The theory of the suspension bridge

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THE THEORY OF THE SUSPENSION BRIDGE.

THÉORIE DES PONTS SUSPENDUS.

THEORIE DER HÄNGEBRÜCKEN.

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Introduction.

In spite of the fact that long-span, stiffened suspension bridges have been built for many years, no apparent attempt has been made to simplify the mathematical equations involved in their analysis and to present them in their simplest forms. It appears ofttimes as if the writers on the subject fail to remove the mathematical falsework they erect during the construction of their equations and thereby almost completely obscure their completed structure. Then, too, these writers have failed to correlate their work to the work of their predecessors so that to the engineer studying the subject for the first time, the theory of the stiffened bridge appears exceedingly complex. Differences in nomenclature, slight variations in apparently innocuous assumptions, and the representation of the same thing in slightly different mathematical forms serve to obscure rather than to clarify the essentially simple theory of the stiffened suspension bridge.

In the following an effort will be made to correlate the work of the different writers and to indicate wherein their methods differ and wherein they are alike. The assumptions made in each method will be scrutinized and their effect upon the final result will be emphasized. A solution will be presented for each method which, it is believed, is simpler than those in present use and the workability of the solution will be shown by application to a numerical problem.

Statement of problem.

The stiffened suspension bridge as built today is a statically indeterminate structure and its analysis would offer no particular difficulty if it were possible to consider the dimensions the same during all conditions of loading and temperature. However, long-span structures suffer such great distortions under live load and temperature variations from the normal that large errors are made in the analysis if these distortions are not considered. These errors effect the economy of the structure but not its safety, and in long-span bridges they are known to be very large. This paper will confine itself to a discussion of the methods of analysis in which these distortions are considered.

There are many types of stiffened suspension bridges, but the theory upon which their analysis is based is fundamentally the same. However, in the analysis of certain types, the volume of algebraic work is less than in others, so here where the object is to stress fundamental relationships the type will be selected in which this volume is reduced to a minimum. This type has unloaded backstays and a two-hinged stiffening truss. The procedure followed in the analysis for this type of structure can be extended so as to be applicable to any type having a horizontal stiffening truss. The only difficulty encountered in such an extension will be algebraic, caused by an increase in the number and length of equations developed.

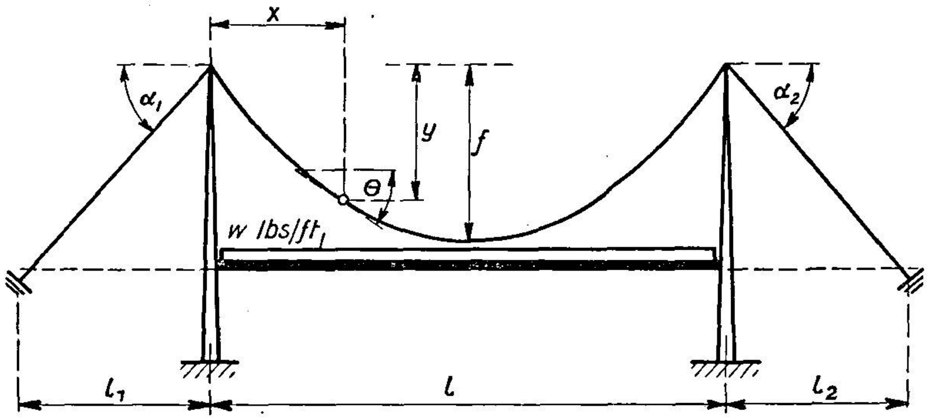

Fig. 1.

Nomenclature.

The type of structure discussed is that shown in Fig. 1 which defines some of the terms employed in the equations. The nomenclature that will be used in addition to that shown in Fig. 1 is as follows:

w = dead load, pounds per lineal foot.
p = total live load, pounds per lineal foot.
q = portion of live load carried by cable, pounds per lineal foot.
H_w = horizontal component of cable stress produced by the dead load w.
H_p = horizontal component of cable stress produced by the live load p.
H_s = horizontal component of cable stress produced by all causes except the dead load.
$H = H_w + H_s$.
η = deflection of cable and stiffening truss.
E = modulus of elasticity of stiffening truss material.
I = moment of inertia of stiffening truss.
A_c = cross-sectional area of cable.
E_c = modulus of elasticity of cable material.
M = bending moment acting on stiffening truss.
V = shear acting on stiffening truss.
M_1 = bending moment acting on stiffening truss calculated as if it were a simple beam independent of the cable.

Additional notation will be defined as introduced.

Fundamental assumptions.

Certain assumptions are usually made in all methods while others are peculiar to the methods themselves. The assumptions which are common to all methods are:

1. The curve assumed by the cable under the action of the dead load is a parabola.

This assumption is correct if the dead load is uniformly distributed on a horizontal plane. The small error in this assumption can be seen from a study of the weight distribution in existing bridges. In the Ambassador Bridge in Detroit[1]) the distribution is as follows:

Uniformly distributed on horizontal plane		85.4 %
„ „	along cable	13.6 %
Variable (hangers)		1.0 %

In the Philadelphia-Camden Bridge[2]) the distribution is approximately

Uniformly distributed on horizontal plane		80 %
„ „	along cable	20 %

2. The elongation of the hangers is so small that it can be neglected.

It has been shown[3]) that the error in this assumption is small. The assumption is equivalent to saying that any deflection suffered by the stiffening truss is transmitted undiminished to the cable.

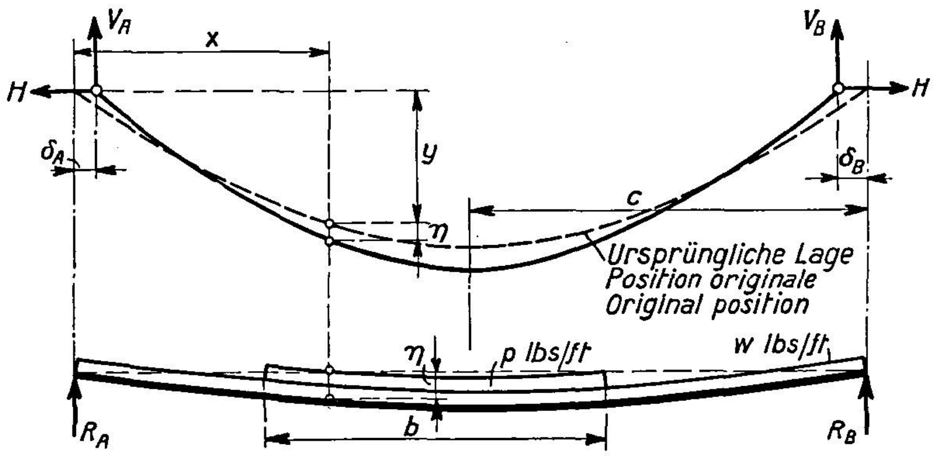

Fig. 2.

3. The spacing of hangers is so small compared with the length of span that the hangers can be considered as forming a continuous sheet.

4. The horizontal component of cable stress in the side spans is assumed equal to that in the main span.

This will be true if the cable is fixed to the top of flexible towers or if the cable is fixed to a movable saddle placed on the top of the towers.

5. The stiffening truss is a beam, simply supported at the ends, that under dead load is initially straight, horizontal, and of constant moment of inertia. This truss is usually constructed so that it receives none of the dead load and is stressed, therefore, by live load and temperature changes only.

Other assumptions that are made are peculiar to the methods themselves and will be noted as they are encountered.

The differential equation method of finding the deflection.

One of the methods used for finding the distorted form of the structure is to solve the differential equation whose result is the equation of the curve assumed by the stiffening truss. This equation will, since the hangers are in-

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[1]) Detroit River Bridge, Mc Clintic-Marshall Co., 1930.

[2]) The Delaware River Bridge, Final Report of Board of Engineers, Ralph Modjeski, Chairman, 1927.

[3]) Page 300. Vol II, Modern Frame Structures, Johnson, Bryan & Turneaure, Ninth Edition.

extensible, give the deflection of the cable from its dead load position and will also give, by differentiation, the equations for slope, bending moment, shear and loading in the truss. This differential equation is

$$EI\frac{d^2\eta}{dx^2} = -M \tag{1}$$

and to solve it, M must first be evaluated. The conditions in the main span due to live load in any position is shown in Fig. 2. The stiffening truss, originally in a horizontal position, deflects to the curved form shown and pulls the cable from its dead load position. The cable in turn pulls in the tower tops, and the points of cable support move inward the amounts δ_A and δ_B respectively. If a section is passed through the structure at any point distant x from the left support, the condition necessary for equilibrium of that portion to the left of such a section is as shown in Fig. 3.

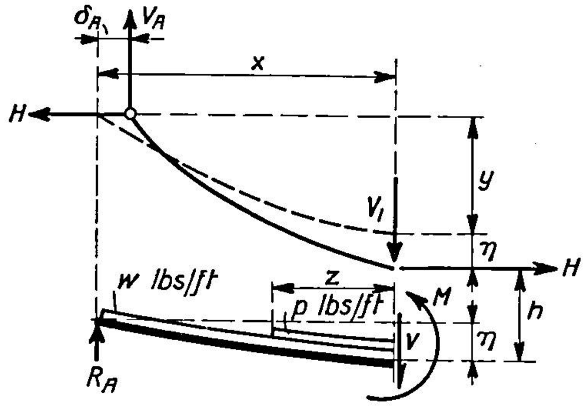

Fig. 3.

