of linear algebraic equations of the type

$$[K]\{q\} = \{P\} \tag{2.14}$$

where $[K]$ is the stiffness matrix, $\{q\}$ is the nodal displacement vector, and $\{P\}$ is the nodal load vector. The nodal displacement vector $\{q\}$ determined from eq. (2.14) the state of stress and deformation of the structure. The stiffness matrix $[K]$ depends on the geometry of the structure, its material properties, and on the geometry of the finite element system. The vector $\{P\}$ represents the external loading on the structure.

If the nonlinear effects of large deformations are taken into

*Additional references can be found in a recent survey paper by O. C. Zienkiewicz [40].

account the systems of equations for $\{q\}$ can be written as

$$([K] + [K_g])\ \{q\} = \{P\} \tag{2.15}$$

where $[K_g]$ is sometimes referred to as the geometrical stiffness matrix. It depends on $\{q\}$; hence, the system of equations (2.15) is nonlinear. Instead of eqs. (2.15), the incremental formulation can be used. For the (small) increments of displacement $\{\Delta q\}_n$ and loading $\{\Delta P\}_n$, from the state of equilibrium $\{q\}_n$ and $\{P\}_n$, the system of equations holds

$$([K] + [K'_g]_n)\ \{\Delta q\}_n = \{\Delta P\}_n \tag{2.16}$$

where the matrix $[K'_g]_n$ is determined at the state $\{q\}_n$. The incremental formulation is especially suitable for nonlinear elastic or inelastic materials with incremental stress-strain relations (e.g. elastic-plastic solids). Then, however, also the matrix $[K]$ depends on $\{q\}$ and, its elements in eq. (2.16) have to be determined at the state $\{q\}_n$. Instead of the systems of equations (2.15) or (2.16), an energy formulation may be used in which the vector $\{q\}$ minimizes the total energy of the system. The methods of mathematical programming are then employed for the determination of $\{q\}$.

3. STABILITY

The problem of stability of a structure is usually formulated as follows. Suppose that the loadings are specified by a vector $\underset{\sim}{p}$. A state of equilibrium is a displacement vector $\underset{\sim}{u}(\underset{\sim}{x},\underset{\sim}{p})$ which satisfies the equations of equilibrium and the boundary conditions of the structure. The fundamental states of equilibrium, or the fundamental path of equilibrium, are the displacements $\underset{\sim}{u}_0(\underset{\sim}{x},\underset{\sim}{p})$ single-valued and continuously differentiable in the components of $\underset{\sim}{p}$, and such that u_0 as $\underset{\sim}{p}\to 0$. In addition to the fundamental path, a structure may have, in general, other states or paths of equilibrium. The points of intersection of different paths of equilibrium are of two types: bifurcation points and limit points. They are shown in Fig. 2 for the simple case of one load component p and one displacement component u.

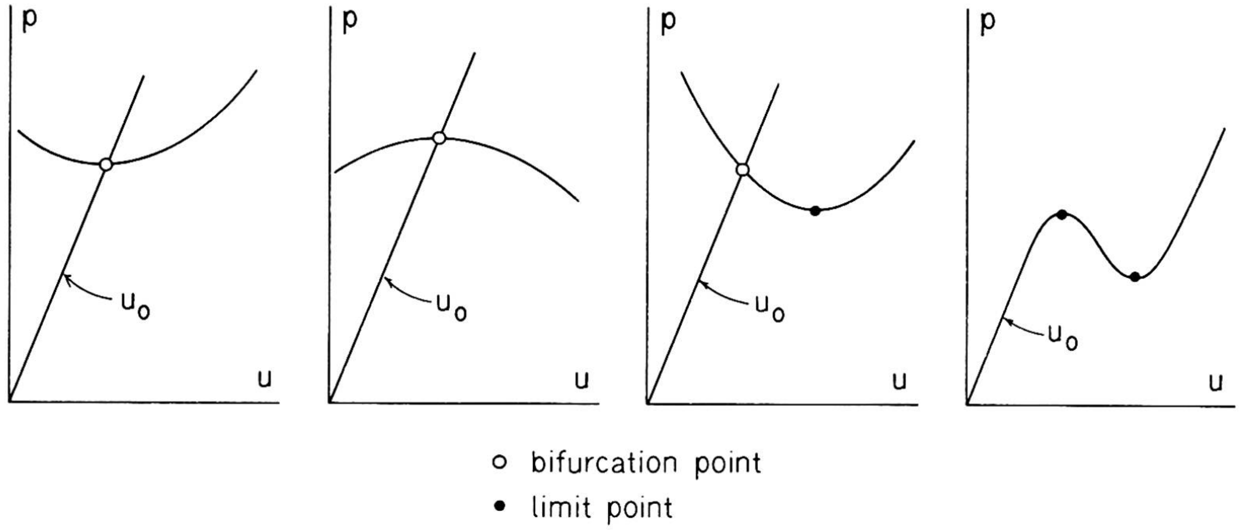

Fig. 2.

