Spherical shells subjected to axial symmetrical bending

SPHERICAL SHELLS SUBJECTED TO AXIAL SYMMETRICAL BENDING.

KUGELSCHALEN, AUF AXIAL-SYMMETRISCHE BIEGUNG BEANSPRUCHT.

COUPOLES MINCES SPHÉRIQUES SOUMISES À UNE FLEXION AXIALE SYMMÉTRIQUE.

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Introduction.

The exact solution of the differential equations for this problem is known and it implies the application of hypergeometric series [1]). Since this exact solution is not well fitted for practical computations, an attempt has been made in this paper to obtain approximate solutions, adaptable to design.

The first of these suggestions was to retain only the highest (second) derivative of the unknown quantities on the left side of the differential equations (A). This simplification is permissible for very thin shells with large angle of opening and gives very simple formulas for the computation of stresses (Approx. I) [2]).

A more accurate approximation (Approx. II) can be obtained by taking into account also the first derivatives (in addition to the second derivatives) and neglecting only the functions themselves on the left side of the differential equations (A) [3]). This Approx. II found application so far only in computing the displacements of the edge of the shell due to edge loads (influence coefficients) [4]).

This paper presents for the first time complete formulas for all the unknown quantities in Approx. II in addition to the already known Approx. I.

Derivation of the Differential Equations for the Problem.

Let us suppose that a segment of a spherical shell is subjected to forces and bending moments uniformly distributed along a hoop circle, notably around the edge of the shell (fig. 1 a). The forces and moments so applied will produce an axial symmetrical bending of the shell.

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[1]) a) REISSNER, H.: Spannungen in Kugelschalen, Müller - Breslau, Festschrift, S. 181, Leipzig, 1912. b) MEISSNER, E.: Das Elastizitätsprob'em für dünne Schalen, Physik Bd. 14 (1913), S. 343. c) MEISSNER, E.: Über Elastizität und Festigkeit dünner Schalen, Vjschr. naturforsch. Ges. Zürich. Bd. 60 (1915), S. 23.

[2]) GECKELER, J. W.: Über die Festigkeit achsen-symmetrischer Schalen, Forsch.-Arb. Ingwes. H. 276, Berlin, 1926.

[3]) BLUMENTHAL, O.: Über die asymptotische Integration, etc. Zeitschr. f. Math. Physik, Bd. 62 (1914), S. 343.

[4]) a) PASTERNAK, P.: Formeln zur raschen Berechnung der Biegebeanspruchung in kreisrunden Behältern, Schweiz. Bauzeit., Bd. 86 (1925), S. 129. b) PASTERNAK, P.: Die praktische Berechnung biegefester Kugelschalen etc., Zeitschr. f. ang. Math. u. Mech., Bd. 6 (1926), S. 1. c) GECKELER, J. W.: Zur Theorie der Elastizität flacher rotationssymmetrischer Schalen, Ing. Archiv., Bd. 1 (1930), S. 255.

Considering the equilibrium of a small rectangular element of this shell, enclosed between two neighboring meridians and two neighboring hoop-circles (fig. 1 b), we find that on the sides of this element normal forces N_1, N_2, bending moments M_1, M_2, and shearing force Q will be acting. In fig. 1 c the positive directions of these forces and moments are shown. Subscripts 1 and 2 refer to the arcs of the hoop and meridional circle respectively and the above forces and moments will be assumed to act on a unit length of arc of the corresponding circle. Other force components will vanish in consequence of the axial symmetry of the loading.

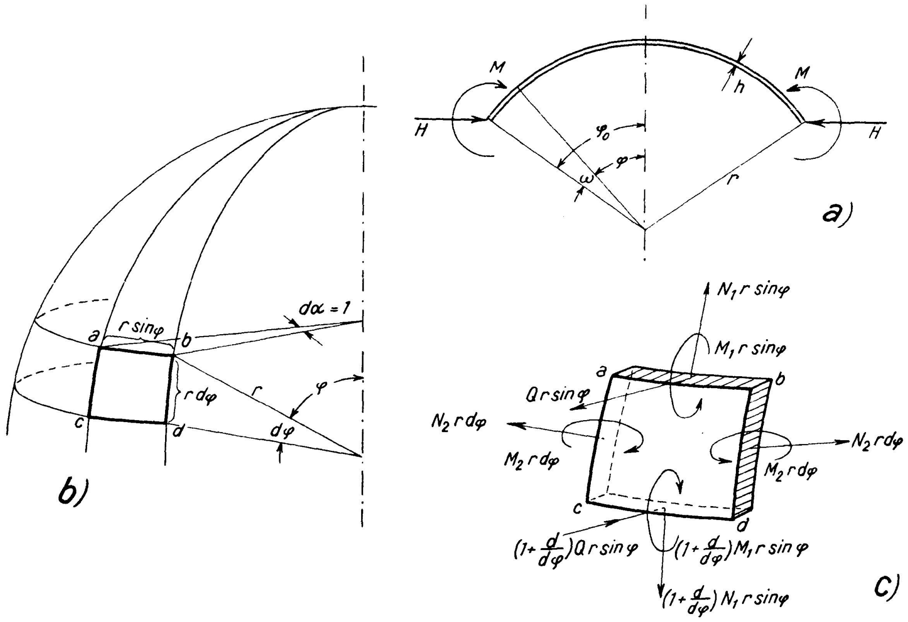

Fig. 1.