Taking a moment center on the cutting section at the neutral axis of the truss, and writing $\Sigma M = 0$,

$$M + H(y+\eta+h) - V_A(x-\delta_A) - R_A(x) + \frac{wx^2}{2} + \frac{pz^2}{2} - Hh = 0.$$

δ_A can be neglected because it is small in comparison with the other dimensions, and $H = H_w + H_s$. The equation becomes

$$M + (H_w + H_s)\eta + H_w y + H_s y - x(V_A + R_A) + \frac{wx^2}{2} + \frac{pz^2}{2} = 0.$$

$$H_w = \frac{wl^2}{8f}; \quad y = \frac{4fx}{l^2}(l-x)\,^{4});$$

and, if b is the length covered by live load and c is the distance to its center of gravity from R_B, then

$$V_A + R_A = \frac{wl}{2} + \frac{(pb)c}{l}$$

$$M + (H_w + H_s)\eta + \frac{wl^2}{8f}\left(\frac{4fx}{l^2}\right)(l-x) + H_s y - x\left(\frac{wl}{2} + \frac{pbc}{l}\right)\frac{wx^2}{2} + \frac{pz^2}{2} = 0$$

$$M + (H_w + H_s)\eta + H_s y - \left(\frac{pbc}{l}x - \frac{pz^2}{2}\right) = 0.$$

[4]) These follow from the assumption that the curve of the cable under dead load is a parabola.

The quantity $\frac{pbc}{l}x - \frac{pz^2}{2}$ can be recognized as the bending moment in the truss computed as if it were a simple beam independent of the cable. This has been designated as M_1. Therefore,

$$M = M_1 - (H_w + H_s)\eta - H_s y \text{ [5]} \qquad (2)$$

Eq. (1) becomes, with this value of M,

$$EI\frac{d^2\eta}{dx^2} = -M_1 + (H_w + H_s)\eta + H_s y$$

Letting

$$a^2 = \frac{H_w + H_s}{EI} \qquad (3)$$

Eq. (1) becomes

$$\frac{d^2\eta}{dx^2} - a^2\eta = -\frac{M_1}{EI} + \frac{H_s y}{EI} \qquad (4)$$

This differential equation can be solved by using either hyperbolic functions or by exponential functions. In most solutions M_1 and y, which are both functions of x, are substituted in eq. (4) before the solution is attempted. This step can be eliminated if the solution is arranged in the following form (using the exponential solution):

$$\eta = C_1 e^{ax} + C_2 e^{-ax} + \frac{1}{EI}\left(\frac{M_1}{a^2} + \frac{1}{a^4}\frac{d^2M_1}{dx^2} + \frac{1}{a^6}\frac{d^4M_1}{dx^4} + \dots\right)$$
$$-\frac{H_s}{EI}\left(\frac{y}{a^2} + \frac{1}{a^4}\frac{d^2y}{dx^2} + \frac{1}{a^6}\frac{d^4y}{dx^4} + \dots\right)$$

The constants C_1 and C_2 are integration constants which must be evaluated for each load position from the end conditions and the conditions of continuity. The equation is simplified by noting that $\frac{d^2y}{dx^2} = -\frac{8f}{l^2}$ and all higher derivatives vanish. Likewise $\frac{d^2M_1}{dx^2} = -p$ for that part of the structure covered by live load and 0 for the rest. Higher derivatives likewise vanish. Making these simplifications,

$$\eta = C_1 e^{ax} + C_2 e^{-ax} + \frac{1}{H_w + H_s}\left(M_1 - \frac{p}{a^2}\right) - \frac{H_s}{H_w + H_s}\left(y - \frac{8f}{a^2 l^2}\right) \qquad (5)$$

In discussions written on this question eq. (5) is given in different forms which are readily obtained from it by algebraic manipulation. The following forms are given by the authorities noted.

According to L. S. Moisseiff [6])

$$\eta = \frac{H_s}{H_w + H_s}\left\{K_1 e^{ax} + K_2 e^{-ax} + \frac{M_1}{H_s} - \frac{p}{H_s a^2} - y + \frac{8f}{a^2 l^2}\right\} \qquad (6)$$

Note: If $\frac{H_s}{H_w + H_s}$ is factored out of the right-hand side of eq. (5) and $C_1\frac{(H_w + H_s)}{H_s}$ is replaced by K_1 and $C_2\frac{(H_w + H_s)}{H_s}$ by K_2, eq. (5) becomes identical with eq. (6).

[5]) If in this equation η is considered small enough to be neglected, the structure can be analyzed as is common in other statically indeterminate structures in which the dimensions before and after loading are considered the same.

[6]) Page 98, The Delaware River Bridge, Final Report of Board of Engineers, R. Modjeski, Chairman.

According to Johnson, Bryan, and Turneaure [7])

$$\eta = \frac{H_s}{H_w + H_s}\left\{K_1 e^{ax} + K_2 e^{-ax} + \frac{M_1}{H_s} + \frac{1}{a^2}\left(\frac{8f}{l^2} - \frac{p}{H_s}\right) - \frac{4fx}{l^2}(l-x)\right\} \quad (7)$$

Note: This can be obtained from Moisseiff's equation replacing y with its equivalent in terms of x.

According to F. Bleich [8])

$$\eta = C_1' \sinh ax + C_2' \cosh ax + \left(\frac{H_w}{EIa^2} - 1\right)\left(y - \frac{8f}{a^2 l^2}\right) + \frac{1}{EIa^2}\left(M' - \frac{p}{a^2}\right) \quad (8)$$

Note: This is the hyperbolic form preferred by most European investigators. It can by obtained from eq. (5) by noting the following identities:

$$e^{ax} = \sinh ax + \cosh ax,$$
$$e^{-ax} = \sinh ax - \cosh ax.$$

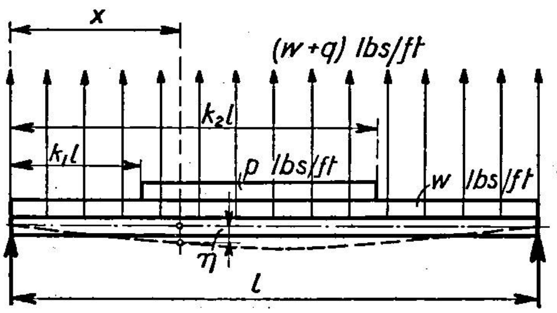

Fig. 4.

If these are substituted in eq. (5), $C_1 - C_2$ replaced by C_1', $C_1 + C_2$ by C_2', H_s by $H - H_w$, then a^2 becomes $\frac{H}{EI}$, and eq. (5) is converted to the form given by Bleich. This form is also given by G. G. Krivoshein [9]).

The various forms of eq. (5) represent the most common methods of approaching the problem of suspension bridge analyses.

The series method of finding the deflection.

Another method for finding the equation of the deflected truss is to represent it by a trigonometric series. This method has been presented by S. Timoshenko [10]) and by G. C. Priester [11]).

If the stiffening truss is considered as a free body it will be in equilibrium under the force system shown in Fig. 4. The truss is acted upon by the dead load w over the entire span, the live load p covering any portion $k_2 l - k_1 l$, and the hanger pull $(w + q)$ [12]).

The deflection curve of the truss can be represented by a trigonometric series, as follows:

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[7]) Page 279, Vol. II, Modern Frame Structures, 9th Edition.

[8]) Page 459, Theorie und Berechnung der eisernen Brücken, F. Bleich.

[9]) Page 280, Simplified Calculation of Statically Indeterminate Bridges, G. G. Krivoshein.

[10]) A. S. C. E. Transactions, Vol. 94 (1930), page 377.

[11]) Engineering Research Bulletin, No. 12, University of Michigan.

[12]) The hanger pull due to dead load is uniform over the entire span and equal to w lb/ft.

$$\eta = a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{2\pi x}{l} + a_3 \sin\frac{3\pi x}{l} + \cdots + a_n \sin\frac{n\pi x}{l}. \qquad (9)$$

The coefficients a_1, a_2, a_3, etc., must be evaluated from the loads producing the deflection in such a way that the series fits the particular case being investigated. This evaluation can be done by a consideration of the energy of the system.

The potential energy stored in a beam when it is deformed by a bending moment is

$$U = \int_0^l \frac{M^2\,dx}{2EI} = \frac{EI}{2}\int_0^l \left(\frac{d^2\eta}{dx^2}\right)^2 dx. \qquad (10)$$

If eq. (9) is substituted in the eq. (10), the expression for potential energy becomes

$$U = \frac{EI\pi^4}{4l^3}\left[1^4 a_1^2 + 2^4 a_2^2 + 3^4 a_3^2 + \cdots + n^4 a_n^2 + \cdots\right] \qquad (11)$$

If a small increase Δa_n is given to one of the coefficients of the series in eq. (9), a small additional deflection $\Delta a_n \sin\frac{n\pi x}{l}$ will be added to those already present. The increase in potential energy produced by this displacement will be

$$\Delta U = \frac{dU}{da_n}\Delta a_n = \frac{EI\pi^4 n^4 a_n}{2l^3}\cdot \Delta a_n. \qquad (12)$$

This increase in potential energy must be equal to the work done by the forces and loads acting on the truss. The dead load w is counter-balanced by the equal and opposite dead load hanger forces, so that the live load p and the live load hanger forces q are the only ones to be considered. The portion of the live load that is carried by the stiffening truss can be found from eq. (2) since it is equal to $-\frac{d^2M}{dx^2}$.

$$-\frac{d^2M}{dx^2} = -\frac{d^2M_1}{dx^2} + (H_w + H_s)\frac{d^2\eta}{dx^2} + H_s\left(\frac{d^2y}{dx^2}\right)$$

$$= p + (H_w + H_s)\frac{d^2\eta}{dx^2} - H_s\left(\frac{8f}{l^2}\right)$$

[concerning p see footnote [13])].

Therefore, $$q = -(H_w + H_s)\frac{d^2\eta}{dx^2} + H_s\left(\frac{8f}{l^2}\right) \qquad (13)$$

If $\beta = \frac{H_s}{H_w}$, and since $H_w = \frac{wl^2}{8f}$

$$q = -H_w(1+\beta)\frac{d^2\eta}{dx^2} + \beta w. \qquad (14)$$

The loads p and q are in pounds per unit length so that the work done, ΔW, by the downward load p and the upward load q acting through the downward displacement

[13]) If the load is being evaluated at a section where there is no loading, $\frac{d^2M_1}{dx^2} = 0$.