A precise and practically meaningful definition of stability is not an easy task. In engineering terms, a state of equilibrium is considered to be stable if sufficiently small perturbations cause arbitrarily small displacements of the structure. A path consisting of stable states of equilibrium is called a stable path. The states at which a path changes from stable to unstable are called critical states (or critical points); the corresponding loads are called the critical loads.

For the above notion of stability, certain criteria, or tests, of stability have been developed.

According to the static criterion, at a critical point two infinitesimally adjacent states of equilibrium exist for the same external loads. If the loads acting on the structure are proportional to a parameter, and if the fundamental state is linear elastic, the static criterion leads to the well known eigenvalue problems, for which an extensive literature is available.

The energy criterion of stability states that in any sufficiently small displacement from the state of equilibrium the internal energy stored or dissipated, ΔE, exceeds the work of the external loads, ΔW. Thus the condition of stable or neutral equilibrium is

$$\Delta E - \Delta W \geq 0 \tag{3.1}$$

In the case of an elastic structure loaded with potential forces, eq. (3.1) implies that the change of the total potential energy is positive or zero,

$$\Delta U \geq 0 \qquad (3.2)$$

The condition (3.2) represents an extension of the Dirichlet criterion of stability of discrete systems.

The dynamic criterion is based on the investigation of small free oscillations about the state of equilibrium. For a stable state of equioibrium of an elastic structure, all the natural frequencies of these oscillations are real and different from zero.

In the cases of conservative systems under conservative loadings, these three criteria lead to the same lowest critical values of the load parameter, and the existing experimental evidence confirms their basic validity; the known examples of substantial discrepancies between the theory and the experimental results can be explained by factors other than faults in the basic concepts of the theory.

The above ideas and criteria form the foundations of the stability, or buckling, analysis in a large number of problems of structural mechanics. This theory is the product of over one-hundred-year effort. Important contributions have been made by R. V. Southwell [41], S. P. Timoshenk [42], G. B. Biezeno and H. Hencky [43], E. Trefftz [44], W. T. Koiter [45], H. Ziegler [46], C. E. Pearson [47], R. Hill [48]. There is a number of excellent books dealing with the methods of analysis and practical applications. To mention some of them: S. P. Timoshenko and J. M. Gere [49], F. Bleich [50], Pflinger [51], C. F. Kollbrunner and M. Meister [52], G. Gerard [53], A. S. Volmir [54], M. Gregory [55], H. Ziegler [56].

The actual application of any of the criteria of stability is based on the equations of small displacements, or vibrations, superimposed on a state of deformation, and on the expressions for the corresponding change of energy. They are given in Ref. [16], [44], and [15] for the general case of a three-dimensional solid.

In Refs. [49] through [56], and in numerous other papers the necessary relations are given for various types of structures.

In the finite element formulation, the conditions leading to the determination of the critical loads are discussed in Refs. [33], [35], [36], [38, [39] and [57] through [61]. It is worth mentioning that in the cases of linear elastic fundamental states, the critical load parameter λ follows either from the equation

$$\det|K + \lambda K_0| = 0 \tag{3.3}$$

or from

$$\det|K(\lambda)| = 0 \tag{3.4}$$

($[K(\lambda)]$ = the stiffness matrix whose elements are known functions of λ). For nonlinear fundamental states, the equation for the critical load is

$$\det|K'(q_0(\lambda))| = 0 \tag{3.5}$$

where $[K'(q_0(\lambda))]$ is the <u>incremental</u> stiffness matrix corresponding to the fundamental state $\{q_0(\lambda)\}$ which, in turn, depends on the load parameter λ.