The complete solution of the problem is obtained if, besides the above mentioned five quantities, also two displacement components are known. From these two additional unknowns will be chosen the change in slope of the meridional circle due to bending: Θ (positive when accompanied by an increase in the radius of curvature) and the horizontal displacement of any point of the meridian: u (positive when accompanied by an increase in the radius of the corresponding hoop circle).

In order to determine these seven unknown quantities we shall have at our disposal three equations of equilibrium, while four additional equations will be furnished by the existing conditions between stresses and deformations. The first equation for equilibrium can be obtained from the consideration that, since the shell is subjected only to the self-balancing edge forces shown in fig. 1 a, along each hoop circle, the vertical resultant of the N_1 and Q forces must vanish, that is:

$$N_1 \sin\varphi + Q \cos\varphi = 0 \quad \text{or} \quad N_1 = -Q \cot\varphi \tag{1}$$

Considering now the element in fig. 1 c and projecting the forces in the direction of the normal to the element, and neglecting small quantities of second order, we have the next condition as

[5]) See also the book by Dr. W. Flügge: Statik und Dynamik der Schalen, Berlin, 1934, Verlag J. Springer.

$$\frac{d}{d\varphi}(Q\sin\varphi) + (N_1 + N_2)\sin\varphi = 0$$

Carrying out the assigned differentiation and substituting the result into eq. (1) we get

$$N_2 = -Q'\,^{6)} \tag{2}$$

The third equation of equilibrium will be obtained by projecting the moments on the plane of the meridian (plane of bending); thus getting

$$\frac{d}{d\varphi}(M_1\sin\varphi) - M_2\cos\varphi - Qr\sin\varphi = 0$$

or

$$M_1' + (M_1 - M_2)\cot\varphi = Qr \tag{3}$$

Proceeding now to the relations between stresses and deformations, we can write

$$M_1 = D(\varrho_1 + \mu\varrho_2) \tag{a}$$
$$M_2 = D(\varrho_2 + \mu\varrho_1) \tag{b}$$

where

$$D = \frac{Eh^3}{12(1-\mu^2)}$$

denotes the flexural rigidity of the element under consideration (μ is Poisson's ratio) and ϱ_1 and ϱ_2 designate the decrease in the principal curvatures due to bending.

Since the change in curvature along the meridian will be the difference between the slopes of two neighboring normals (at b and d in fig. 1 b) we have

$$\varrho_1 = \frac{d\Theta}{r\,d\varphi} = \frac{\Theta'}{r} \tag{c}$$

A similar expression can be obtained in the other direction, by considering that on the hoop circle the two neighboring normals (at points a and b on fig. 1 b) will change in consequence of the bending of their angle of intersection from

$$\frac{\sin\varphi}{r\sin\varphi} \quad \text{to} \quad \frac{\sin(\varphi+\Theta)}{r\sin\varphi}$$

This gives a change of curvature of the hoop circle:

$$\varrho_2 = \frac{\sin(\varphi+\Theta) - \sin\varphi}{r\sin\varphi} = \Theta\frac{\cot\varphi}{r} \tag{d}$$

where the assumption was made that Θ being small we can put $\sin\Theta \sim \Theta$ and $\cos\Theta \sim 1$.

Substituting the above expressions for ϱ_1 und ϱ_2 into equations (a) and (b), we get:

$$M_1 = D\frac{1}{r}(\Theta' + \mu\Theta\cot\varphi) \tag{4}$$

and

$$M_2 = D\frac{1}{r}(\Theta\cot\varphi + \mu\Theta') \tag{5}$$

[6]) The symbols for derivatives will be understood with respect to the variable φ (measured in radians).

Two additional equations can be obtained by making use of the correlations between normal forces and the corresponding strain components:

$$N_1 = \frac{Eh}{1-\mu^2}(\varepsilon_1 + \mu\,\varepsilon_2) \tag{e}$$

$$N_2 = \frac{Eh}{1-\mu^2}(\varepsilon_2 + \mu\,\varepsilon_1) \tag{f}$$

The above strain components ε_1 and ε_2 can be expressed in terms of the v tangential and w radial displacements. In order to find this relationship let us consider first an infinitesimal AB portion of the meridional circle with an arclength ds_1 (fig. 2). Assuming v and w displacements for point A, also $(v + v'd\varphi)$ and $(w + w'd\varphi)$ displacements for point B, we find that the change of ds_1 is

