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Autor: Templeman, A.B.

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Optimality Criteria and Dual Methods in Truss Design

Critères d'optimisation et méthodes duales dans le dimensionnement de treillis

Optimierungskriterien und Dualmethoden in der Berechnung von Fachwerken

A.B. TEMPLEMAN
Department of Civil Engineering, The University
Liverpool, England

1. INTRODUCTION

In the Introductory Report of the 10th Congress of IABSE Gellatly and Dupree[1] describe the optimality criteria approach to the optimum design of large structural systems. In handling large structural systems the direct solution approach by numerical mathematical programming methods is often excessively slow and cumbersome as a result of the large numbers of variables which must be optimized. The optimality criteria approach is intended to overcome the difficulties posed by having large numbers of variables. Gellatly and Dupree consider the optimality approach to the design of structures in which element mass and stiffness are proportional. Such structures include those composed of axial force bars, membrane plates and shear panels. For this class of structures Gellatly and Dupree derive an optimality criterion, their equation (2), for the minimum weight design of a truss subject to a single displacement constraint. They then use this optimality criterion, (2), to develop a recursion relationship, (8), which allows any arbitrary set of member areas to be modified iteratively so as to eventually produce an optimal set of member sizes. The important time-saving feature of this approach is that at each iteration the existing set of member sizes is altered by applying the simple relationship (8) to each area in turn. There is no complicated numerical search involved.

Gellatly and Dupree then continue to describe a large computer program, OPTIM II, in which this optimality criterion and redesign formula is used to design structures with multiple displacement constraints (stiffness requirements) and also individual member size constraints. They point out that neither the optimality criterion itself nor the redesign formula is valid for anything other than a single displacement constraint but, despite this lack of rigour, OPTIM II still obtains remarkably good numerical results very quickly. This is not disputed here; OPTIM II is an efficient program, but its lack of rigour is perplexing and it makes it difficult to interpret and identify those occasional cases in which OPTIM II performs poorly.

The purpose of this paper is to examine a new dual formulation of optimum design problems for this class of structures. In particular the problem of how best to handle multiple constraints is examined and an interpretation of the dual problem is presented which has considerable relevance in the development of improved optimum design algorithms for large structural systems.

2. THE OPTIMUM DESIGN PROBLEM

For simplicity of notation a truss structure composed only of axial force bars is considered, being typical of the general class of structures with member stiffness proportional to member mass. The minimum weight (minimum volume) design problem can be posed as that of finding the set of member areas A_i, $i = 1, \ldots, N$, which

$$\text{Minimize } W = \sum_{i=1}^{N} L_i A_i \tag{1}$$

subject to M independent nodal displacement constraints (Gellatly and Dupree consider only a single generalised stiffness constraint),

$$g_m \equiv \sum_{i=1}^{N} \left(\frac{FUL}{E\delta}\right)_{mi} \frac{1}{A_i} \leqslant 1 \qquad m = 1, \ldots, M \tag{2}$$

and subject also to N member size constraints, one for each member

$$g_{M+i} \equiv \frac{\overline{A_i}}{A_i} \leqslant 1 \qquad i = 1, \ldots, N \tag{3}$$

In constraints (2) F and U are the member actual forces and virtual forces associated with unit displacement in the direction of the nodal constraint. δ_m is the maximum permitted displacement of a node in constraint m, m = 1, ..., M. E is the elastic modulus, and each of constraints (2) is derived from specific applied loads and virtual force systems. In constraints (3) $\bar{A}_i$ is the minimum permissible size of member i, derived either from maximum member stress limits or from fabricational considerations.

In the above formulation it is assumed that F and U are constants, hence $\bar{A}_i$ is also constant. This assumption is valid for statically determinate trusses. It is strictly invalid for indeterminate trusses, however, F, U and hence $\bar{A}_i$ do not usually alter appreciably as members sizes alter and it is common to assume them constant, obtain an altered set of member sizes in some way, update the values of F, U and $\bar{A}_i$, solve again and continue in this iterative fashion until the process converges to an optimum solution. This iterative solution technique is used by both mathematical programming and optimality criterion devotees, the essential difference between them being only the way in which the altered set of member sizes is obtained. It is assumed here that this iterative method for indeterminate structures is used and so in the above formulation F, U, L, E, δ and $\bar{A}$ are all known constants. Our problem is how best to find the optimal set of member sizes.