$\Delta a_n \sin \frac{n\pi x}{l}$ is

$$\Delta W = p\,\Delta a_n \int_{k_1 l}^{k_2 l} \sin \frac{n\pi x}{l}\,dx - \Delta a_n \int_0^l \left[\beta w - H_w(1+\beta)\frac{d^2\eta}{dx^2}\right] \sin \frac{n\pi x}{l}\,dx.$$

This, when evaluated, becomes

$$\Delta W = \frac{p\,\Delta a_n l}{n\pi}(\cos n\pi k_1 - \cos n\pi k_2)$$
$$-\Delta a_n\left[\frac{\beta w l}{n\pi}(1-\cos n\pi) - H_w(1+\beta)a_n\frac{n^2\pi^2}{2l}\right]. \qquad (15)$$

If the increase in external work given by eq. (15) is set equal to the corresponding increase in potential energy, eq. (12), the expression for the general coefficient a_n is obtained.

$$a_n = \frac{\frac{q\,l}{n\pi}(\cos n\pi k_1 - \cos n\pi k_2) - \frac{l\beta}{n\pi} w(1-\cos n\pi)}{\frac{n^4 E I \pi^4}{2l^3} + H_w(1+\beta)\frac{\pi^2 n^2}{2l}}. \qquad (16)$$

The deflection curve is now completely found, for the coefficients a_1, a_2, a_3, etc. can be evaluated from eq. (16) by letting $n = 1, 2, 3$, etc. It will be shown later that the series shown by eq. (9) converges fast enough to make its use practical.

Discussion of preceding equations.

If the series eq. (9) with its evaluated coefficients is compared with eq. (5) one of the greatest advantages of the series form is apparent. When the coefficient a_n was found the problem of finding η was complete, but in eq. (5) there is still left the problem of finding the constants C_1 and C_2. In addition, eq. (9) holds for all values of x, while eq. (5) holds only for those values of x in which M_1 has the same algebraic form. When eq. (5) is applied to the type of loading shown in Fig. 4 there will be three expressions for M_1 so that it will be necessary to evaluate six constants of integration.

In both of the equations developed for the deflection of the stiffening truss there appears the unknown H_s which must be found before the deflections can be numerically evaluated for any particular case. There are two methods for finding H_s. One method can be called the "Energy Method" and the other the "Geometric Method".

The energy method of finding H_s[14]).

This method consists of equating the work done by the hanger forces moving through the deflection suffered by the cable to the internal work done by the internal stress in the cable moving through the deformation suffered by the cable. The force system acting on the cable as a free body is shown in Fig. 5.

To evaluate the external work done it must be assumed that the deflection is directly proportional to the load. This is not apparent from the equations

[14]) Originally by Melan, „Eiserne Bogenbrücken und Hängebrücken", 1888, and by D. B. Steinmann, "A Practical Treatise on Suspension Bridges", 1922.
L. S. Moiseiff, "The Delaware River Bridge", Final Report of Board of Engineers, R. Modjeski, Chairman.
Johnson, Bryan & Turneaure, "Modern Frame Structures", Vol. II.

for η but it will be shown that it is, for all practical purposes, true. Before the live load and temperature change acted, the load on the cable was w, after they have acted the load is $(w + q)$. The average load is therefore

$$\frac{w + (w + q)}{2} = \left(w + \frac{q}{2}\right).$$

Since the load on an element dx in length is $\left(w + \frac{q}{2}\right) dx$ the external work becomes

$$W_e = \int_0^l \left(w + \frac{q}{2}\right) \eta \, dx. \tag{17}$$

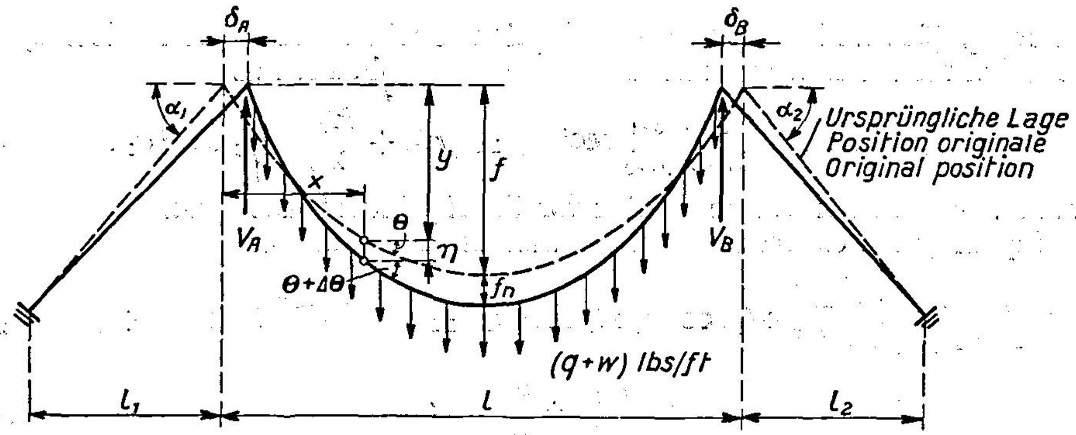

Fig. 5.

For the main span, the stress at any section under dead load is H_w sec Θ, under live load and temperature change it increases to $(H_w + H_s)$ sec $(\Theta + \Delta\Theta)$. The average stress is

$$\frac{H_w \sec\Theta + (H_w + H_s)\sec(\Theta + \Delta\Theta)}{2} \tag{18}$$

If the length of the element whose horizontal projection is dx be called ds, then the change in the length of the element becomes

$$H_w\{\sec(\Theta + \Delta\Theta) - \sec\Theta\}\frac{ds}{A_c E_c} + H_s \sec(\Theta + \Delta\Theta)\frac{ds}{A_c E_c} \pm \omega t ds. \tag{19}$$

Here ω is the coefficient of linear expansion and t is the temperature change from the normal.

If eq. (18) is multiplied by eq. (19) the expression for internal work is obtained. The change in angle $\Delta\Theta$ is neglected because of its minuteness so that

$$\Delta W_{i_1} = \left(H_w + \frac{H_s}{2}\right)\sec\Theta\left[H_s \sec\Theta \frac{ds}{A_c E_c} \pm \omega t ds\right]$$

sec Θ, however, $= \dfrac{ds}{dx}$

$$\Delta W_{i_1} = \left(H_w + \frac{H_s}{2}\right)\frac{H_s}{A_c E_c}\frac{ds^3}{dx^2} \pm \left(H_w + \frac{H_s}{2}\right)\omega t \frac{ds^2}{dx}$$

and,
$$W_{i_1} = \left(H_w + \frac{H_s}{2}\right)\left(\frac{H_s}{A_c E_c}\right)\int_0^l \frac{ds^3}{dx^2} \pm \left(H_w + \frac{H_s}{2}\right)\omega t \int_0^l \frac{ds^2}{dx}. \tag{20}$$

For the left-hand side span, the dead load stress is $H_w \sec \alpha_1$. The live load and temperature change increases it to $(H_w + H_s) \sec(\alpha_1 + \varDelta\alpha_1)$. The average stress in then

$$\frac{H_w \sec \alpha_1 + (H_w + H_s) \sec(\alpha_1 + \varDelta \alpha_1)}{2} \tag{21}$$

The change in length of the side span becomes

$$H_w \left\{\sec(\alpha_1 + \varDelta \alpha_1) - \sec \alpha_1\right\} \frac{l_1 \sec \alpha_1}{A_c E_c} + H_s \sec(\alpha_1 + \varDelta \alpha_1) \frac{l_1 \sec \alpha_1}{A_c E_c} \pm \omega t l_1 \sec \alpha_1 \tag{22}$$

Now if eq. (21) and eq. (22) are multiplied together and $\varDelta\alpha_1$ is neglected, the expression for internal work in the side spans is obtained as

$$W_{i_2} = \left(H_w + \frac{H_s}{2}\right) \frac{H_s l_1}{A_c E_c} \sec^3 \alpha_1 \pm \left(H_w + \frac{H_s}{2}\right) \omega t l_1 \sec^2 \alpha_1 \,. \tag{23}$$

A similar expression can be written for the right-hand side span.

Equating the expressions for internal and external work

$$\left(H_w + \frac{H_s}{2}\right)\left[\frac{H_s}{A_c E_c}\left(\int_0^l \frac{ds^3}{dx^2} + l_1 \sec^3 \alpha_1 + l_2 \sec^3 \alpha_2\right) \pm \omega t \left(\int_0^l \frac{ds^2}{dx} l_1 \sec^2 \alpha_1 + l_2 \sec^2 \alpha_2\right)\right] = \int_0^l \left(w + \frac{q}{2}\right) \eta \, dx \,. \tag{24}$$

To condense the equations let

$$\int_0^l \frac{ds^3}{dx^2} + l_1 \sec^3 \alpha_1 + l_2 \sec^3 \alpha_2 = L$$

and

$$\int_0^l \frac{ds^2}{dx} + l_1 \sec^2 \alpha_1 + l_2 \sec^2 \alpha = L_t \,.$$

Eq. (24) becomes with these substitutions

$$\left(H_w + \frac{H_s}{2}\right)\left\{\frac{H_s L}{A_c E_c} \pm \omega t L_t\right\} = \int_0^l \left(w + \frac{q}{2}\right) \eta \, dx \,. \tag{25}$$

Since $q = -H_w (1 + \beta) \dfrac{d^2 \eta}{dx^2} + \beta_w$, eq. (14), the right side of eq. (25) becomes

$$\int_0^l \left[w - \frac{H_w (1 + \beta)}{2} \cdot \frac{d^2 \eta}{dx^2} + \frac{\beta_w}{2}\right] \eta \, dx \tag{26}$$

or, since $\beta = \dfrac{H_s}{H_w}$ and $w = \dfrac{8 f H_w}{l^2}$

$$\frac{8f}{l^2}\left(H_w + \frac{H_s}{2}\right)\int_0^l \eta \, dx - \frac{H_w}{2}(1 + \beta) \int_0^l \frac{d^2 \eta}{dx^2} \cdot \eta \, dx \,. \tag{27}$$

Finally, eq. (24) becomes

$$\frac{H_s L}{A_c E_c} \pm \omega t L_t = \frac{8f}{l^2} \int_0^l \eta \, dx - \frac{H_w + H_s}{2 H_w + H_s} \int_0^l \frac{d^2 \eta}{dx^2} \cdot \eta \, dx \,. \tag{28}$$

The geometric method of finding H_s.