$$\varepsilon_1\, ds_1 = (ds_1 + v'\,d\varphi)\frac{r+w}{r} - ds_1$$

and omitting, as before, small quantities of second order this gives

$$\varepsilon_1 = \frac{v'+w}{r} \tag{g}$$

From the same fig. 2 we can see that due to the same v and w displacements the radius of the hoop circle through A will increase from $r\sin\varphi$ to $r\sin\varphi + (v\cos\varphi + w\sin\varphi)$ and the corresponding unit strain ε_2 in the hoop circle be therefore

$$\varepsilon_2 = \frac{v\cot\varphi + w}{r} \tag{h}$$

With the aid of v and w we can express also the formerly introduced Θ, the change in slope of the meridian due to bending; namely, during the displacement of point A to A'' the slope changes from φ to $\varphi + \frac{v}{r}$. At point A' it will be modified again by $\frac{w'd\varphi}{r}$ so that the total change in slope at point A, while it displaces to A', will be

$$\Theta = \frac{w'-v}{r} \tag{i}$$

The fact that Θ as well as ε_1 and ε_2 can be expressed as functions of the u and v displacements, offers a possibility to eliminate the latter quantities, getting thus

$$\Theta = \varepsilon_1\cot\varphi + \varepsilon_2' \tag{k}$$

Expressing now ε_1 and ε_2 in terms of N_1 and N_2 from equations (e) and (f) we have a new correlation between Θ and the normal forces as

$$\Theta = \frac{1}{Eh}\{(N_2 - \mu N_1)' + (N_2 - N_1)(1+\mu)\cot\varphi\} \tag{6}$$

When deriving the above expression we made only partial use of the correlations implied in equations (e) to (h). One more independent equation can be obtained from the connection between the horizontal displacement u

and the normal forces; namely, on the basis of eq. (h) we can write directly that:

$$u = \frac{r \sin \varphi}{E h} (N_2 - \mu N_1) \tag{7}$$

In equations (1) to (7) we have at our disposal all the necessary data for solving the problem [7]). It is seen that all the unknown quantities can be expressed by two variables Θ and Q. By doing this we get two simultaneous differential equations:

$$\left.\begin{aligned} \Theta'' + \Theta' \cot \varphi - \Theta(\cot^2 \varphi + \mu) &= \frac{r^2}{D} Q \\ Q'' + Q' \cot \varphi - Q(\cot^2 \varphi - \mu) &= - E h \, \Theta \end{aligned}\right\} \tag{A}$$

which express in concise form the problem of axial-symmetrical bending of spherical shells.

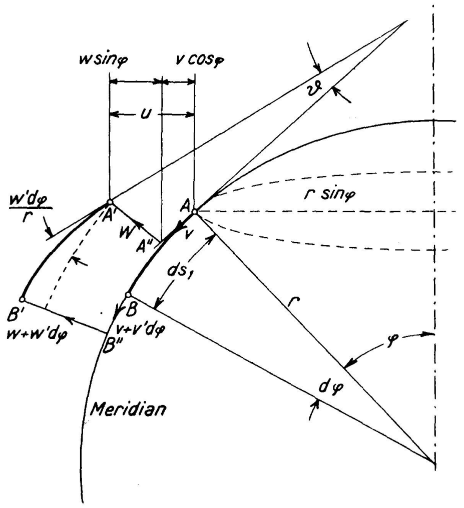

Fig. 2.

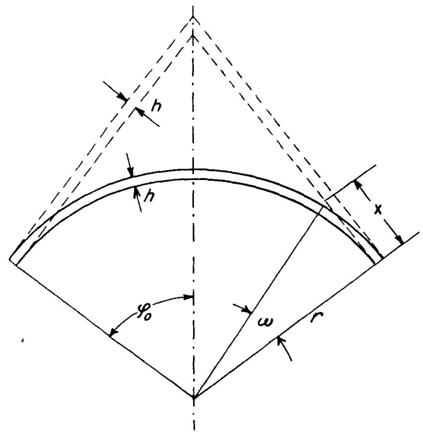

Fig. 3.

It has been shown by E. Meissner that equations (A) can be transformed into such form that their exact solution is obtainable by means of hypergeometric series (see references 1 and 5). But the use of these non-tabulated power series involves considerable computing work and on the other hand the thickness (h) appearing as negative exponent makes the series less convergent, more particularly for thin walled shells which are most important in practice. These points made it necessary to establish some useable approximate method for analysing the problem.

In the following two approximate solutions will be discussed. The first one was established by J. W. Geckeler (reference 2) and is submitted here

[7]) The v and w displacements can be determined also from equations (e) to (h), getting

$$v = \frac{r}{E h} \sin \varphi \cdot J \quad \text{and} \quad w = \frac{r}{E h} (N_2 - \mu N_1 - \cos \varphi \cdot J)$$

where
$$J = \int_0^{\varphi} \frac{1 + \mu}{\sin \varphi} (N_1 - N_2) \, d\varphi$$

In these formulas the vertex ($\varphi = 0$) is considered as the immovable origin to which the displacements are referred.

only for the sake of comparison with the second one which is here presented in complete form for the first time to the writer's knowledge.