In spite of great achievements of the classical theory of stability, there are certain areas which require more general approaches or, at least, refinements of the existing methods. The fallacy of the static methods in the case of nonconservative systems has been discovered long ago. A comprehensive review of this question and of recent contributions has been given by G. Herrmann [62]. Another area of practical interest is the stability, or buckling, under dynamic loading. Also, it has been pointed out (R. T. Shield and A. E. Green [63], R. J. Knops and E. W. Wilkes [64]), that, in general, the uniqueness of a path of equilibrium and the energy and the dynamic criteria not necessarily assure boundedness of the displacements, velocities, and strains.

An important step in answering certain fundamental questions is the development of the theory of stability of continuous media (A. A. Movchan [68], [69], [70] and Ref. [64]) along the lines of Liapunov's general theory of stability of motion [65] . (Modern accounts of this theory can be found in the books by N. G. Chetaev

[66] and W. Hahn [67]). The major points of this work are: precise definitions of "initial perturbations", rational measures of the magnitudes of the initial perturbations and the ensuing perturbed motions, rigorous definitions of stability, and derivation of the corresponding stability criteria. It has been shown, incidentally, that the classical energy criterion corresponds to the stability, or boundedness, in the mean square value of the displacements (but not to the boundedness of the maxima of the displacements).

Further studies on the foundations of the theory of stability are presented in Refs. [71] to [83].

A relatively new and important problem of the theory of stability (and in the post-critical buckling) is the effect of initial imperfections. It has been investigated originally to explain the discrepancies between the theoretical predictions and the experimental data in buckling of shells (L. H. Donnell and C.C. Wan [83], W. H. Horton and S. C. Durham [84]); recently, it became a part of more general studies on structural stability (W. T. Koiter [71], J. M. T. Thompson [85], [87], J. Roorda [86]; also the survey paper [76] by B. Budiansky and J. W. Hutchinson). The effect of initial imperfection may be one of the following: (a) The equilibrium paths and the critical points are essentially of the same type as in the perfect structure, with the displacements and the critical loads slightly influenced by the initial imperfections. (b) The bifurcation point vanishes; the pre-

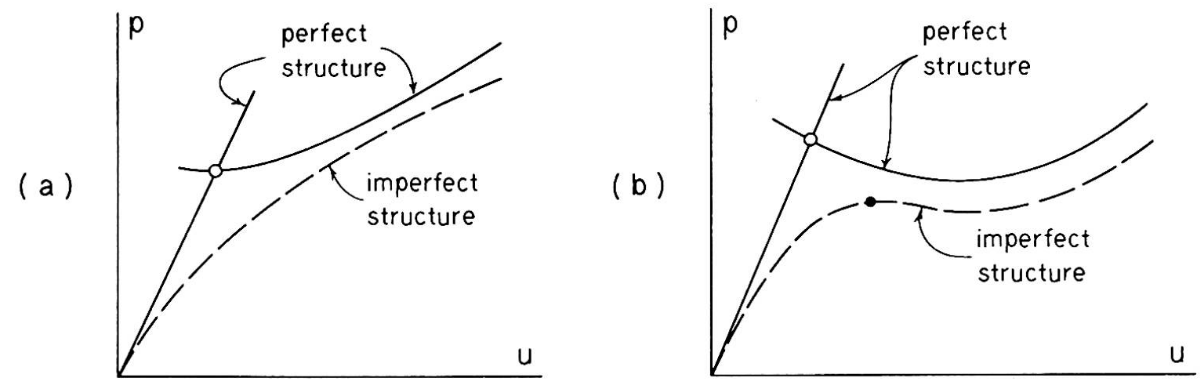

Fig. 3.

buckling and the postbuckling states form a continuous path (Fig. 3a). (c) The bifurcation point vanishes and a limit point appears, usually at much smaller magnitudes of the loadings (Fig. 3b).

The essential factor in the analysis of the effects of initial imperfections is their shape and size. It is, of course, possible to assume some unfavorable configuration of the initial imperfections and to determine the corresponding buckling load and the postbuckling behavior. Sometimes, conceivably, specific information concerning the initial imperfections in a structure may be available. Basically, however, the initial imperfections are errors of fabrication, of random magnitude and random distribution over the structure. Accordingly, the statistical methods appear to be the most rational approach. The analyses of buckling of various structures with random imperfections are presented in the works by V. V. Bolotin [88] and [89], W. E. Boyce [90], J. M. T. Thompson [91], B. Budiansky and W. B. Fraser [92], J. C. Amazigo [93], and J. Roorda [94]. There is a close relation between the statistical approach to the buckling loads and the statistical methods in structural safety.