The geometric method has been presented by Fr. Bleich[15]) and by G. G. Krivoshein[16]). Bleich makes certain assumptions in his method that are unnecessary and do not simplify the resulting equations, therefore, the method of Krivoshein is presented here. If an element of the cable is removed as in Fig. 6, the following relationships are obvious.

$$ds^2 = dx^2 + dy^2 \tag{29}$$

$$(ds + \Delta\, ds)^2 = (dx - \Delta\, dx)^2 + (dy + \Delta\, dy)^2 \tag{30}$$

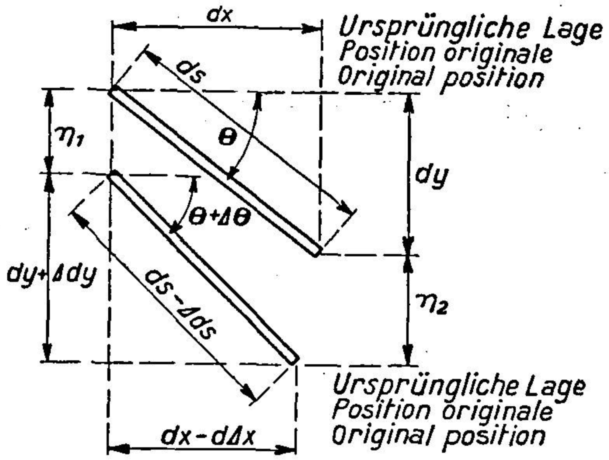

Fig. 6.

If from eq. (30), eq. (29) is subtracted and the terms $(\Delta\, ds)^2$, $(\Delta\, dx)^2$ and $(\Delta\, dy)^2$ are neglected, one obtains

$$2\, ds (\Delta\, ds) = -2\, dx (\Delta\, dx) + 2\, dy (\Delta\, dy) \tag{31}$$

$$\Delta\, ds \frac{ds}{dx} + \Delta\, dx = + \Delta\, dy \left(\frac{dy}{dx}\right). \tag{32}$$

But $\Delta\, ds = H_s \sec(\Theta + \Delta\,\Theta) \dfrac{ds}{A_c E_c} \pm \omega t\, ds,\ \ \Delta\, dy = d\eta,$

$\dfrac{dy}{dx} = \dfrac{4f}{l} - \dfrac{8fx}{l^2}$, and if $\Delta\,\Theta$ is neglected as before,

eq. (32) becomes

$$\frac{H_s}{A_c E_c} \frac{ds^3}{dx^2} \pm \omega t \frac{ds^2}{dx} + \Delta\, dx = \frac{4f}{l} d\eta - \frac{8fx}{l^2} d\eta \tag{33}$$

and for the entire cable

$$\frac{H_s}{A_c E_c} \int_0^l \frac{ds^3}{dx^2} \pm \omega t \int_0^l \frac{ds^2}{dx} + (\delta_A + \delta_B) = \frac{4f}{l} \int_0^l d\eta - \frac{8f}{l^2} \int_0^l x\, d\eta. \tag{34}$$

$\delta_A + \delta_B =$ the total inward movement of the towers. Under the assumption that the horizontal component of cable stress in the side spans is equal to that in the main spans this quantity can be evaluated from the conditions in the side spans. Referring to Fig. 7, the elongation along the length of the side span

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[15]) Page 459, Theorie und Berechnung der eisernen Brücken.
[16]) Page 281, Simplified Calculation of Statically Indeterminate Bridges.

cables, neglecting the small change $\Delta\alpha_1$, is

$$\left[\frac{H_s \sec\alpha_1}{A_c E_c}\right] l_1 \sec\alpha_1 \pm \omega t l_1 \sec\alpha_1 .$$

Therefore,

$$\delta_A = \left[\left(\frac{H_s l_1 \sec^2\alpha_1}{A_c E_c}\right) \pm \omega t l_1 \sec\alpha_1\right] \sec\alpha_1 . \tag{35}$$

For the right-hand span a similar equation can be written for evaluating δ_B.

If eq. (35) and the corresponding equation for the right-hand span is substituted in eq. (34) it becomes, since $\int_0^l d\eta = 0$,

$$\frac{H_s L}{A_c E_c} \pm \omega t L_t = -\frac{8f}{l^2}\int_0^l x\, d\eta = \frac{8f}{l^2}\int_0^l \eta\, dx . \tag{36}$$

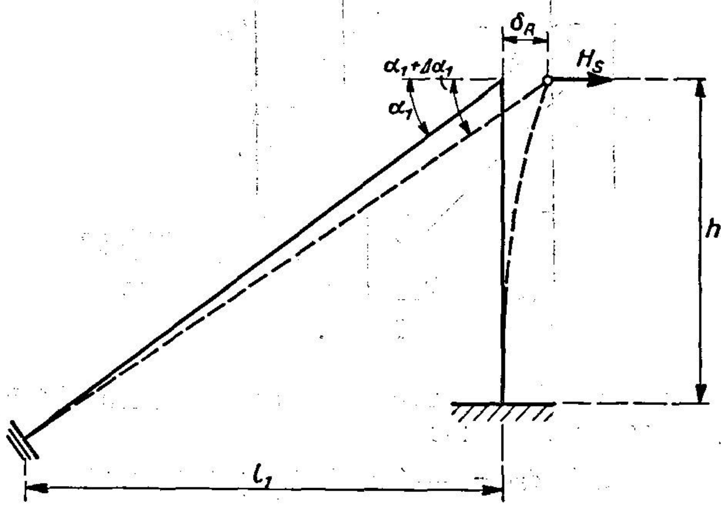

Fig. 7.

Discussion of eqs. (28) and (36).

It is apparent that eqs. (36) and (28) would be identical if it were not for the second term on the right-hand side of eq. (28). This term has a simple meaning. If when q was evaluated, the effect of deflection on moment had been neglected, eq. (2) would have been

$$M = M_1 - H_s y \tag{37}$$

Then the portion of the live load carried by the cable would have been

$$q = H_s\left(\frac{8f}{l^2}\right) \tag{38}$$

With this value of q eq. (28) would be identical with eq. (36). This shows that the energy method gives more accurate results than the geometric method because it takes into consideration the correct distribution of live load to the cable. It also shows that neglecting the terms $(\Delta ds)^2$, $(\Delta dx)^2$ and $(\Delta dy)^2$ is equivalent to the assumptions that the deflection is proportional to the load and that the live load is distributed to the cables as a uniform load.

In order to complete the equations for H_s it is necessary to take the equation for η as expressed by eqs. (5) or (9) and substitute it in eqs. (28) or (36). A little consideration will show that the algebraic work encountered in solving eq. (28) with η as given by eq. (5) is prohibitive, and, whenever this solution has been made, graphical integration has been used. Fortunately the effect of the true distribution of live load to the cable is for all practical purposes negligible[17]). Therefore, if eq. (5) is used for η, it is used in eq. (36). If, however, η as given by eq. (9) is used, either eq. (28) or (36) can be readily solved. This fact gives the series method a theoretical advantage over the differential equation method.

Three methods are available for obtaining formulae for H_s. These are: 1. eq. (5) in eq. (36); 2. eq. (9) in eq. (36); and 3. eq. (9) in eq. (28). These will be called

1. Melan method.
2. Timoshenko, approximate method.
3. Timoshenko, exact method.

Working formulae for H_s.

Formulae by the three methods will be developed for H_s for the general loading condition shown in Fig. 8. The live load p extends from $x = k_1 l$ to $x = k_2 l$ where $k_1 l$ and $k_2 l$ are any two distances.

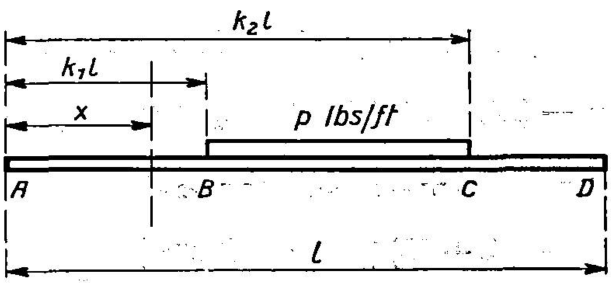

Fig. 8.

1. The Melan Method.

Since the value of η, eq. (5), holds only for those values of x in which M_1 has the same algebraic form, three equations must be written for η, as follows:

η_1 for values of x between 0 and $k_1 l$

$$\eta_1 = C_1 e^{ax} + C_2 e^{-ax} + \frac{1}{H_w + H_s}(M_1)_1 - \frac{H_s}{H_w + H_s}\left(y - \frac{8f}{a^2 l^2}\right) \tag{39}$$

in which

$$(M_1)_1 = pl(k_2 - k_1)\left(1 - \frac{k_1}{2} - \frac{k_2}{2}\right)x \tag{40}$$

η_2 for values of x between $k_1 l$ and $k_2 l$

$$\eta_2 = C_3 e^{ax} + C_4 e^{-ax} + \frac{1}{H_w + H_s}\left[(M_1)_2 - \frac{P}{a^2}\right] - \frac{H_s}{H_w + H_s}\left(y - \frac{8f}{a^2 l^2}\right) \tag{41}$$

and

$$(M_1)_2 = pl(k_2 - k_1)\left(1 - \frac{k_1}{2} - \frac{k_2}{2}\right)x - \frac{p(x - k_1 l)^2}{2} \tag{42}$$

η_3 for values of x between $k_2 l$ and l

$$\eta_3 = C_5 e^{ax} + C_6 e^{-ax} + \frac{1}{H_w + H_s}(M_1)_3 - \frac{H_s}{H_w + H_s}\left(y - \frac{8f}{a^2 l^2}\right) \tag{43}$$

and

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[17]) This will be shown in the numerical problem.

$$(M_1)_s = p\,l\,(k_2 - k_1)\left(1 - \frac{k_1}{2} - \frac{k_2}{2}\right)x - p\,(k_2\,l - k_1\,l)\left[x - \left(\frac{k_2 l - k_1 l}{2} + k_1\,l\right)\right]. \quad (44)$$

Therefore, eq. (36) becomes

$$\frac{H_s L}{A_c E_c} \pm \omega t L_t + \frac{8f}{l^2}\left[\int_0^{k_1 l} x\,d\eta_1 + \int_{k_1 l}^{k_2 l} x\,d\eta_2 + \int_{k_2 l}^{l} x\,d\eta_3\right] = 0. \quad (45)$$

Before the values of η_1, η_2, and η_3 can be substituted in eq. (45) the six constants of integration C_1, C_2, C_3, C_4, C_5 and C_6 must be evaluated from the end conditions and the conditions of continuity.