Both of these approximate solutions are based on the fact that in thin shells the effect of bending by edge loads is rapidly diminishing when progressing from the edge toward the vertex of the shell.

The consequence of this fact is that on the left side of equations (A) the higher derivatives will be of much greater magnitude than the lower ones and so it is to be expected that in some cases even the entire omission of the latter ones may furnish approximate results not differing appreciably from the exact solution.

Approximation I.

For very thin shells, with a large angle of opening (φ_0) it is permissible to retain only the second derivatives on the left side of equations (A) and neglect all the other terms. The range of this approximation and the amount of the involved errors will be discussed later; first we shall present the formulas through this simplification.

Equations (A) will now reduce to

$$\Theta'' = \frac{r^2}{D} Q \quad \text{and} \quad Q'' = -Eh\,\Theta \tag{A$_\mathrm{I}$}$$

from which eliminating Θ we get

$$Q^{IV} + 4\lambda^4 Q = 0 \tag{8}$$

where
$$\lambda^4 = 3(1-\mu^2)\left(\frac{r}{h}\right)^2$$

The general solution of eq. (8) is

$$Q = C_1 e^{\lambda\varphi}\cos\lambda\varphi + C_2 e^{\lambda\varphi}\sin\lambda\varphi + C_3 e^{-\lambda\varphi}\cos\lambda\varphi + C_4 e^{-\lambda\varphi}\sin\lambda\varphi \tag{9}$$

From the existing conditions at the vertex (namely that at $\varphi = 0$; $Q = 0$ and $\Theta = -\frac{1}{Eh}Q'' = 0$) we find that $C_1 = -C_3$ and $C_2 = C_4$[8]).

In determining the two remaining integration constants it is of convenience to introduce in the place of φ a new variable ω (see fig. 1 a) such that $\varphi = \varphi_0 - \omega$. Thus, after some trigonometrical transformation the solution of (8) can be put in the form

$$Q = Ce^{-\lambda\omega}\sin(\lambda\omega + \psi) \tag{Ia}$$

where C and ψ are constants of integration. The above equation represents a damped wave with the period $\lambda\omega$ and this is the reason why λ is often called characteristic or damping factor. This λ, as well as $\lambda\cdot\omega$, represents absolute numbers, ω being measured in radians.

In a similar way we can obtain also Θ from the simplified equations (A).

$$\Theta = \frac{2\lambda^2}{Eh} Ce^{-\lambda\omega}\cos(\lambda\omega + \psi) \tag{Ib}$$

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[8]) If the shell is open at the top we must deal with four different integration constants. The result of it will be that we shall get, also, for the upper edge, expressions similar to the ones derived here for the lower edge of the shell. The same procedure holds also for Approx. II.

Substituting the above expressions of Q and Θ into equations (1) to (7), and considering at every place only the highest derivatives, we get the formulas for the other unknowns as:

$$\left.\begin{aligned} N_1 &= -\cot(\varphi_0 - \omega)\, C e^{-\lambda\omega} \sin(\lambda\omega + \psi) \\ N_2 &= -\lambda\sqrt{2}\, C e^{-\lambda\omega} \sin\left(\lambda\omega + \psi - \frac{\pi}{4}\right) \\ M_1 &= \frac{r}{2\lambda}\, C e^{-\lambda\omega} \sin\left(\lambda\omega + \psi + \frac{\pi}{4}\right) \\ M_2 &= \mu M_1 \\ u &= -\frac{r\sin(\varphi_0 - \omega)}{Eh}\, \lambda\sqrt{2}\, C e^{-\lambda\omega} \sin\left(\lambda\omega + \psi - \frac{\pi}{4}\right) \end{aligned}\right\} \quad (\mathrm{Ic-g})$$

When designing such shells it is convenient to know the rotation and displacement of the edge due to the unity of the respective edge loading (see fig. 1 a). These displacement components are easily obtained from the above equations. Denoting by

a_{11} = rotation of the edge due to $M = 1$ in. lb./in.,
a_{12} = horizontal displacement of the edge due to $M = 1$,
a_{21} = rotation of the edge due to $H = 1$ lb./in. and
a_{22} = horizontal displacement of the edge due to $H = 1$,
we find that:

$$a_{11} = \frac{4\lambda^3}{Erh}; \quad a_{12} = a_{21} = \frac{2\lambda^2 \sin\varphi_0}{Eh}; \quad a_{22} = \frac{2\lambda r \sin^2\varphi}{Eh} \quad (\mathrm{Ih})$$

The present approximation is based on eq. (8) which is in turn the exact differential equation for the axial-symmetrical bending of cylindrical shells. From this we find the physical interpretation of Approx. I; namely, that here the original spherical shell was replaced by a cylindrical one. As a consequence of this, the angle of opening of the spherical shell (φ_0) has no effect on the stress distribution calculated on the basis of this approximation.

Approximation II.