Recently the present author[2] has shown that there is a dual formulation of the problem expressed in relationships (1), (2) and (3). Derivation of the dual problem is accomplished by exploiting the fact that the Lagrangian function of the above problem has a saddle point as a stationarity condition. A full proof of the dual formulation is given in reference[2] and here it is merely stated as

$$\left.\begin{aligned} &\text{Maximise } V = \sum_{i=1}^{N} L_i \left\{ \sum_{m=1}^{M} \left(\frac{FU}{E\delta}\right)_{mi} \lambda_m + \frac{\overline{A_i}}{L_i} \lambda_{M+i} \right\}^{\frac{1}{2}} \\ &\text{subject to } \sum_{m=1}^{M+N} \lambda_m = 1 \\ &\qquad\qquad \lambda_m \geqslant 0 \qquad\qquad m = 1, \ldots, M+N \end{aligned}\right\} \tag{4}$$

The solution of (4) is equivalent exactly to the solution of the primal problem, (1), (2) and (3). At the solution point (minimum of W, maximum of V) the following transformation relationships hold, with superscript asterisk denoting optimal values,

$$\left.\begin{aligned} &(\text{Minimum}) \quad W^* \equiv V^{*2} \quad (\text{Maximum}) \\ &A_i^* \equiv V^* \left\{\sum_{m=1}^{M} \left(\frac{FU}{E\delta}\right)_{mi} \lambda_m^* + \frac{\overline{A_i}}{L_i} \lambda_{M+i}^*\right\}^{\frac{1}{2}} \qquad i = 1, \ldots, N \end{aligned}\right\} \tag{5}$$

The dual variables in dual problem (4) are the λ_m, $m = 1, \ldots, M + N$ and it will be noted that there is a dual variable λ_m for each of the primal constraints (2) and (3). The dual variables are therefore similar to the unknown Lagrange multipliers of the primal problem. All λ's must be non-negative; any value of $\lambda = 0$ denotes that the primal constraint to which it corresponds is inactive at the optimum. The single constraint in dual problem (4) requires that all λ's sum to unity.

3. PROBLEMS WITH ONLY DISPLACEMENT CONSTRAINTS

Gellatly and Dupree[1] consider only a single displacement constraint and their equations (2) and (8) represent an optimality criterion and a resizing formula for this problem. Their equation (2) contains a single unknown Lagrange multiplier corresponding to the single constraint. This unknown multiplier may be eliminated by substitution into the constraint which must perforce be active; consequently their resizing formula (8) contains no unknown multipliers. A major difficulty is encountered if this method is extended to multiple displacement constraints. In this case there will be M unknown Lagrange multipliers, one for each constraint, and since it is not known *a priori* which of the multiple displacement constraints are active and which are slack at the optimum it is not possible to eliminate the unknown multipliers by substitution. Consequently when a member resizing formula for multiple constraints is developed corresponding to Gellatly and Dupree's equation (8) it contains all the M unknown Lagrange multipliers. In order to use the resizing formula it is necessary to supply values to all the unknown Lagrange multipliers but there is no way of knowing what these values should be. This constitutes the major difficulty of using optimality criteria methods for multiple constraints. In order to get round this difficulty OPTIM II uses the envelope method which resizes each member according to the single constraint resize formula for each displacement constraint and then selects the largest resulting size. This process seems intuitively logical but has no theoretical rigour.

If the dual approach is examined for multiple displacement constraints only, the dual problem becomes

$$\left.\begin{aligned} &\text{Maximise } V = \sum_{i=1}^{N} L_i \left\{\sum_{m=1}^{M} \left(\frac{FU}{E\delta}\right)_{mi} \lambda_m\right\}^{\frac{1}{2}} \\ &\text{subject to } \sum_{m=1}^{M} \lambda_m = 1 \\ &\qquad\qquad \lambda_m \geqslant 0 \qquad\qquad m = 1, \ldots, M \end{aligned}\right\} \tag{6}$$

At the optimum, we have

$$\left.\begin{aligned} &(\text{Minimum}) \quad W^* = V^{*2} \quad (\text{Maximum}) \\ &A^* = V^* \left\{\sum_{m=1}^{M} \left(\frac{FU}{E\delta}\right)_{mi} \lambda_m^*\right\}^{\frac{1}{2}} \qquad i = 1, \ldots, N \end{aligned}\right\} \tag{7}$$

Problem (6) consists of maximizing V, a non-linear function of the M dual variables λ_m subject only to a single linear equality constraint and non-negativity of the dual variables. This is easily done by classical optimization methods. Once λ_m^*, m = 1, ..., M are known, relationships (7) give the minimum weight and optimal member sizes directly.