The expressions for the bending moments and shears in the three sections are obtained from the equations for the elastic curve, eqs. (39), (41) and (43), since from mechanics

$$EI\frac{d^2\eta}{dx^2} = -M \quad (46)$$

$$\frac{dM}{dx} = V. \quad (47)$$

Substituting η_1, η_2, and η_3 respecfively in these two equations

$$M_1 = -(H_w + H_s)(C_1 e^{ax} + C_2 e^{-ax}) - \frac{8 H_s f}{a^2 l^2} \quad (48)$$

$$V_1 = -a\,(H_w + H_s)(C_1 e^{ax} - C_2 e^{-ax}) \quad (49)$$

$$M_2 = -(H_w + H_s)(C_3 e^{ax} + C_4 e^{-ax}) + \frac{P}{a^2} - \frac{8 H_s f}{a^2 l^2} \quad (50)$$

$$V_2 = -a\,(H_w + H_s)(C_3 e^{ax} - C_4 e^{-ax}) \quad (51)$$

$$M_3 = -(H_w + H_s)(C_5 e^{ax} + C_6 e^{-ax}) - \frac{8 H_s f}{a^2 l^2} \quad (52)$$

$$V_3 = -a\,(H_w + H_s)(C_5 e^{ax} - C_6 e^{-ax}) \quad (53)$$

From the end conditions and conditions of continuity, the following six relationships are available for evaluating the six constants

(1) When $x = 0$, $\eta_1 = 0$
(2) " $x = l$, $\eta_3 = 0$
(3) " $x = k_1\,l$, $M_1 = M_2$
(4) " $x = k_1\,l$, $V_1 = V_2$
(5) " $x = k_2\,l$, $M_2 = M_3$
(6) " $x = k_2\,l$, $V_2 = V_3$.

These relationships give

$$C_1 = \frac{8 f H_s}{l^2 (H_w + H_s)}\left[\frac{e^{-al} - 1}{a^2 (e^{al} - e^{-al})}\right] + \frac{p}{2 (H_w + H_s)}\left[\frac{e^{al}(e^{-ak_2 l} - e^{-ak_1 l}) + e^{-al}(e^{ak_2 l} - e^{ak_1 l})}{a^2 (e^{al} - e^{-al})}\right] \quad (54)$$

$$C_2 = -\frac{8 f H_s}{a^2 l^2 (H_w + H_s)} - C_1 \quad (55)$$

$$C_3 = \frac{p\,e^{-ak_1 l}}{2 a^2 (H_w + H_s)} + C_1 \quad (56)$$

$$C_4' = \frac{p\,e^{ak_1 l}}{2a^2(H_w+H_s)} + C_2 \tag{57}$$

$$C_5 = -\frac{8fH_s e^{-al}}{a^2 l^2(H_w+H_s)} + e^{-2al}\left(\frac{p\,e^{ak_2 l}}{2a^2(H_w+H_s)} - C_4\right) \tag{58}$$

$$C_6 = \frac{-p\,e^{ak_2 l}}{2a^2(H_w+H_s)} + C_4. \tag{59}$$

These constants are then substituted in the equations for η and the resulting values of η_1, η_2, and η_3 are in turn substituted in eq. (45) to obtain the following expression:

$$H_s^2\left[\frac{L}{A_c E_c}\right] + H_s\left[\frac{H_w L}{A_c E_c} \pm \omega t L_t + \frac{16f^2}{3l} - \frac{64f^2K_2}{l_4}\right] + \left[\frac{pflK_1}{3} \pm \omega t L_t + H_w + \frac{8fpK_3}{l^2}\right] = 0. \tag{60}$$

In which K_1, K_2 and K_3 have the following values:

$$K_1 = k_2^2(4k_2 - 6) - k_1^2(4k_1 - 6) \tag{61}$$

$$K_2 = \frac{4 + al(e^{al} - e^{-al}) - 2(e^{al} + e^{-al})}{a^3(e^{al} - e^{-al})} \tag{62}$$

$$K_3 = \frac{(e^{al}-1)(e^{-ak_2 l} - e^{-ak_1 l}) + (e^{-al}-1)(e^{ak_2 l} - e^{ak_1 l})}{a^3(e^{al} - e^{-al})} + \frac{(e^{al} - e^{-al})(ak_2 l - ak_1 l)}{a^3(e^{al} - e^{-al})}. \tag{63}$$

2. Timoshenko — Approximate Method.

If eq. (9) is substituted in eq. (36) a solution which is equivalent to that given by eq. (60) ist obtained.

$$\frac{H_s L}{A_c E_c} \pm \omega t L_t - \frac{16f}{l\pi}\left[a_1 + \frac{a_3}{3} + \frac{a_5}{5} + \frac{a_7}{7} + \cdots\right] = 0. \tag{64}$$

In comparison with eq. (60) this equation is very readily obtained and the time involved is infinitely less.

3. Timoshenko — Exact Method.

This solution, which takes into consideration the actual distribution of live load to the cable is obtained by substituting eq. (9) into eq. (28). This yields

$$\frac{H_s L}{A_c E_c} \pm \omega t L_t - \frac{16f}{l\pi}\left(a_1 + \frac{a_3}{3} + \frac{a_5}{5} + \frac{a_7}{7} + \cdots\right)$$
$$\left(\frac{H_w + H_s}{2H_w + H_s}\right)\left(\frac{\pi^2}{2l}\right)(a_1^2 + 2a_2^2 + 3a_3^2 + \cdots) = 0. \tag{65}$$

By comparing eq. (65) with eq. (64) the effect of the actual distribution of live load is seen to be represented by the last term.

None of the three equations developed for finding H_s can be solved directly because they all contain unknown terms. The terms K_2 and K_3 in eq. (60) contain "a" which equals $\sqrt{\frac{H_w + H_s}{EI}}$ and the coefficients of eq. (64) and (65) contain "β" which equals $\frac{H_s}{H_w}$. The solutions must therefore be made by successive approximations. In the numerical example which follows, methods are given in which the number of approximations made are reduced to a minimum.

Illustrative problem.

For purposes of comparing the workability of the different methods and for showing the differences involved in them the Detroit-Windsor Bridge has been selected for analysis. This structure is one for which the equations developed are directly applicable, for it has unloaded backstays and a two-hinged stiffening truss. In what follows, the effect of temperature has not been considered, so that H_p is the unknown instead of H_s.

The dimensions of the structure, as taken from the report of the Mc Clintic-Marshall Company, are as follows:

$$f = 205.6 \text{ ft} \qquad A_c = 240.89 \text{ sqin}$$
$$l = 1850 \text{ ft} \qquad E_c = 27{,}000{,}000 \text{ lbs/sqin}$$
$$l_1 = 984.2 \text{ ft} \qquad E = 30{,}000{,}000 \text{ lbs/sqin}$$
$$l_2 = 833.9 \text{ ft}$$

From this data and from information taken from the report, it is found that

$$\alpha_1 = 20^0\text{—}32 \text{ ft}$$
$$w = 6200 \text{ lbs/ft. for the East Cable}^{18)} \qquad \alpha_2 = 24^0$$
$$H_w = 12{,}920{,}000 \text{ lbs for the East Cable}^{18)} \qquad I = 113.71 \text{ ft.}^{19)}$$

From the preceding development of equations for H_p there are available for analysis three methods, represented by eqs. (60), (64) and (65). In the derivation of eqs. (60) and (64) it was assumed that the deflection of the cable was proportional to the hanger forces acting upon it and that the live load produced a hanger force whose magnitude was constant from one end of the structure to the other. The only difference in these two equations is that in eq. (60), η as given by eq. (5) was used and in eq. (64), η as given by eq. (9) was employed. Since the equations differ only in the representation of η any numerical problem that is worked out by both methods will show how many terms of the series must be employed in eq. (64) to give results comparable with eq. (60). However, the question of how many terms must be used is one of practical importance and is worthy of presentation. Eq. (65) makes one less assumption than the other two, since it takes into consideration the actual distribution of live load to the cable.

The Detroit-Windsor Bridge will be analyzed by eqs. (60) and (64) for live loads of 200, 400, 600, 800, 1000, 1200, 1400, 1600, 1800 and 2000 lbs. per foot. This analysis will be for values of H_p and η and will determine how these quantities vary with the live load; and will show what, if any, error there is in the assumption that the deflection is proportional to the load. The distribution of these live loads will be as follows:

1. Over entire span, $k_1 = 0$, $k_2 = 1$
2. End half, $k_1 = 0$, $k_2 = .5$
3. Center half, $k_1 = .25$, $k_2 = .75$
4. End quarter, $k_1 = 0$, $k_2 = .25$
5. Quarter nearest center, $k_1 = .25$, $k_2 = .50$
6. Center quarter, $k_1 = .375$, $k_2 = .625$.

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[18]) There being only one sidewalk on this bridge, the loading on the East Cable is greater than on the West.

[19]) This calculation is based on the average areas of the top and bottom chords. No allowance was made for the effect of web members.