This approximation, which can be considered as an improved form of the former one, takes into account not only Θ'' and Q'' but also Θ' cot φ and Q' cot φ, on the left side of equations (A). This can be done by substituting:

$$\Theta = \frac{1}{\sqrt{\sin\varphi}}\bar{\Theta} \quad \text{and} \quad Q = \frac{1}{\sqrt{\sin\varphi}}\bar{Q}$$

into equations (A). As a result of this substitution the terms containing Θ' and Q' will cancel out; now we will have to neglect only the terms of Θ and Q in order to bring equations (A) in the simplified form:

$$\begin{aligned} \bar{\Theta}'' &= \frac{r^2}{D}\bar{Q} \\ \bar{Q}'' &= -Eh\bar{\Theta} \end{aligned} \quad (\mathrm{A_{II}})$$

The solution of the above equations can be obtained in the same manner in which it was done in Approx. I. Returning then to the original Q and Θ quantities and using ω as variable we get now:

$$Q = C \frac{e^{-\lambda \omega}}{\sqrt{\sin(\varphi_0 - \omega)}} \sin(\lambda \omega + \psi) \tag{IIa}$$

and

$$\Theta = \frac{2 \lambda^2}{Eh} C \frac{e^{-\lambda \omega}}{\sqrt{\sin(\varphi_0 - \omega)}} \cos(\lambda \omega + \psi) \tag{IIb}$$

Substituting these expressions into equations (1) to (7), and taking into account not only the derivatives as before, but also the functions themselves, we shall have the formulas for the remaining unknowns as:

$$\left.\begin{aligned}
N_1 &= -\cot(\varphi_0 - \omega)\, C \frac{e^{-\lambda \omega}}{\sqrt{\sin(\varphi_0 - \omega)}} \sin(\lambda \omega + \psi) \\
N_2 &= C \frac{e^{-\lambda \omega}}{2\sqrt{\sin(\varphi_0 - \omega)}} [2 \cos(\lambda \omega + \psi) - (k_1 + k_2) \sin(\lambda \omega + \psi)] \\
M_1 &= \frac{r}{2 \lambda} C \frac{e^{-\lambda \omega}}{\sqrt{\sin(\varphi_0 - \omega)}} [k_1 \cos(\lambda \omega + \psi) + \sin(\lambda \omega + \psi)] \\
M_2 &= \frac{r}{4 \mu \lambda} C \frac{e^{-\lambda \omega}}{\sqrt{\sin(\varphi_0 - \omega)}} \{[(1+\mu^2)(k_1 + k_2) - 2 k_2] \cos(\lambda \omega + \psi) + 2 \mu^2 \sin(\lambda \omega + \psi)\} \\
u &= \frac{r \sin(\varphi_0 - \omega)}{Eh} C \frac{\lambda e^{-\lambda \omega}}{\sqrt{\sin(\varphi_0 - \omega)}} [\cos(\lambda \omega + \psi) - k_2 \sin(\lambda \omega + \psi)]
\end{aligned}\right\} \text{IIc–g}$$

where:

$$k_1 = 1 - \frac{1 - 2 \mu}{2 \lambda} \cot(\varphi_0 - \omega)$$

and

$$k_2 = 1 - \frac{1 + 2 \mu}{2 \lambda} \cot(\varphi_0 - \omega)$$

The previously discussed influence coefficients will take, now, the modified form [9]:

$$a_{11} = \frac{4 \lambda^3}{Erh} \frac{1}{k_1}; \quad a_{12} = a_{21} = \frac{2 \lambda^2 \sin \varphi_0}{Eh} \frac{1}{k_1}$$

and

$$a_{22} = \frac{\lambda r \sin^2 \varphi_0}{Eh} \left(k_2 + \frac{1}{k_1}\right) \tag{IIh}$$

Since in Approx. I the original sphere was substituted by a cylindrical one, in an analogous way it can be expected that Approx. II will be useable for analysing conical shells, the geometrical similarity between sphere and

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[9]) P. Pasternak, in his quoted publication (ref. 4a and 4b) derives formulas for the influence factors using this Approx. II, but his expression for a_{22} contains an error, namely the second term in the parenthesis should be multiplied by s.
The correct formula (with his notations) will be

$$E a_{22} = \frac{r}{h s} \sin \alpha \left[\frac{\omega_2}{\omega_1} \sin \alpha - \nu s \cos \alpha\right],$$

which differs only by a small quantity of higher order from the expression of a_{22} given here by equation (IIh).

cone being even greater than the one between sphere and cylinder. It is only necessary to put $\omega = \frac{x}{r}$ (fig. 3) into formulas II a—h and they can be used for approximate calculation of conical shells. (See also ref. 4 a.)

Example.

Let us consider a spherical shell, fixed along the edge, having the dimensions $r = 90$ in.; $h = 3$ in.; $\varphi_0 = 35^0$ and assume that this shell is subjected to a $p = 1$ lb./sq. in. uniformly distributed radial pressure (fig. 4 a). For the material we shall take reinforced concrete with Poisson's ratio $\mu = 1/6$.

