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CONTRIBUTION TO THE EXACT THEORY OF THICK CYLINDRICAL SHELLS.

BEITRAG ZUR GENAUEN THEORIE DICKWANDIGER ZYLINDRISCHER ROHRE.

CONTRIBUTION À LA THÉORIE EXACTE DES ENVELOPPES CYLINDRIQUES ÉPAISSES.

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I.

Let us consider a hollow circular cylinder subjected to the radial pressure of a fluid, the surface of which stands at a vertical distance h above the horizontal axis X of the cylinder (Fig. 1). The radial pressure acting on the interior surface of the shell is

$$p = \gamma (h - y) = p_0 - p_1,$$

where $p_0 = \gamma h$ is a constant pressure corresponding to the axis of the cylinder and $p_1 = -\gamma y = -\gamma r_1 \cos \varphi$ is a radial pressure varying linearly with the coordinate y; γ is the spectific weight of the fluid.

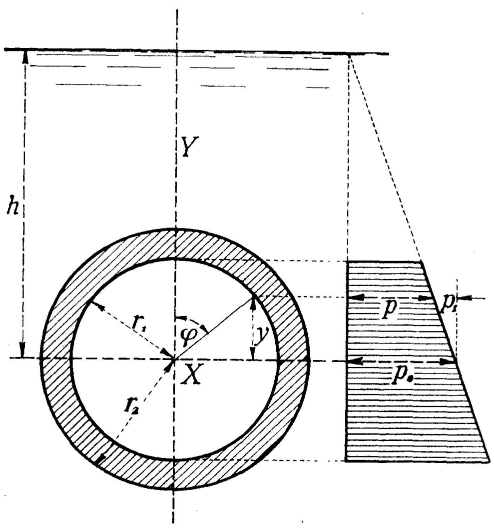

Fig. 1

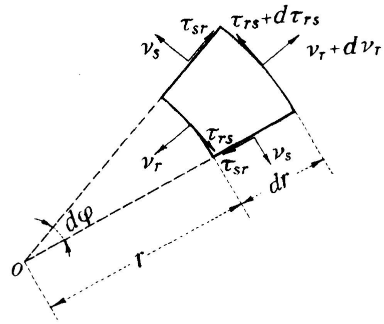

Fig. 2

If we consider a portion of the cylinder between the supports and disregard the dependence of stresses and strains on the coordinate x (distance along the axis of the cylinder), the fundamental equations for this case follow from general equations for cylindrical shells [1]) by neglecting all derivatives with respect to x. The radial directions are the principal directions in which

[1]) See author's paper „Théorie exacte des enveloppes cylindriques épaisses". Mémoires de l'Assoc. Intern. des Ponts et Charpentes, 4e v., p. 131.

there is no tangential stress component: $\tau_{xr} = \tau_{xs} = 0$. There remains only one tangential component $\tau_{rs} = \tau_{sr} = \tau$ (Fig. 2).

In the general equation (1) all the terms vanish, and this is true also for equation (9) derived from (1). The fundamental equations for this case are deducible from equations (10), (11) by neglecting body forces and all derivatives with respect to x:

$$2\left[(m-1)\frac{\partial^2\varrho}{\partial r^2}+\frac{m-1}{r}\frac{\partial\varrho}{\partial r}-\frac{m-1}{r^2}\left(\varrho+\frac{\partial\sigma}{\partial\varphi}\right)+\frac{1}{r}\frac{\partial^2\sigma}{\partial r\,\partial\varphi}\right]+(m-2)\left(\frac{1}{r^2}\frac{\partial^2\varrho}{\partial\varphi^2}+\frac{1}{r}\frac{\partial^2\sigma}{\partial r\,\partial\varphi}-\frac{1}{r^2}\frac{\partial\sigma}{\partial\varphi}\right)=0,$$

$$\frac{2}{r}\left[\frac{\partial^2\varrho}{\partial r\,\partial\varphi}+\frac{m-1}{r}\left(\frac{\partial\varrho}{\partial\varphi}+\frac{\partial^2\sigma}{\partial\varphi^2}\right)\right]+(m-2)\left[\frac{1}{r}\left(\frac{1}{r}\frac{\partial\varrho}{\partial\varphi}+\frac{\partial\sigma}{\partial r}-\frac{\sigma}{r}\right)+\frac{1}{r}\frac{\partial^2\varrho}{\partial r\,\partial\varphi}+\frac{\partial^2\sigma}{\partial r^2}\right]=0,$$