4. METHODS OF ANALYSIS OF POST-CRITICAL STATES

The primary objective of the analysis is the determination of a stable state (or states) of equilibrium of the structure for a given system of loadings exceeding the lowest critical level. In the case of non-unique solutions, the accessibility of each state via a realistic path (history) of loading should be evaluated. The determination of the ultimate loading capacity of the structure is also an important part of the problem. Frequently, the complete path (or paths) of equilibrium from zero to the ultimate loading capacity is required.

The difficulties connected with the nonlinear equations of the post-critical behavior necessitate the use of approximate and numerical methods. The presently available solutions are usually based on one or a combination of the following methods: perturbation methods, successive approximations, Ritz' method, Galerkin's method, finite differences and finite element method. The perturbation methods and the methods of successive approximations reduce the solution of the original nonlinear differential equations to repeated solutions of linear differential equations. The Ritz, Galerkin, finite difference, and finite element methods result in a system of nonlinear algebraic equations.

There is an extensive literature on the approximate and numerical methods; see, for example, Refs. [96], [97], [98], [99], [100]. The solution of the nonlinear algebraic equations connected with some of these methods is a formidable task in itself (Refs. [101] and [102]). The numerical treatment of the eigenvalue problems, which arise in the course of this analysis, is described in Refs. [103], [104], and [105].

K. O. Friedrichs and J. J. Stoker [106] analyse a supported circular plate subjected to radial aedge compression p, and described by von Kármán's equations. For this problem, they develope and appraise three methods suitable for three ranges of the ratio P/p_{cr} (where p_{cr} is the lowest buckling load): a perturbation method for $1<p/p_{cr}<2.5$, a power series solution for $2.5<p/p_{cr}<25$, an asymptotic solution for $p/p_{cr}\to\infty$ with a perturbation method for very large values of p/p_{cr}.

W. T. Koiter ([45] and [71]) discusses the energy method for the analysis of the initial post-buckling behavior of an arbitrary elastic structure. The determination of the critical points and buckling modes is accomplished with the aid of a stability criterion. The initial post-buckling deflections in the vicinity of a bifurcation point result from the minimizing of the energy increment functional. The stability at the critical point and the stability of the post-buckling states is investigated in terms of the asymptotic expansion of the energy in the vicinity of the critical point. Koiter's work includes also the effect of initial imperfections.

An extension of the well-known Newton's method for calculating roots of algebraic equations to the differential equations of nonlinear mechanics has been presented by G. A. Thurston [107], [108].

In general, the problem of determination of stable states of postbuckling equilibrium is not an easy one. The difficulties of the analysis beyond the first bifurcation or limit points exist even if the task is reduced to a system of nonlinear algebraic equations or a discrete system is dealt with, such as in the finite element method. The works by A. H. Chilver [109], M. J. Sewell [110], and J. M. T. Thompson [111], [112] contain further research on the methods of analysis.

The difficulties in the tests of stability and in the search for critical points make the linearization of the pre-buckling states an extremely tempting step. The consequences of this linearization are discussed in a paper by A. D. Kerr and M. T. Soifer [113].

5. APPLICATIONS

Trusses and Frames

For certain types of statically indeterminate, pin-jointed plane or space trusses, the post-buckling behavior can be determined in a relatively elementary manner, with the assumptions that the bars in their pre-buckling states are linearly elastic, the compressive forces in buckled bars remain constant, and the changes of the geometry of the truss are negligible. If successive bucklings of individual bars occur at the load levels $\lambda_1, \ldots \lambda_n$, the load-deflection history is as shown in Fig. 4. E. F. Masur [114] has derived the lower and upper bounds for the ultimate loads of redundant trusses in post-critical states. There is an analogy between this phenomenon and the elastic-plastic behavior of certain frames.

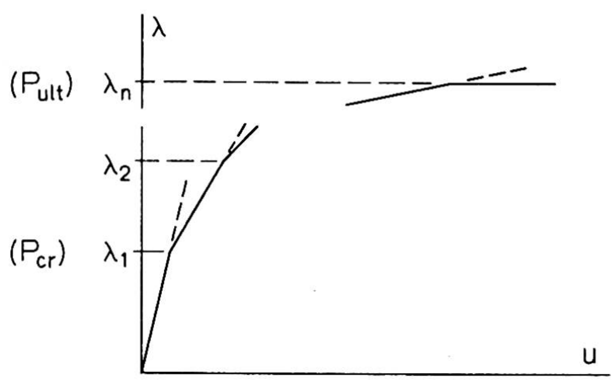

Fig. 4.