A. Analysis by Eq. (60).

Eq. (60) becomes, when the numerical data are substituted for the values of L[20]), A_c, E_c, etc.,

$$H_p^2(.0000006670) + H_p(130.48 - .0000002309\,K_2) + p\,(126{,}786{,}67\,K_1 + .00048058\,K_3) = 0. \qquad (66)$$

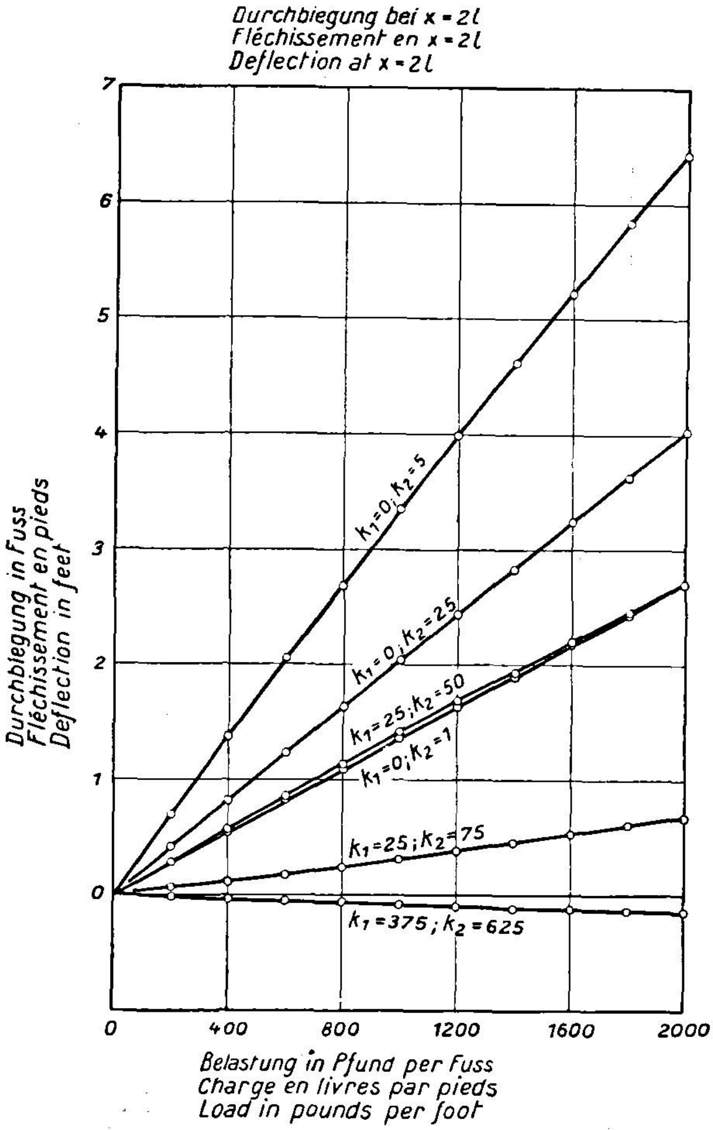

Fig. 9.

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20) $L = \int_0^l \frac{ds^3}{dx^2} + l_1 \sec^3 \alpha_1 + l_2 \sec^3 \alpha_2$, in which

$$\int_0^l \frac{ds^3}{dx^2} = \int_0^l \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} dx^3}{dx^2} = 2\int_0^{l/2}\left[1+\frac{64 f^2 x^2}{l^4}\right]^{3/2} dx$$

$$= l\left\{\frac{1}{4}\left(\frac{5}{2}+\frac{16 f^2}{l^2}\right)\left(1+\frac{16 f^2}{l^2}\right)^{1/2} + \frac{3l}{32 f}\,Log_e\left[\frac{4f}{l}+\left(1+\frac{16 f^2}{l^2}\right)^{1/2}\right]\right\}$$

From eq. (61),

When $k_1 = 0$, and $k_2 = 1$, $K_1 = -2.0$
„ $k_1 = 0$, „ $k_2 = .5$, $K_1 = -1.0$
„ $k_1 = .25$, „ $k_2 = .75$, $K_1 = -1.375$
„ $k_1 = 0$, „ $k_2 = .25$, $K_1 = -0.3125$
„ $k_1 = .25$, „ $k_2 = .50$, $K_1 = -0.6875$
„ $k_1 = .375$, „ $k_2 = .625$, $K_1 = -0{,}734375$.

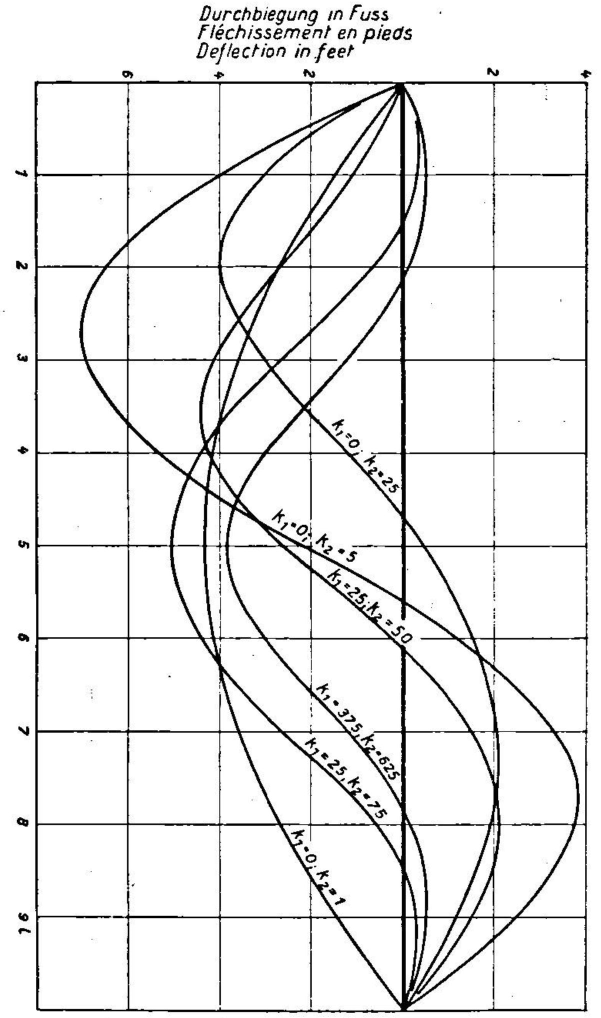

Fig. 10.

Charakteristische Biegungslinien. Belastung 2000 Pfund pro Fuß.
Courbes caractéristiques de fléchissement. Charge 2000 livres par pieds.
Typical defection curves. Load 2000 pounds per Foot.

With these values of K the last term of eq. (66) becomes for the 6 cases of distribution

1. $p\,(-253{,}573.34 + .00048058\,K_3)$
2. $p\,(-126{,}786.67 + .00048058\,K_3)$
3. $p\,(-174{,}331.67 + .00048058\,K_3)$
4. $p\,(-\ 39{,}620.83 + .00048058\,K_3)$
5. $p\,(-\ 87{,}165.83 + .00048058\,K_3)$
6. $p\,(-\ 93{,}108.96 + .00048058\,K_3)$

Since K_2 and K_3 depend upon "a" for their value and "a" depends upon H_p, a cut and try procedure must be resorted to for solution. However, a ready procedure suggests itself. The coefficient of K_2 is so small that K_2 must be very large to materially affect the value of the second term of eq. (66). Actually K_2 had values ranging between 44,251,200 and 54,798,800, but when multiplied by its coefficient the product added to the second term amounts ranging between 10.22 and 12.49. If eq. (66) is solved in the usual quadratic form where $A = .0000006670$, $B = 130.480 - .0000002309 K_2$ and $C = 126{,}786.67 K_1 + 00048058 K_3$ so that

$$H_p = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

then it is apparent that although the coefficient of K_3 is much larger than that of K_2 the value of H_p cannot be changed very much by any value of K_3 because of the small value of A. If K_2 is considered zero then $B^2 = 17{,}025.24$. It was found that although K_3 hat values ranging between 54,104,300 and 10,483,400,4 AC ranged between 121.43 and 184.53. This shows that K_3 cannot materially change the value of $\sqrt{B^2 - 4AC}$. Therefore, as a first approximation, K_2 and K_3 were assumed equal to zero and the equation was solved for H_p. Then K_2 and K_3 were computed with this approximate value of H_p. These values of K_2 and K_3 were then used in eq. (66) to get another value of H_p. It was found that this second value of H_p was, for all practical purposes, exact. The value obtained by considering K_2 and K_3 equal to zero was very close to the final value and the second approximation could not be improved upon. The results of this method of solution are given in Table I. The values of H_p found by considering K_2 and K_3 equal to zero (H_p') are compared with the exact values and the maximum difference is shown. The almost exact proportionality between H_p and p is apparent from the figures.

Table I

Values of H_p in 1000 Pound Units

H_p' = Approximate Value, H_p = Exact Value

Live Load	$k_1 = 0$, $k_2 = 1$		$k_1 = 0$, $k_2 = \frac{1}{2}$		$k_1 = .25$, $k_2 = .75$		$k_1 = 0$, $k_2 = .25$		$k_1 = .25$, $k_2 = .50$		$k_1 = .375$, $k_2 = .625$	
	H_p'	H_p	H_p'	H_p	H_p'	H_p	H_p'	H_p	H_p'	H_p	H_p'	H_p
200	388	385	194	193	267	269*	61	59	134	134	143	144
400	774	768	388	385	533	536	121	117	267	268	285	288
600	1159	1150	581	577	798	803	182	175*	400	402	427	432*
800	1542	1531	774	768	1063	1069	243	234	533	536	569	575
1000	1924	1910	967	960	1327	1334	303	292	666	670	711	718
1200	2305	2287*	1159	1150	1590	1598	364	350*	798	803*	853	861
1400	2684	2664	1351	1341	1853	1862	424	409	931	936	994	1004
1600	3062	3038	1543	1531*	2115	2125	485	467	1063	1068	1135	1146
1800	3438	3412	1734	1721	2376	2387	545	525	1195	1201	1276	1288
2000	3812	3784	1924	1910	2637	2648	605	583	1327	1334	1417	1430
		*0.78		*0.78		*0.74		*4.00		*0.62		*1.16

* Maximum difference — % of Exact Value

After the evaluation of H_p the calculation of the deflection is made by use of eqs. (39), (41) and (43) with the proper integration constants evaluated from eqs. (54) to (59) inclusive. Calculations were made for each 10th point of the span, but only the results of the 0.2 point are shown, see Fig. 9. In Fig. 10 are shown typical deflection curves for the entire span, the live load in each instance being 2000 lbs/ft.[21]).