or

$$2(m-1)r^2\frac{\partial^2\varrho}{\partial r^2}+2(m-1)r\frac{\partial\varrho}{\partial r}+(m-2)\frac{\partial^2\varrho}{\partial\varphi^2}-2(m-1)\varrho-(3m-4)\frac{\partial\sigma}{\partial\varphi}+mr\frac{\partial^2\sigma}{\partial r\,\partial\varphi}=0, \qquad (1)$$

$$mr\frac{\partial^2\varrho}{\partial r\,\partial\varphi}+(3m-4)\frac{\partial\varrho}{\partial\varphi}+2(m-1)\frac{\partial^2\sigma}{\partial\varphi^2}+(m-2)r^2\frac{\partial^2\sigma}{\partial r^2}+(m-2)r\frac{\partial\sigma}{\partial r}-(m-2)\sigma=0. \qquad (2)$$

Here ϱ and σ represents the deformation in the direction of the radius r and the arc $s = r\varphi$ respectively, m is Poisson's constant. From ϱ and σ the stress components follow in equations (7), (8). Omitting all derivatives with respect to x, we obtain the normal stresses

$$\left.\begin{aligned}
\nu_x &= \frac{Em}{(m+1)(m-2)}\left[\frac{\partial\varrho}{\partial r}+\frac{1}{r}\left(\varrho+\frac{\partial\sigma}{\partial\varphi}\right)\right],\\
\nu_r &= \frac{Em}{(m+1)(m-2)}\left[(m-1)\frac{\partial\varrho}{\partial r}+\frac{1}{r}\left(\varrho+\frac{\partial\sigma}{\partial\varphi}\right)\right],\\
\nu_s &= \frac{Em}{(m+1)(m-2)}\left[\frac{\partial\varrho}{\partial r}+\frac{m-1}{r}\left(\varrho+\frac{\partial\sigma}{\partial\varphi}\right)\right]
\end{aligned}\right\} \qquad (3)$$

and the tangential stress

$$\tau_{rs} = \tau_{sr} = \tau = G\left(\frac{1}{r}\frac{\partial\varrho}{\partial\varphi}+\frac{\partial\sigma}{\partial r}-\frac{\sigma}{r}\right); \qquad (4)$$

$E\,(G)$ is the modulus of elasticity in tension (shear). The equations (3) give

$$\nu_r + \nu_s = \frac{Em}{(m+1)(m-2)}\left[m\frac{\partial\varrho}{\partial r}+\frac{m}{r}\left(\varrho+\frac{\partial\sigma}{\partial\varphi}\right)\right] = m\nu_x,$$

which corresponds to a plane deformation, where there is no elongation in the direction X.

For the solution of equations (1) and (2) it is necessary to transform them into two independent equations corresponding to each of the functions ϱ and σ. To eliminate the function σ from equations (1), (2) we can use (1), (2) and their first and second derivatives with respect to r and φ. The equations necessary for the elimination of σ may best be determined by noting in a table which of the equations (1), (2) and their derivatives contains the function σ and its derivatives

$$\frac{\partial\sigma}{\partial r},\ \frac{\partial\sigma}{\partial\varphi},\ \ldots\ \frac{\partial^4\sigma}{\partial r^4},\ \frac{\partial^4\sigma}{\partial r^3\,\partial\varphi},\ \ldots\ \frac{\partial^4\sigma}{\partial\varphi^4},$$

without differentiating the equations (1), (2). The equations not necessary for the elimination of σ may then be recognised as those which alone contain σ or any derivative of it. Omitting successive equations in accordance with this condition we finally obtain equations containing σ and each derivative of it at least twice, which can be used to eliminate σ. In our case these are the equations:

$$(1)\ldots\ 2(m-1)r^2\frac{\partial^2\varrho}{\partial r^2}+2(m-1)r\frac{\partial\varrho}{\partial r}+(m-2)\frac{\partial^2\varrho}{\partial\varphi^2}-2(m-1)\varrho+mr\frac{\partial^2\sigma}{\partial r\,\partial\varphi}-(3m-4)\frac{\partial\sigma}{\partial\varphi}=0,$$

$$\frac{\partial(1)}{\partial r}\ldots\ 2(m-1)r^2\frac{\partial^3\varrho}{\partial r^3}+6(m-1)r\frac{\partial^2\varrho}{\partial r^2}+(m-2)\frac{\partial^3\varrho}{\partial r\,\partial\varphi^2}+mr\frac{\partial^3\sigma}{\partial r^2\,\partial\varphi}-2(m-2)\frac{\partial^2\sigma}{\partial r\,\partial\varphi}=0,$$

$$\frac{\partial^2(1)}{\partial r^2}\ldots 2(m-1)r^2\frac{\partial^4\varrho}{\partial r^4}+10(m-1)r\frac{\partial^3\varrho}{\partial r^3}+6(m-1)\frac{\partial^2\varrho}{\partial r^2}+(m-2)\frac{\partial^4\varrho}{\partial r^2\partial\varphi^2}+mr\frac{\partial^4\sigma}{\partial r^3\partial\varphi}-(m-4)\frac{\partial^3\sigma}{\partial r^2\partial\varphi}=0,$$

$$\frac{\partial^2(1)}{\partial\varphi^2}\ldots 2(m-1)r^2\frac{\partial^4\varrho}{\partial r^2\partial\varphi^2}+2(m-1)r\frac{\partial^3\varrho}{\partial r\,\partial\varphi^2}+(m-2)\frac{\partial^4\varrho}{\partial\varphi^4}-2(m-1)\frac{\partial^2\varrho}{\partial\varphi^2}-(3m-4)\frac{\partial^3\sigma}{\partial\varphi^3}+mr\frac{\partial^4\sigma}{\partial r\,\partial\varphi^3}=0,$$

$$\frac{\partial(2)}{\partial\varphi}\ldots mr\frac{\partial^3\varrho}{\partial r\partial\varphi^2}+(3m-4)\frac{\partial^2\varrho}{\partial\varphi^2}+2(m-1)\frac{\partial^3\sigma}{\partial\varphi^3}+(m-2)r^2\frac{\partial^3\sigma}{\partial r^2\partial\varphi}+(m-2)r\frac{\partial^2\sigma}{\partial r\partial\varphi}-(m-2)\frac{\partial\sigma}{\partial\varphi}=0,$$

$$\frac{\partial^2(2)}{\partial r\partial\varphi}\ldots mr\frac{\partial^4\varrho}{\partial r^2\partial\varphi^2}+4(m-1)\frac{\partial^3\varrho}{\partial r\,\partial\varphi^2}+2(m-1)\frac{\partial^4\sigma}{\partial r\,\partial\varphi^3}+(m-2)r^2\frac{\partial^4\sigma}{\partial r^3\partial\varphi}+3(m-2)r\frac{\partial^3\sigma}{\partial r^2\partial\varphi}=0.$$

To eliminate σ from these equations, we multiply the first with α_1, the second with $\alpha_2\ldots$, the sixth with α_6 and add them all together; the coefficients α are to be determined from the conditions that σ and all its derivatives vanish from the resulting equation. The following are the conditions to be satisfied:

the factor of $\frac{\partial^4\sigma}{\partial r^3\partial\varphi}$ is $mr\alpha_3+(m-2)r^2\alpha_6=0,$

" " " $\frac{\partial^4\sigma}{\partial r\partial\varphi^3}$ " $mr\alpha_4+2(m-1)\alpha_6=0,$

" " " $\frac{\partial^3\sigma}{\partial r^2\partial\varphi}$ " $mr\alpha_2-(m-4)\alpha_3+(m-2)r^2\alpha_5+3(m-2)r\alpha_6=0,$

" " " $\frac{\partial^3\sigma}{\partial\varphi^3}$ " $-(3m-4)\alpha_4+2(m-1)\alpha_5=0,$

" " " $\frac{\partial^2\sigma}{\partial r\,\partial\varphi}$ " $mr\alpha_1-2(m-2)\alpha_2+(m-2)r\alpha_5=0,$

" " " $\frac{\partial\sigma}{\partial\varphi}$ " $-(3m-4)\alpha_1-(m-2)\alpha_5=0.$

We thus have 6 linear homogeneous equations for 6 coefficients $\alpha_1\ldots\alpha_6$; one equation is superfluous, but the equations are not mutually contradictory and they can be solved.