The assumptions listed above are not always satisfied; in numerous structural systems the changes of geometry influence the pre-buckling and post-buckling behavior, which become strongly nonlinear. A typical example is the truss shown in Fig. 5, with similar conditions existing in shallow reticulated shells. Large deflections cause also non-negligible secondary bend-moments; consequently, a truss must be analyzed as a frame (unless, of course, pin joints are actually constructed).

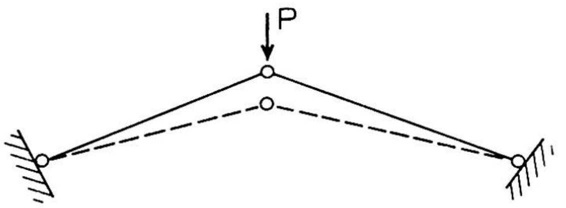

Fig. 5.

The classical works (Bleich [50]) in the area of stability of frames reduce the problem to a system of equations which are linear with respect to the joint displacements and rotations and joint forces and moments. The coefficients of these equations are known functions of the load parameter λ. While this approach has been successful in predicting the critical loads in many practical applications, it is inadequate for dealing with the post-critical behavior or for the cases when the pre-buckling behavior is non-linear.

The analysis of the post-critical behavior and the fundamental states, and the determination of the critical loading conditions of a plane or space frameworks should take into account the following effects: (a) The influence of the axial forces acting in the individual bars on their stiffness characteristics. (b) The non-linearities of the force-displacement relations in the bars (the effect of bending curvature on the relative axial displacement of the ends, large bending deformations, material nonlinearities, etc.) (c) The effect of the changes of geometry on the equations of equilibrium. An extensive discussion of the nonlinear effects in the frame behavior may be found in the works by R. K. Livesly [115], M. R. Horne [116], [117], S. J. Britvec and A.H. Chilver [118], S. A. Saafan [119], R. W. Williams [120], J. H. Argyris [33], and

R. H. Mallett and P. V. Marcal [39]; Th. V. Galambos, G. C. Driscoll, and L.-W. Lu report some related experimental research [121].

The results of the analysis which takes the above effects into account differ considerably from the stability problem of pin-joint trusses of linearized rigid-joint frames, not only quantitatively but also qualitatively. For example, the presence of primary and secondary bending moments will remove at least some of the bifurcation points in the equilibrium path (such as in Fig. 4) and the behavior up to the ultimate load may consist of the fundamental path only. The absence, however, of the bifurcation points should not be assumed in advance. (To this effect, see H. L. Schreyer and E. F. Masur [122]). Also, a nonlinear analysis may reveal the existence of snap-buckling at certain load levels (Fig. 1b), which is entirely beyond any linearized theory.

Under certain circumstances, some of the effects listed above may be disregarded. For example, J. H. Argyris [33] proposes a method of analysis which neglects the influence of axial forces on the element stiffness. Accordingly, the buckling phenomena within individual elements cannot be predicted with this theory. If, however, the elements are relatively short and stiff, and the problem is such that the buckling "waves" extend over several elements, the analysis should yield satisfactory results, with considerable simplification of the numerical work. The theories presented by R. K. Livesley [175], J. D. Renton [122], S. J. Britvec and A. H. Chilver [118], R. J. Aguilar and T.-A. Huang [124], S. J. Britvec [125], S. S. Tezcan and B. Ovunc [126], J. J. Connor, R. D. Logcher, and S.-C. Chan [127] assume flexible elements to which the linearized theory of bending with axial loading is applicable. The stiffness coefficients of these elements contain trigonometric or hyperbolic functions of the axial forces. With the equilibrium equations written in the deformed configuration, the problems of local buckling, overall buckling (including snap-buckling), and post-buckling behavior are analyzed. The systems of nonlinear equations of these theories are solved by either iterative or incremental schemes. The analyses presented by G. Lobel [128], F. W. Williams [120], C. N. Kerr [129], S. -L. Lee, F. S. Manuel, and E. C. Rossow [130] utilize the nonlinear bending theory of beams. They are capable to deal with the problems in which the post-buckling deflections are of the order of the column or girder lengths.