Fig. 9 shows how closely the deflections are proportional to the load. The scale of the figure is so small, however, that the proportionality appears almost perfect. Table II gives the deflections in feet for $x = .2\,l$ and is representative of the other points. It shows some deviation from perfect proportionality.

Table II

Live Load	$k_1 = 0$ $k_2 = 1$	$k_1 = 0$ $k_2 = .5$	$k_1 = .25$ $k_2 = .75$	$k_1 = 0$ $k_2 = .25$	$k_1 = .25$ $k_2 = .50$	$k_1 = .375$ $k_2 = .625$
200	0.2746	0 6926	0.0572	0.4108	0.2844	-0.0197
400	0.5463	1.3730	0.1174	0.8189	0.5663	-0.0383
600	0.8170	2.0422	0.1818	1.2262	0.8443	-0.0557
800	1.0878	2.6721	0.2488	1.6306	1.1183	-0.0724
1000	1.3579	3.3488	0.3157	2.0319	1.3903	-0.0879
1200	1.6226	3.9868	0.3852	2.4313	1.6602	-0.1021
1400	1.8936	4.6080	0.4554	2.8294	1.9282	-0.1150
1600	2.1617	5.2334	0.5274	3.2268	2.1915	-0.1264
1800	2.4278	5.8428	0.6080	3.6196	2.4521	-0.1371
2000	2.6927	6.4437	0.6782	4.0169	2.7104	-0.1464

B. Analysis by Eq. (64).

Eq. (64) becomes, when the proper values are used for L, A_c, E_c, etc.,

$$H_p = 848{,}602.69\left(a_1 + \frac{a_3}{3} + \frac{a_5}{5} + \frac{a_7}{7} + \cdots\right) \qquad (67)$$

Since this equation contains only the odd numbered coefficients of the trignometric series it was decided to carry out the solution with the coefficients a_1, a_3 and a_5. It will be shown that this gave values of H_p that compared favorably with the results of eq. (66). In order to show clearly how much each term contributed to the value of H_p, eq. (67) was used in the following form:

$$H_p = 848{,}602.69\, a_1 + 282{,}867.55\, a_3 + 169{,}720.53\, a_5. \qquad (68)$$

The coefficients a_1, a_3 and a_5 which are evaluated from eq. (16) contain $\beta = \frac{H_p}{H_w}$ so again a method of successive approximations must be used. After some experimentation a procedure was developed that gave very satisfactory results. First, a value of H_p or β was assumed, and then this value of β was used in eq. (16) to get values of a_1, a_3 and a_5. These in turn were substituted in eq. (68) to find H_p. This computed value of H_p dit not agree with the assumed value unless by accident the correct value of H_p had been guessed. This procedure was repeated again so that finally for two assumed values of β, (β_1 and β_2) two calculated values of H_p, (H_{p1} and H_{p2}) were obtained.

21) This loading is larger than one truss of the Detroit-Windsor Bridge carries or is designed to carry.

The next step in finding the correct value of H_p was graphical although an analytical procedure could have been used. The straight line $H_p = \beta H_w = 12{,}920{,}000\ \beta$ was drawn, see Fig. 11, and the points β_1, H_{p1} and β_2, H_{p2} plotted. When these two points were connected by a straight line, the intersection of this line with the line $H_p = 12{,}920{,}000\ \beta$ gave the correct value of H and β. As many as six points were plotted and always the calculated value of H_p lay on a straight line. This procedure was used in calculating 60 values of H_p, each of which was checked by finding the correct values of a_1, a_3 and a_5. These values are shown in Table III. The values of H_p from the graph are the values most nearly correct since the check values of H_p often changed several thousand pounds when H_p from the graph was changed a few hundred.

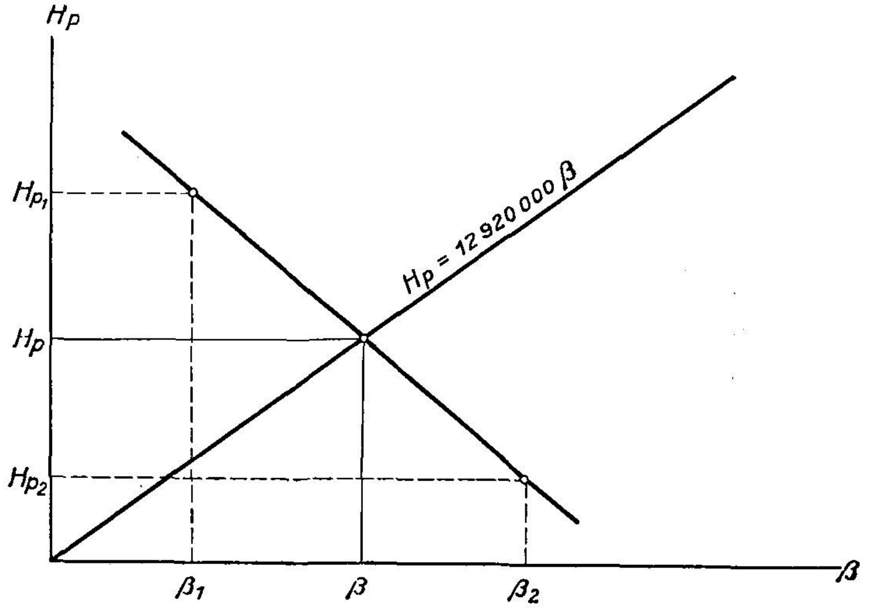

Fig. 11.

Table III

Values of H_p in 1000 Pound Units

H_p = Value from graph, H_p' = Check value

Live Load	$k_1 = 0$, $k_2 = 1$		$k_1 = 0$, $k_2 = \frac{1}{2}$		$k_1 = .25$, $k_2 = .75$		$k_1 = 0$, $k_2 = .25$		$k_1 = .25$, $k_2 = .50$		$k_1 = .375$, $k_2 = .625$	
	H_p	H_p'	H_p	H_p'	H_p	H_p'	H_p	H_p'	H_p	H_p'	H_p	H_p'
200	385	386	193	193	269	268	59	59	134	135	144	146*
400	770	765	385	383	536	538	117	117	269	268	289	287
600	1151	1160*	578	575	803	811*	176	174*	403	403	432	433
800	1534	1524	769	770	1069	1079	234	236	537	534	576	576
1000	1912	1915	961	956	1336	1331	292	293	671	666*	719	720
1200	2291	2281	1152	1149	1600	1601	351	352	804	804	863	857
1400	2668	2658	1343	1341	1863	1879	409	413	937	938	1005	1006
1600	3041	3054	1534	1524*	2127	2139	468	465	1070	1070	1148	1145
1800	3416	3419	1723	1722	2391	2383	526	526	1203	1203	1291	1286
2000	3789	3786	1913	1906	2651	2661	584	587	1335	1337	1433	1422
		*0.78		*0 65		*1.00		*1.15		*0.75		*1.39
* Maximum difference — % of H_p												

Table IV

Live Load — 2000 lbs/ft.

k_1	k_2	β (from graph)	H_p (from graph)	848,602,69 a_1	282,867.55 a_3	169,720.53a_5	H_p (Check)
0	1	.29328	3,789,000	3,755,177.05	28,525.47	2,089.30	3,785,791.88
0	.5	.14806	1,913,000	1,891,382.90	13,756.47	978.66	1,906,118.03
.25	.75	.20520	2,652,000	3,131,472.32	- 438,788.13	-31,587.26	2,661,096.93
0	.25	.04520	584,000	317,660.67	251,406.23	17,486.03	586,552.93
.25	.50	.10336	1,335,000	1,584,538.69	- 230,891.03	-16,268.04	1,337,379.62
.375	.625	.11095	1,433,000	1,840,594.54	- 432,763.79	13,971.38	1,421,802.13

The contribution of the various terms of eq. (68) to the value of H_p is shown in Table IV. The values are for a live load of 2000 lbs/ft. and are representative. It is seen that the term containing a_5 is contributing only about 1% to the final answer in most cases.

In calculating the deflections it was found necessary, especially in cases of unsymmetrical loading, to use five terms of eq. (9) to get values comparable with eq. (5). The values compare so well with five terms that to the scale of Figures 9 and 10 no differences would be noted. However, in Table V the deflections in feet are given for $x = .2\,l$ so that they can be compared directly with those in Table II.

The rate of convergence of the series can be seen from Table VI where the complete calculations for $x = .2\,l$ and $x = .3\,l$ are shown for the 2000 lbs/ft. loading. It is seen that $a_5 \sin \frac{5\pi x}{l}$ is contributing deflections of over one-tenth of a foot at $x = .3\,l$. The term $a_2 \sin \frac{2\pi x}{l}$ which appears in the unsymmetrical loading cases is large in comparison with the others, so that it must be obtained with great accuracy. Any discrepancies between Tables II and V may be accounted for by the use of insufficient terms in the series and by slight differences in the values of H_p.