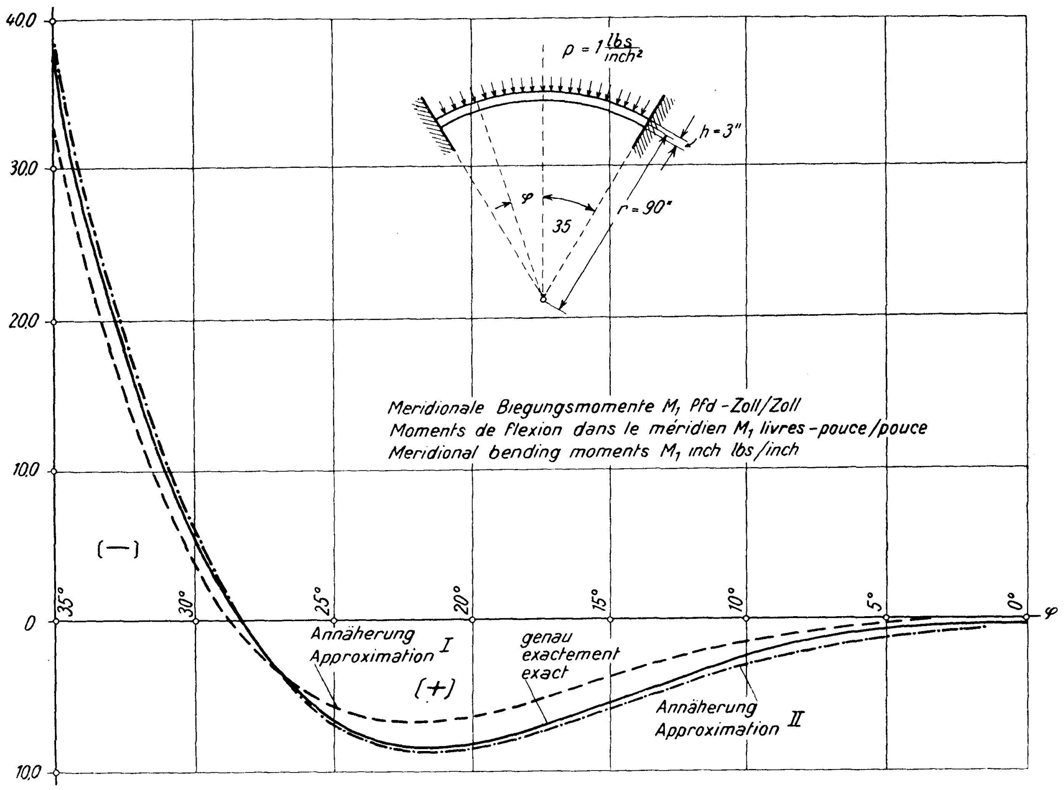

Fig. 4.

If the edge were free, then only constant $N_1 = N_2 = \frac{pr}{2} = 45$ lb./in. membrane forces would be produced over the entire surface of the shell, and this would result at the edge in a $-Eu = \frac{r \sin \varphi_0}{h} (N_2 - \mu N_1) = 645.03$ lb./in. horizontal displacement.

Due to the restraint of the edge any rotation or displacement is prevented there and, as a consequence of this, bending will be set up.

This bending can be calculated by the presented approximate methods using the edge conditions $\Theta = 0$ and $Eu = 645.03$ for determining the constants of integration.

The computation has been carried out by Approximations I and II and also, for sake of comparison, by the exact method with hypergeometric series. The amount of the computing work, when using these three different methods, was, roughly, of the ratio 1 : 2 : 20. With the exact method 10 terms of the series had to be considered to obtain sufficient accuracy. The obtained values for M_1 and N_2 are given in the tables below and also shown in figures 4 and 5.

Meridional Moments M_1 in. lb./in.

φ^0	Approx. I	Approx. II	Exact
35	-32.924	-37.978	-37.675
30	- 3.992	- 5.958	- 5.756
25	5.973	6.826	6.687
20	6.333	8.538	8.135
15	3.789	6.022	5.451
10	1.476	3.079	2.364
5	0.195	1.273	0.377
0	- 0.254	∞ [10])	0.294