Several features of the dual problem can be noted. Firstly the number of dual variables is M, the number of displacement constraints. This means that the dimensionality of the original problem, which had N member size variables, is greatly reduced. Thus a large structure with perhaps 1000 members to be sized and 5 displacement constraints has a dual problem which consists of maximizing a non-linear function V of only 5 variables. In most large structural problems there are usually many more members than displacement constraints so the reduction in dimensionality afforded by the dual problem is of considerable advantage. Secondly, the dual problem itself is of a convenient form for rapid solution. The single linear equality constraint may be eliminated by substitution, converting the problem to one of unconstrained form with non-negativity requirements. First and second derivatives can be easily evaluated which makes solution comparatively simple. Thirdly, the result gives immediate information about which constraints in the primal problem are active and which are slack since a value of $\lambda_m = 0$ corresponds to a slack constraint. Finally the dual approach has the theoretical rigour which is lacking in the envelope method.

A physical interpretation of the primal/dual problems in terms of structural behaviour is illuminating. Consider a structure constrained by M independent displacement constraints, i.e.

$$\left.\begin{array}{ll} \text{Minimize } W & \\ \text{Subject to } g_m \leqslant 1 & m = 1, \ldots, M \end{array}\right\} \quad (8)$$

If each of the M constraints in (8) is multiplied by a multiplier λ_m, m = 1, ..., M, such that the sum of the λ_m's is unity, and all the constraints are then summed into a single surrogate constraint we have

$$\left.\begin{array}{l} \text{Minimize } W \\ \text{Subject to } \sum_{m=1}^{M} \lambda_m g_m \leqslant 1 \end{array}\right\} \quad (9)$$

Examination of the dual problems corresponding to (8) and (9) shows them to be identical providing the λ_m's in (9) solve problem (6) optimally. This demonstrates that in responding to multiple constraints the structure apportions its member sizes as if all the independent constraints were surrogated into a single generalised stiffness requirement. The structure therefore responds to a single fictitious surrogated stiffness requirement and, since the λ_m must solve (6), the surrogate stiffness requirement is such that the independent sitffness requirements are combined together in such a way as to maximize their constraining potential.

This physical interpretation may partly help to explain the good results often obtained by the envelope method as used in OPTIM II. The envelope method resizes a a member by applying a single resize formula to each constraint in turn and selects the highest resulting member size. These highest sizes form a resized set. By this means the constraining potential of all the constraints is maximized. This is in the same spirit as the more rigorous dual approach outlined above but is mathematically different and is not rigorous. However, it may be conjectured that the good results obtained by OPTIM II correspond to problems in which the enveloping and surrogation approaches are similar and that the occasional poor performance of

OPTIM II corresponds to problems in which the member sizes obtained by enveloping are very different from those which satisfy the more correct surrogated constraint in (9).

4. PROBLEMS WITH DISPLACEMENT AND MEMBER SIZE CONSTRAINTS

As Gellatly and Dupree demonstrate, a displacement constraint governs the distribution of material throughout the structure. A member stress or size constraint only controls the material in an individual member. Difficulties arise when both types of constraints are present together since the distribution of material required to optimally satisfy a displacement constraint may violate the amount of material required to satisfy one or more of the individual member constraints. There is no optimality criterion of practical use for combined types of constraints. Somewhat *ad hoc* methods are usually used such as active/passive sets of variables as in OPTIM II to handle both types of constraints.