Table V

Live Load	$k_1 = 0$ $k_2 = 1$	$k_1 = 0$ $k_2 = .5$	$k_1 = .25$ $k_2 = .75$	$k_1 = 0$ $k_2 = .25$	$k_1 = .25$ $k_2 = .50$	$k_1 = .375$ $k_2 = .625$
200	0.2744	0.6962	0.0594	0.4122	0.2879	- 0.0214
400	0.5440	1.3267	0.1229	0.7951	0.5445	- 0.0448
600	0.8249	2.0758	0.1908	1.1888	0.8622	- 0.0646
800	1.0842	2.7167	0.2582	1.5838	1.1277	- 0.0829
1000	1.3448	3.2418	0.3170	2.0394	1.3385	- 0.1000
1200	1.6241	3.8637	0.3906	2.3625	1.6768	- 0.1211
1400	1.8926	4.4757	0.4723	2.7514	1.8599	- 0.1322
1600	2.1746	5.0835	0.5435	3.1326	2.1137	- 0.1499
1800	2.4355	5.6719	0.6056	3.5173	2.3668	- 0.1634
2000	2.6870	6.2535	0.6937	4.0276	2.7363	- 0.1801

C. Analysis by Eq. (65).

Since it is probable that the effect of the actual distribution of the hanger loads is greatest with non-symmetrical loading, eq. (65) was used for the loading conditions where $k_1 = 0$; $k_2 = .5$ and $k_1 = 0$; $k_2 = .25$. The procedure for finding the correct value of H_p was exactly the same as when eq. (64) was used, and the results were satisfactory.

In Table VII are shown the values of H_p in 1000 lb. units obtained by eq. (65) and the comparison with the values of H_p from Table I which were obtained from eq. (60). It is apparent that the actual distribution of live load did not increase the value of H_p more than 1 %.

Table VI

Deflections for $x = .2\,l$, Load — 2000 lbs/ft.

k_1	k_2	$a_1 \sin \frac{\pi x}{l}$	$a_2 \sin \frac{2\pi x}{l}$	$a_3 \sin \frac{3\pi x}{l}$	$a_4 \sin \frac{4\pi x}{l}$	$a_5 \sin \frac{5\pi x}{l}$	η
0	1	2.60105	0	0.08591	0	0	2.6870
0	.5	1.31008	4.89721	0.04625	0	0	6.2535
.25	.75	2.16903	0	-1.47530	0	0	0.6937
0	.25	0.22003	2.73806	0.84528	0.22423	0	4.0276
.25	.50	1.09754	2.63478	-0.77630	-0.21966	0	2.7363
.375	.625	1.27490	0	-1.45504	0	0	-0.1801

Deflections for $x = .3\,l$, Load — 2000 lbs/ft.

k_1	k_2	$a_1 \sin \frac{\pi x}{l}$	$a_2 \sin \frac{2\pi x}{l}$	$a_3 \sin \frac{3\pi x}{l}$	$a_4 \sin \frac{4\pi x}{l}$	$a_5 \sin \frac{5\pi x}{l}$	η
0	1	3.58002	0	0.03116	0	-0.01231	3.5989
0	.5	1.80316	4.89721	0.01503	0	-0.00577	6.6916
.25	.75	2.98541	0	-0.47936	0	0.18611	2.6922
0	.25	0.30284	2.73806	0.27465	-0.22423	-0.10303	2.9884
.25	.50	1.51063	2.63478	-0.25224	0.21966	0.09586	4.2087
.375	.625	1.75474	0	-0.47277	0	-0.08232	1.1996

Table VII

Live Load	$k_1 = 0$; $k_2 = .5$			$k_1 = 0$; $k_2 = .25$		
	Eq. (65)	Eq. (60)	% Diff.	Eq. (65)	Eq. (60)	% Diff.
200	193	193	0	59	59	0
400	386	385	0.26	117	117	0
600	579	577	0.35	176	175	0.57
800	773	768	0.65	235	234	0.43
1000	966	960	0.62	294	292	0.68
1200	1159	1150	0.78	353	350	0.86
1400	1352	1341	0.82	412	409	0.73
1600	1545	1531	0.92	472	467	1.07
1800	1738	1721	0.99	531	525	1.14
2000	1931	1910	1.10	591	583	1.37
	Average difference 0.65			Average difference 0.68		

Conclusions.

It is not possible to draw any far reaching conclusions from the analysis of one structure, but from the analysis of the Detroit-Windsor Bridge the following conclusions may be drawn:

1. Eq. (60) which is based on the Melan Methods leads to values of the horizontal component of cable stress with a minimum of effort. The results show that a value of H_p, that is exact enough for purposes of estimating costs, may be obtained directly, and that in one additional step the result may be obtained exactly. It is felt that this equation is much easier to use than others derived for the same purpose.

2. Eq. (60) shows, since the moment of inertia of the stiffening truss is involved only in the terms K_2 and K_3 which can be called zero with little error, that the size of the truss has little effect on the value of H_p. This also means, since the truss is the distributing medium of the live load, that consideration of the actual live load distribution to the cable can produce only small changes in H_p except perhaps in the case of concentrated loads. That this was true was shown by the numerical problem.

3. In the series methods of Professor Timoshenko it was found necessary to use three terms, and coefficients a_1, a_3 and a_5 to obtain satisfactory values of H_p; and five terms, and coefficients a_1, a_2, a_3, a_4 and a_5 to get satisfactory values of η. The procedure used for finding H_p is quite direct.

4. The advantages and disadvantages of the two methods of solution can be summarized as follows:

a) The complete derivation of the equations for deflection and for H_p is a great deal easier and quicker by the series method than by the Melan method.

b) After the equations for H_p have been found, it is simpler to find the exact values of H_p for any loading condition by the Melan method than by the the series method.

c) However, when H_p is once determined in any given numerical case the deflections can be found more readily by the series method than by the Melan method.

d) The series method ist the only practical way in which the effect of the actual load distribution can be found.

5. A method of solution that works out quickly for finding deflections in the stiffening truss for any loading condition is as follows:

1. Find H_p by eq. (60).
2. Find η by eq. (9).
3. Find bending moments by the substitution of the numerical values of η in eq. (2).

Summary.

The author undertakes to simplify the existing methods of calculation and formulae in use for suspension bridges. He is of opinion that often inadequate attention has been given to the works of earlier writers on this subject and that the nomenclature should be standardised. The author by correlating his work with earlier publications shows at the same time how the various methods differ. He scrutinises the assumptions forming the basis of existing methods with the object of finding their effects on the final result. For each method of calculation a shorter solution is given.

The stiffened suspension bridges as built today are statically indeterminate systems which would not cause much difficulty in calculation if it were possible to start by choosing such dimensions as would satisfy all conditions of loading and changes in temperature. Wide-span constructions are subject to strong distorsions due to live loads and temperature changes, so that errors will be committed if such influences are left without sufficient consideration in the method of calculation. These errors affect the economy of the structures more than its safety. Special mention is made of methods which reduce the amount of algebraic work. The procedure adopted by the author can be applied to all suspension bridges stiffened with a horizontal truss. The only difficulties are to be found in extra algebraic work.

Résumé.

L'auteur cherche à simplifier les méthodes de calcul et formules existantes concernant le calcul des ponts suspendus. Il est d'avis que l'on ne s'est le plus souvent pas référé aux travaux précédents et que l'on ne s'est pas basé sur ces derniers; il estime que les notations doivent être unifiées. C'est pour cette raison que l'auteur s'efforce de mettre son travail en rapport avec des publications précédentes, et de montrer où les diverses méthodes diffèrent. Il examine à fond les diverses hypothèses qui sont à la base des méthodes de calcul, afin de déterminer l'influence qu'elles exercent sur le résultat final. Pour chaque méthode de calcul, l'auteur donne un développement un peu plus court de la solution, comparé à ce qui a été fait jusqu'à présent.

Les ponts suspendus raidis, tels qu'on les construit actuellement, sont des constructions hyperstatiques, dont le calcul ne présenterait pas de grandes difficultés, si l'on pouvait choisir dès le début des dimensions telles qu'elles satisfassent à toutes les surcharges et variations de température. Les constructions de grande portée sont sujettes à de grandes variations dues aux surcharges mobiles et aux changements de température; si l'on ne tient pas suffisamment compte de ces influences, il en résulte de grandes erreurs. Ces dernières concernent la rentabilité plus que la sécurité. L'auteur fait ressortir surtout les méthodes de calcul qui, selon son opinion, donnent moins à calculer. Le procédé utilisé s'applique à tous les ponts suspendus raidis par des poutres horizontales à treillage. Les seules difficultés qu'il présente résident dans le calcul algébrique plus long.

Zusammenfassung.

Der Verfasser setzt sich zur Aufgabe, eine Vereinfachung der bestehenden Berechnungsmethoden und Formeln zur Berechnung der Hängebrücken herbeizuführen. Er ist der Ansicht, daß des öftern keine Bezugnahme und kein Aufbau auf frühere Arbeiten stattgefunden hat und daß die Bezeichnungen vereinheitlicht werden sollen. Der Verfasser bemüht sich daher, seine Arbeit mit früheren Veröffentlichungen in Verbindung zu bringen und dabei zu zeigen, worin sich die verschiedenen Methoden unterscheiden. Die den verschiedenen Berechnungsannahmen zu Grunde gelegten Annahmen werden eingehender untersucht, um festzustellen, was für einen Einfluß sie auf das Endresultat haben. Für jede Berechnungsmethode wird eine etwas kürzere Fassung der Lösung als bis anhin angegeben.

Die versteiften Hängebrücken, wie sie heute gebaut werden, sind statisch unbestimmte Bauwerke, deren Berechnung keine großen Schwierigkeiten hätten,

könnte man von Anfang an solche Dimensionen wählen, die allen Belastungen und Temperaturänderungen genügen.

Weitgespannte Konstruktionen sind großen Veränderungen durch Verkehrslasten und Temperaturänderungen unterworfen, sodaß große Irrtümer begangen werden, wenn solche Einflüsse nicht genügend Berücksichtigung finden. Diese beziehen sich mehr auf die Wirtschaftlichkeit als auf die Sicherheit. Diejenigen Berechnungsmethoden, die nach Ansicht des Verfassers weniger Rechenarbeit erfordern, sind besonders hervorgehoben. Das angewendete Verfahren kann auf alle Hängebrücken mit horizontalen Gitter-Versteifungsträgern angewendet werden. Die einzigen Schwierigkeiten sollen in der algebraischen Mehrarbeit bestehen.