Some of the problems discussed in this report occur in the analysis of tall buildings and have been reviewed in the preliminary report by Professors Steinhardt and Beer [131] (8th Congress, 1968), which contains a very extensive list of references.

Plates

The papers by A. van der Neut [132], G. Winter[2], and Ch. Massonnet [4] contain comprehensive reviews of previous works on the post-critical behavior of plates. There are also two recent books on thin-walled structures (Refs. [133] and [134]), where the papers by W. J. Supple and A. H. Chilver [135], A. C. Walker [136], J. B. Dwight and A. T. Ractliffe [137], T. R. Graves Smith [138], and others, deal with plates in post-critical states.

Among most recent contributions, J. W. Dwight and K. E. Moxham [139] describe their research on welded steel plates in compression. The work reported by them is obviously a necessary step without which a full practical utilization of the post-buckling strength of plates would not be acceptable. K. R. Rushton [140] analyzes the post-critical state of tapered plates. The problem of a plate with three edges simply supported and one edge attached to a stiffener has been investigated by K. Klöppel and B. Unger [141], the analysis is based on the von Kármán equations and the energy method. The computed deflections agree very well with the results of a test program. An analysis of the post-critical behavior of thin plates, employing the finite element method has been presented by D. W. Murray and E. L. Wilson [142], [143]. The effect of creep deformation on the post-critical behavior of compressed plates has been investigated by I. M. Levi and N. J. Hoff [144].

Shells

The problems of post-critical behavior become probably most interesting and difficult in the theory of shells. They have received ample attention from many researchers, and a list of publications in this field could easily reach a few hundred positions. A review of the research on shell buckling has been made by Y. C. Fung and E. E. Sechler in 1960 [145]. The book by A. S. Volmir [54] contains numerous Russian contributions.

The load-displacement behavior of cylindrical shells under axial compression has been analyzed by Th. von Kármán and H. S. Tsien [146], W. T. Koiter [147], B. O. Almroth [148], N. J. Hoff, W. A. Madsen, and J. Mayers [149], R. L. de Neufville and J. J. Connor [150], and others (see also N. J. Hoff [151]). The load-displacement relations are of the type in Fig. 1b, with strong imperfection-sensitivity at the critical point A. The ultimate shape of the buckled shell (Fig. 6) is referred to as Yoshimura pattern [152]. (This shape, together with considerable loading capacity of a buckled cylinder, prompted K. Miura's suggestion of a shell structure resembling Yoshimura's pattern [153]).

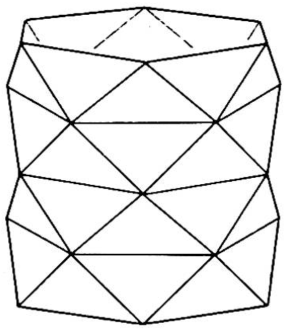
Fig. 6.

The problem of stiffened and barreled shells is analyzed by J. W. Hutchinson and J. C. Frauenthal [154].

The nonlinear buckling problem and the initial post-buckling behavior of a complete spherical shell has been analyzed by W. T. Koiter [155], who also gives a review and assessment of previous work in this area. The papers by J. R. Fitch [156] and J. R. Fitch and B. Budiansky [157] deal with a similar problem for spherical caps. Ref. [157] clarifies, in particular, the effect of the load distribution, and of the shell thickness, on the type of buckling (bifurcation or snap-through) and on the post-buckling load carrying capacity.

An analysis of the initial post-buckling behavior of the toroidal shell segments has been given by J. W. Hutchinson [158]. The paper by G. A. Greenbaum and D. C. Conroy [159] contains an example of an efficient numerical analysis of a conical shell.

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SUMMARY

This report reviews the problems related to the analysis of post-critical behavior of structures, including formulation of the basic equations, methods of their solution, and criteria of stability. Recent applications to trusses, frames, plates and shells are outlined.

RÉSUMÉ

Ce rapport traite de l'analyse du comportement post-critique des structures, avec formulation des équations fondamentales, les méthodes de leur résolution et les crytères de stabilité. On présente aussi les applications récentes aux treillis, aux cadres, aux plaques et aux voiles.

ZUSAMMENFASSUNG

Dieser Bericht bahandelt die Probleme der Berechnung des überkritischen Verhaltens von Tragwerken, einschliesslich die Formulierung von Grundgleichungen, der Methode ihrer Lösung, und der Stabilitätskriterien. Neue Anwendungen auf Stabwerke, Platten und Schalen sind beschrieben.

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