The primal problem concerning us here is that given in (1), (2) and (3) and the corresponding dual problem is given in (4) and (5). On examining the dual problem it at first appears that its dimensionality, (M + N), is greater than that of the primal problem, N. This would negate the advantage which the dual approach has of reducing problem dimensionality. Fortunately, very recent research has shown that the N dual variables corresponding to member size constraints may be effectively eliminated by an iterative process. A brief summary of this now follows.

Consider dual problem (4) for a single displacement constraint (with dual variable λ_0) and a full set of N member size constraints. If we write

$$\overline{W_i} = L_i \overline{A_i} \qquad \overline{\delta_i} = \left(\frac{FUL}{\overline{AE}}\right)_i$$

$$\overline{W} = \sum_{i=1}^{N} \overline{W_i} \qquad \overline{\delta} = \sum_{i=1}^{N} \overline{\delta_i}$$

and if δ is the maximum permissible nodal displacement, dual problem (4) is

$$\left.\begin{aligned} &\text{Maximize } V = \sum_{i=1}^{N} \sqrt{\overline{W_i}} \left\{\frac{\overline{\delta_i}}{\delta} \lambda_0 + \lambda_i\right\}^{\frac{1}{2}} \\ &\text{Subject to } \sum_{i=0}^{N} \lambda_i = 1 \\ &\qquad\qquad \lambda_i \geq 0 \qquad\qquad i = 0, \ldots, N \end{aligned}\right\} \qquad (10)$$

Necessary conditions for a constrained maximum of V with respect to the N member size dual variables only are that

$$\frac{\partial V}{\partial \lambda_i} = 0 \qquad\qquad i = 1, \ldots, N$$

This leads to

$$\lambda_i^* = \frac{\overline{W_i}}{\overline{W}} \left(1 + \frac{\lambda_0}{\delta} \left[\overline{\delta} - \delta - \frac{\overline{W}}{\overline{W_i}} \overline{\delta_i}\right]\right) \qquad\qquad i = 1, \ldots, N \qquad (11)$$

Substituting (11) into V of (10) gives

$$V = \sqrt{\overline{W}}\,\{1 + \lambda_0 \left[\frac{\bar{\delta} - \delta}{\delta}\right]\} \tag{12}$$

If $\bar{\delta} < \delta$ this denotes that member sizes evaluated from the member size constraints alone will satisfy the displacement constraint and hence λ_0 will be zero. We are interested in the case where $\bar{\delta} > \delta$ and the displacement constraint must be active. In this case V as given in (12) is maximized by as large a value of λ_0 as is possible. However, λ_0 may not increase to a value such as to drive any of the λ_i*, i = 1, ..., N in (11) below zero. The highest possible value of λ_0 is therefore that value which first puts any λ_i* equal to zero, i.e.

$$\lambda_0 = \underset{i=1,\ldots,N}{\text{Min}} \left\langle (1 - \frac{\bar{\delta}}{\delta} + \frac{\overline{W}}{\overline{W}_i}\,\frac{\overline{\delta_i}}{\delta}) \right\rangle \tag{13}$$

This value of λ_0 drives one of the λ_i* to zero. Let the variable driven to zero be $\lambda_N * = 0$. This is now eliminated as a slack member size constraint.

A new dual problem may now be formed with λ_N eliminated. This replaces problem (10) and is

$$\left.\begin{aligned} &\text{Maximize } V = \sum_{i=1}^{N-1} \sqrt{\overline{W}_i}\,\{\frac{\overline{\delta_i}}{\delta}\lambda_0 + \lambda_i\}^{\frac{1}{2}} + \sqrt{\overline{W}_N}\,\{\frac{\overline{\delta_N}}{\delta}\lambda_0\}^{\frac{1}{2}} \\ &\text{Subject to } \sum_{i=o}^{N-1} \lambda_i = 1 \\ &\qquad\qquad \lambda_i \geq 0 \qquad\qquad\qquad i = 0, \ldots, N = 1 \end{aligned}\right\} \tag{14}$$

Problem (14) is treated in a similar way to problem (10). Relationships similar to (11) are established for the λ_i*, this time for i = 1, ,.., N - 1. An expression for V similar to (12) is found and a new value of λ_0 is determined as (13). If the new value of λ_0 is greater than its previous value another of the λ_i* is eliminated, another problem similar to (14) but with (N − 2) values of λ_i is set up and the process is continued in this iterative fashion until the value of λ_0 reduces. The previous iteration's results for all the λ's are then optimal. Relationships (5) then give the minimum weight and optimal member sizes.