The sixth equation can be satisfied with

$$\alpha_1=-(m-2),\ \alpha_5=3m-4;$$

from the fifth it follows that $\alpha_2 = (m - 2)\, r,$
from the fourth $\alpha_4 = 2\,(m - 1),$
from the second $\alpha_6 = -\, m\, r$
and from the first $\alpha_3 = (m - 2)\, r^2.$

These values of α satisfy also the third equation.

Multiplying the foregoing equations by $\alpha_1, \alpha_2, \ldots \alpha_6$ and adding them together, we eliminate σ and obtain for ϱ the independent equation

$$r^4 \frac{\partial^4 \varrho}{\partial r^4} + 2\, r^2 \frac{\partial^4 \varrho}{\partial r^2 \partial \varphi^2} + \frac{\partial^4 \varrho}{\partial \varphi^4} + 6 r^3 \frac{\partial^3 \varrho}{\partial r^3} + 2\, r \frac{\partial^3 \varrho}{\partial r \partial \varphi^2} + 5 r^2 \frac{\partial^2 \varrho}{\partial r^2} + 2 \frac{\partial^2 \varrho}{\partial \varphi^2} - r \frac{\partial \varrho}{\partial r} + \varrho = 0. \tag{5}$$

Eliminating the elongation ϱ in a similar way from equations (1) and (2), we find for σ an equation identical with (5). This means that by integrating we get for ϱ and σ the same function of r and φ; the only difference between them being in the constants of integration.

II.

Another method of solution starts from the s t r e s s e s. The conditions of equilibrium of an element [2]) are here

$$\frac{1}{r} \frac{\partial (r \nu_r)}{\partial r} + \frac{1}{r} \frac{\partial \tau}{\partial \varphi} - \frac{\nu_s}{r} = 0, \tag{6}$$

$$\frac{1}{r} \frac{\partial \nu_s}{\partial \varphi} + 2 \frac{\tau}{r} + \frac{\partial \tau}{\partial r} = 0. \tag{7}$$

Equations (6), (7), the former equations (3) for ν_r, ν_s and the equation (4) for τ give 5 equations for 5 functions ν_r, ν_s, τ, ϱ, σ. By eliminating the stresses ν_r, ν_s, τ from these equations we obtain the equations (1) and (2) for the elongations ϱ and σ. But we can also from these 5 equations eliminate ϱ, σ and two components of stresses and thus find an independent equation for one stress component only.

From equations (3) for ν_r, ν_s we obtain

$$\frac{\partial \varrho}{\partial r} = \frac{(m^2 - 1)\, \nu_r - (m + 1)\, \nu_s}{E m^2}, \tag{8}$$

$$\varrho + \frac{\partial \sigma}{\partial \varphi} = \frac{(m^2 - 1)\, r \nu_s - (m + 1)\, r \nu_r}{E m^2}.$$

Differentiating the last equation, we find

$$\frac{\partial \varrho}{\partial r} + \frac{\partial^2 \sigma}{\partial r \partial \varphi} = \frac{m^2 - 1}{E m^2} \cdot \frac{\partial (r \nu_s)}{\partial r} - \frac{m + 1}{E m^2} \cdot \frac{\partial (r \nu_r)}{\partial r}$$

and substituting for $\frac{\partial \varrho}{\partial r}$ from (8), we have

$$\frac{\partial^2 \sigma}{\partial r \partial \varphi} = \frac{m^2 - 1}{E m^2} \left(\frac{\partial (r \nu_s)}{\partial r} - \nu_r \right) - \frac{m + 1}{E m^2} \left(\frac{\partial (r \nu_r)}{\partial r} - \nu_s \right). \tag{9}$$

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[2]) See equations (2), (3) in the author's article „Théorie exacte . . .“.