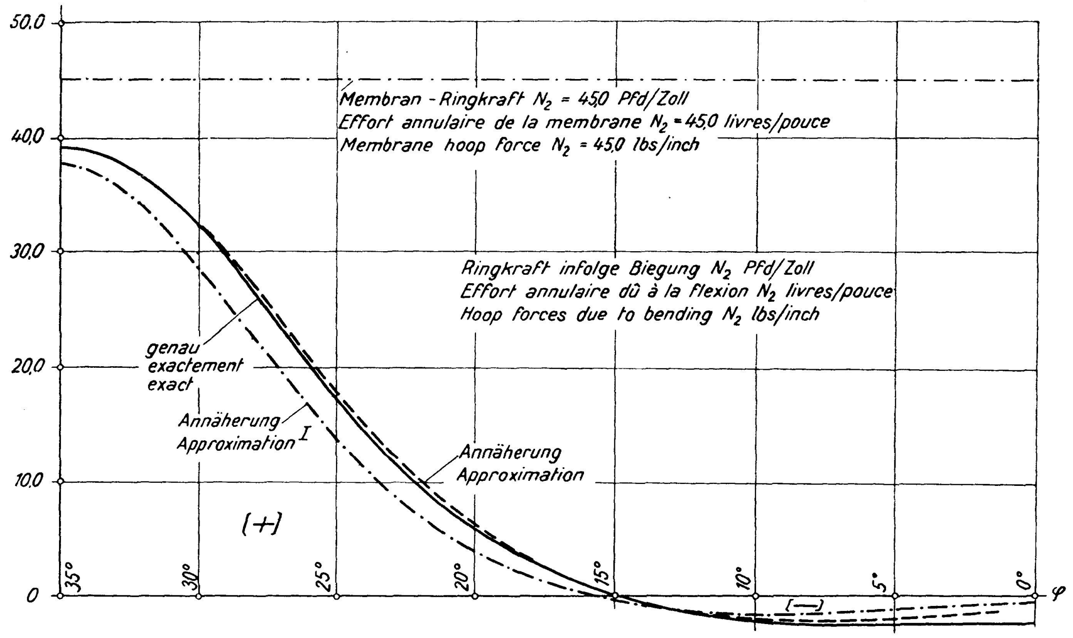

Fig. 5.

Hoop Forces due to Bending N_2 lb./in.

φ^0	Approx. I	Approx. II	Exact
35	37.486	38.926	38.920
30	28.021	32.184	31.900
25	13.592	17.582	17.258
20	3.774	6.131	5.950
15	- 0.618	0.043	- 0 021
10	- 1.619	- 2.024	- 2.166
5	- 1.229	- 1.922	- 2.497
0	- 0.600	∞ [10])	- 2.456

The selected example has a relatively small angle of opening ($\varphi_0 = 35^0$) and with a $\frac{r}{h} = 30$ ratio it represents the thickest type of shell which occurs at modern reinforced concrete domes or containers. From the good accuracy obtained even in this case through Approx. II, we can conclude that this method can be adapted for the design of any reinforced concrete shell possibly encountered in the present structural practice.

[10]) The vertex ($\varphi = 0$) is a singular point in Approx. II but its local character does not affect the accuracy of the results at other points.

Range of Applicability of the Approximate Solutions.

From Approx. I (eq. (8)) we find that the ratio between Q and its derivatives in $\frac{Q''}{Q'} = \frac{Q'}{Q} = \lambda \sqrt{2}$. Assuming that the same ratios will hold also for the exact solution we have means with which to estimate the magnitude of errors involved in the approximate solutions.

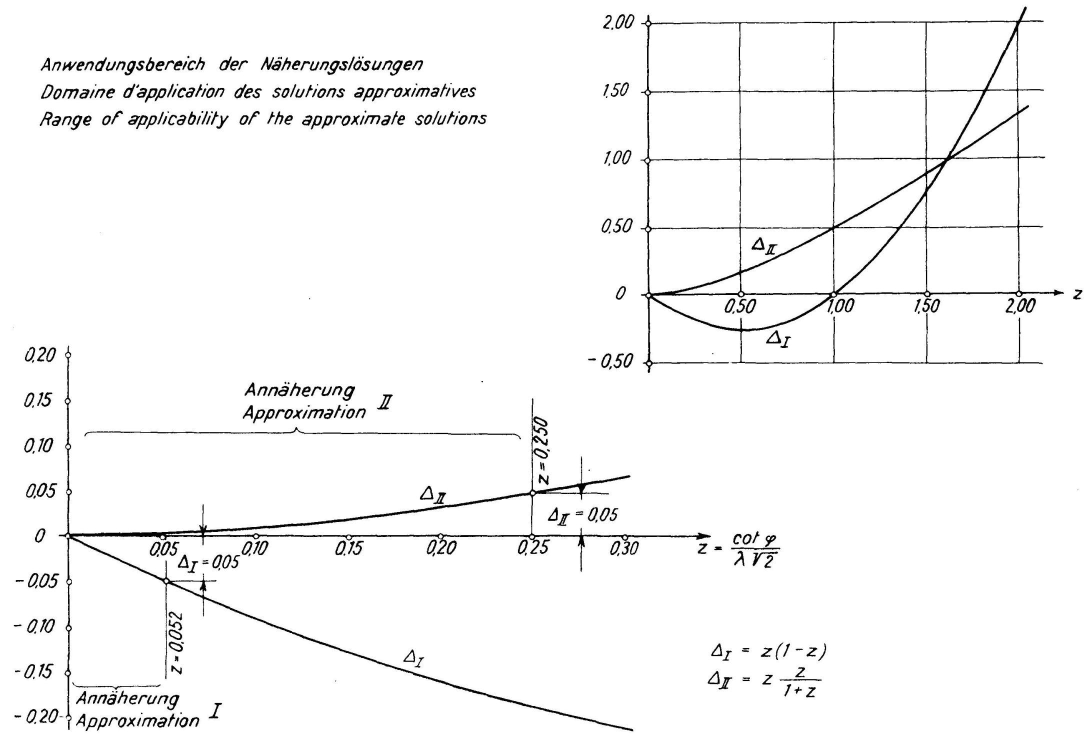

Fig. 6.