The iterative procedure described above forms into a very simple algorithm since the relationships of the types of (11), (12) and (13) are very concise in nature. Using this iterative dual approach the interactions of member size constraints and a displacement constraint may be optimized very rapidly, the dimensionality of the method being essentially unity. An advantage of the method is that it starts essentially with a fully-stressed design (all member size dual variables active and $\lambda_0 = 0$). The activity level of the displacement constraint, λ_0, is then progressively increased, knocking out member size constraints as they become slack. In many practical design situations a first requirement is to examine the fully-stressed design and check it against possible displacement limitations. If the displacements are excessive the fully-stressed design needs to be altered in some way so as to optimally satisfy displacement limitations. This is precisely how the dual approach outlined above tackles the problem and it is therefore well suited to implementation in practical optimum design programs.

The treatment above is limited to the combination of a single displacement constraint and member size constraints. If multiple displacement constraints are present the iteration algorithm is more complex and has not yet been fully investigated. However, it has already been shown in this paper that multiple displacement constraints behave as a single surrogated constraint. This suggests a possible solution algorithm in which the multiple constraints are first solved separately and the single surrogate constraint formed and then the above algorithm used to handle the interactions of the surrogate constraint and the member size constraints. This remains to be further investigated.

5. CONCLUSIONS

This paper has examined a dual approach to the optimum design of structures whose elements have stiffness proportional to mass. It has shown that a study of duality gives insight and rationale for some of the successful, non-rigorous approaches to truss design such as the optimality criterion approach used in OPTIM II. It would have been more satisfying to give numerical results confirming the speed and efficiency of the dual algorithms suggested in this paper but space limitations preclude this. Nevertheless it can be stated that the dual approach does provide a means of very rapidly solving optimum design problems for large structural systems. The reduction in dimensionality and the ease with which the dual problems may be manipulated and solved makes the approach a very serious competitor to the much-used, less rigorous optimality criteria methods. From a practical structural engineering point of view it should be stressed that although duality theory and the associated algebra may seem unnecessarily complicated and abstract, the algorithms which may be developed from it are rigorous and are very simple to operate, giving practically useful results very rapidly. Furthermore the dual-based algorithms often tend to be similar to those suggested by engineering intuition. This is very satisfying and a firmer theoretical basis for intuitive design approaches adds considerable strength to them.

As the present author has commented in the Introductory report to the 10th IABSE Congress[3] a major advantage of a study of dual methods is that it sheds new light on well-known problems and enables the nature of the problems to be understood more deeply. Sometimes, as in the case here, this extra insight allows new solution algorithms to be developed. The ultimate usefulness of these algorithms remains to be fully investigated in a continuing program of research.

6. REFERENCES

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SUMMARY

The paper examines a new dual approach to the optimum design of trusses with multiple displacement and member size constraints. Comparison is made with optimality criteria approaches to the same problem. Reductions in problem dimensionality and simple solution algorithms arise from casting the problem into dual space, which also gives insight into some ad hoc, intuitive artifices often employed in the solution of these problems.

RESUME

Une nouvelle méthode duale est présentée pour le dimensionnement optimal de treillis, soumis à des contraintes de déplacements multiples et de types de profils. Une comparaison est faite avec la méthode des critères d'optimisation. Des réductions de la dimension des problèmes ainsi que des algorithmes simples pour leur résolution sont obtenus en situant le problème dans l'espace dual, ce qui permet également d'analyser quelques artifices de calcul souvent utilisés dans la solution de tels problèmes.

ZUSAMMENFASSUNG

Der Bericht behandelt eine neue Dualmethode für die Optimierung von Fachwerken mit mehrfachen Formänderungs- und Formgebungsrestriktionen. Die Ergebnisse werden mit der Methode der Optimalitätskriterien verglichen. Eine Abminderung der Komplexität und einfache Lösungsalgorithmen resultieren aus der Problemprojektion in einem Dualraum, was auch Einblick in gewisse intuitive Verfahren gewährt, die bei der Lösung solcher Probleme oft angewendet werden.