From equation (4) it follows

$$r\tau = G\left(\frac{\partial \varrho}{\partial \varphi} + r\frac{\partial \sigma}{\partial r} - \sigma\right)$$

and differentiating this once with respect to r and once with respect to φ, we find

$$\frac{\partial^2 (r\tau)}{\partial r\,\partial \varphi} = G\left(\frac{\partial^3 \varrho}{\partial r\,\partial \varphi^2} + r\frac{\partial^3 \sigma}{\partial r^2\,\partial \varphi}\right).$$

Differentiating equation (8) twice with respect to φ, we have

$$\frac{\partial^3 \varrho}{\partial r\,\partial \varphi^2} = \frac{m^2-1}{Em^2}\cdot\frac{\partial^2 \nu_r}{\partial \varphi^2} - \frac{m+1}{Em^2}\cdot\frac{\partial^2 \nu_s}{\partial \varphi^2}$$

and differentiating equation (9) with respect to r, we find

$$\frac{\partial^3 \sigma}{\partial r^2\,\partial \varphi} = \frac{m^2-1}{Em^2}\left(\frac{\partial^2 (r\nu_s)}{\partial r^2} - \frac{\partial \nu_r}{\partial r}\right) - \frac{m+1}{Em^2}\left(\frac{\partial^2 (r\nu_r)}{\partial r^2} - \frac{\partial \nu_s}{\partial r}\right).$$

Substituting the last two values and $G = \frac{Em}{2(m+1)}$ into $\frac{\partial^2 (r\tau)}{\partial r\,\partial \varphi}$, we obtain

$$\frac{\partial^2 (r\tau)}{\partial r\,\partial \varphi} = \frac{m-1}{2m}\left(\frac{\partial^2 \nu_r}{\partial \varphi^2} + r\frac{\partial^2 (r\nu_s)}{\partial r^2} - r\frac{\partial \nu_r}{\partial r}\right) - \frac{1}{2m}\left(\frac{\partial^2 \nu_s}{\partial \varphi^2} + r\frac{\partial^2 (r\nu_r)}{\partial r^2} - r\frac{\partial \nu_s}{\partial r}\right). \quad (10)$$

The equation (6) gives

$$\frac{\partial \tau}{\partial \varphi} = \nu_s - \frac{\partial (r\nu_r)}{\partial r} \quad \text{and} \quad \frac{\partial^2 \tau}{\partial r\,\partial \varphi} = \frac{\partial \nu_s}{\partial r} - \frac{\partial^2 (r\nu_r)}{\partial r^2};$$

therefore

$$\frac{\partial^2 (r\tau)}{\partial r\,\partial \varphi} = r\frac{\partial^2 \tau}{\partial r\,\partial \varphi} + \frac{\partial \tau}{\partial \varphi} = r\frac{\partial \nu_s}{\partial r} - r\frac{\partial^2 (r\nu_r)}{\partial r^2} + \nu_s - \frac{\partial (r\nu_r)}{\partial r}.$$

This gives with respect to (10) a relation between ν_r, ν_s which can be put into the form

$$\frac{\partial^2 \nu_r}{\partial \varphi^2} - r\frac{\partial \nu_r}{\partial r} - 2\frac{\partial (r\nu_r)}{\partial r} + r\frac{\partial^2 (r\nu_s)}{\partial r^2} + 2\frac{\partial^2 \nu_s}{\partial \varphi^2} + 2\nu_s = 0. \quad (11)$$

A second equation for ν_r, ν_s can be obtained by eliminating τ from equations (6) and (7). By differentiating equation (7) with respect to φ, we find

$$\frac{\partial^2 \nu_s}{\partial \varphi^2} + 2\frac{\partial \tau}{\partial \varphi} + r\frac{\partial^2 \tau}{\partial r\,\partial \varphi} = 0;$$

substituting here for $\frac{\partial \tau}{\partial \varphi}$ and $\frac{\partial^2 \tau}{\partial r\,\partial \varphi}$ the values already obtained from equation (6), we have

$$\frac{\partial^2 \nu_s}{\partial \varphi^2} + 2\nu_s - 2\frac{\partial (r\nu_r)}{\partial r} + r\frac{\partial \nu_s}{\partial r} - r\frac{\partial^2 (r\nu_r)}{\partial r^2} = 0. \quad (12)$$