Investigating first the ratio of the omitted terms to the ones which were taken into account we get (from equations (A), neglecting μ) for Approx. I:

$$Q''\left(1 + \frac{Q' \cot\varphi - Q \cot^2\varphi}{Q''}\right) = Q''(1 + \Delta_I)$$

and for Approx. II:

$$(Q'' + Q' \cot\varphi)\left(1 - \frac{Q \cot^2\varphi}{Q'' + Q' \cot\varphi}\right) = (Q'' + Q' \cot\varphi)(1 - \Delta_{II})$$

Here $-\Delta_I$ and $+\Delta_{II}$ represent, in percentage, the error involved in Approximations I and II, respectively.

Putting now in the above equations

$$Q'' = Q' \lambda \sqrt{2} = Q\, 2\, \lambda^2$$

we have:

$$\left.\begin{aligned} \Delta_I &= -z(1-z) \\ \text{and} \qquad \Delta_{II} &= z \frac{z}{1+z} \end{aligned}\right\} \qquad (11)$$

where $$z = \frac{\cot\varphi}{\lambda\sqrt{2}}$$

A graphical representation of these Δ error-functions is given in fig. 6. If we take $\Delta_{I,II} = 5$ per cent as permissible deviation from the exact results, we find, from equations (11), that the corresponding limit of application will be $z_I = 0.052$ and $z_{II} = 0.250$ for Approximations I and II respectively.

Using the results of the previously presented numerical example, we have, also, an opportunity to compare the actual errors with the ones predicted from eq. (11).

Such comparison is shown in the following table.

Meridional Moments M_1.

	Approximation I		Approximation II	
φ^0	Actual error %	Estimated error Δ_I %	Actual error %	Estimated error Δ_{II} %
35	-12.60	-12.10	0.81	1.74
30	-30.60 [11])	-14.20	3 51 [11])	2.50
25	-10.70	-16.70	2.08	3.70
20	-22.20	-19.75	4.96	5.78
15	-30.50	-23 20	10.50	9.90
10	-37.50	-24.70	30.10	20.10

For the N_2 values the approximations were more accurate than could be predicted though the ratio between the results of I and II remained about the same.

It is seen from fig. 6 that for large z values both approximate solutions fail equally. Large z values may occur either on account of the smallness of λ (thick or flat shells) or on account of the smallness of φ (at points close to the top of the shell). But such cases are rare in the structural practice and they do not detract from the usefulness of these approximate solutions.

Summary.

This paper presents two approximate solutions, both of which are based on neglecting certain terms in the exact differential equations for the problem. The first of these approximations (Approx. I) was already known and is presented here only for comparison with the second one (Approx. II) which in complete form has not been discussed in the literature so far.

The accuracy of both of these approximations is shown in an example of comparison with results obtained by the exact method. Finally the range of applicability is discussed and a scheme given to estimate the probable errors involved in the approximate solutions.

Zusammenfassung.

In dieser Arbeit werden zwei Näherungslösungen gezeigt, beide auf der Weglasssung gewisser Ausdrücke fußend, die sonst in den exakten Differentialgleichungen des Problems zu finden sind. Die erste der Näherungslösungen ist allgemein bekannt und wird hier nur zum Vergleich mit der

[11]) No mistake could be detected in the computation of these ordinates. Possibly the approximate solutions, consisting of simple functions, could not follow so well the exact curve at these points, where, on account of the small ordinates and large tangents, a little deviation can produce great percentual deviation in the results.

zweiten Näherungsmethode, die bis jetzt in der Literatur nie in vollständiger Form gebracht wurde, beigezogen.

Die Genauigkeit der beiden Näherungslösungen wird an einem Beispiel gezeigt und ihre Resultate werden mit der genauen Lösung verglichen. Ferner wird noch der Anwendungsbereich der Näherungslösungen diskutiert. Schließlich ist ein Schema der zu erwartenden wahrscheinlichen Fehler dieser Näherungsmethoden beigefügt.

Résumé.

Dans ce travail l'auteur présente deux solutions approximatives basées sur l'abandon de certaines expressions qui se trouvent dans les équations différentielles exactes du problème. La première de ces solutions approximatives est connue et n'est introduite ici que pour la comparer avec la seconde qui n'a jamais été publiée au complet.

L'exactitude de ces deux solutions approximatives est montrée par un exemple et les résultats acquis sont comparés aux résultats fournis par la solution exacte. L'auteur discute ensuite les possibilités d'application des solutions approximatives et pour terminer il donne un schéma de l'erreur vraisemblable de ces deux méthodes approximatives.

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