Subtracting equation (12) from (11), we find

$$\frac{\partial^2 \nu_r}{\partial \varphi^2} - r\frac{\partial \nu_r}{\partial r} + r\frac{\partial^2 (r\nu_s)}{\partial r^2} + \frac{\partial^2 \nu_s}{\partial \varphi^2} - r\frac{\partial \nu_s}{\partial r} + r\frac{\partial^2 (r\nu_r)}{\partial r^2} = 0.$$

It is $$\frac{\partial (r v_r)}{\partial r} = r \frac{\partial v_r}{\partial r} + v_r, \qquad \frac{\partial^2 (r v_r)}{\partial r^2} = r \frac{\partial^2 v_r}{\partial r^2} + 2 \frac{\partial v_r}{\partial r}$$

and analogically for v_s. The foregoing equation becomes

$$r^2 \frac{\partial^2 v_r}{\partial r^2} + r \frac{\partial v_r}{\partial r} + \frac{\partial^2 v_r}{\partial \varphi^2} = - r^2 \frac{\partial^2 v_s}{\partial r^2} - r \frac{\partial v_s}{\partial r} - \frac{\partial^2 v_s}{\partial \varphi^2} \tag{13}$$

and the equation (12)

$$r^2 \frac{\partial^2 v_r}{\partial r^2} + 4 r \frac{\partial v_r}{\partial r} + 2 v_r = r \frac{\partial v_s}{\partial r} + 2 v_s + \frac{\partial^2 v_s}{\partial \varphi^2}. \tag{14}$$

From the equations (13), (14) we eliminate v_s analogically, as we have eliminated σ from the equations (1) and (2). We use equations (13), (14) and their first and second derivatives with respect to r and φ. Omitting each equation which alone contains any derivative of v_s, we get the following equations for the elimination of v_s:

$$(13) \ldots r^2 \frac{\partial^2 v_r}{\partial r^2} + r \frac{\partial v_r}{\partial r} + \frac{\partial^2 v_r}{\partial \varphi^2} = - r^2 \frac{\partial^2 v_s}{\partial r^2} - r \frac{\partial v_s}{\partial r} - \frac{\partial^2 v_s}{\partial \varphi^2},$$

$$\frac{\partial (13)}{\partial r} \ldots r^2 \frac{\partial^3 v_r}{\partial r^3} + 3 r \frac{\partial^2 v_r}{\partial r^2} + \frac{\partial v_r}{\partial r} + \frac{\partial^3 v_r}{\partial r \partial \varphi^2} = - r^2 \frac{\partial^3 v_s}{\partial r^3} - 3 r \frac{\partial^2 v_s}{\partial r^2} - \frac{\partial v_s}{\partial r} - \frac{\partial^3 v_s}{\partial r \partial \varphi^2},$$

$$\frac{\partial^2 (13)}{\partial \varphi^2} \ldots r^2 \frac{\partial^4 v_r}{\partial r^2 \partial \varphi^2} + r \frac{\partial^3 v_r}{\partial r \partial \varphi^2} + \frac{\partial^4 v_r}{\partial \varphi^4} = r^2 \frac{\partial^4 v_s}{\partial r^2 \partial \varphi^2} - r \frac{\partial^3 v_s}{\partial r \partial \varphi^2} - \frac{\partial^4 v_s}{\partial \varphi^4},$$

$$\frac{\partial (14)}{\partial r} \ldots r^2 \frac{\partial^3 v_r}{\partial r^3} + 6 r \frac{\partial^2 v_r}{\partial r^2} + 6 \frac{\partial v_r}{\partial r} = r \frac{\partial^2 v_s}{\partial r^2} + 3 \frac{\partial v_s}{\partial r} + \frac{\partial^3 v_s}{\partial r \partial \varphi^2},$$

$$\frac{\partial^2 (14)}{\partial r^2} \ldots r^2 \frac{\partial^4 v_r}{\partial r^4} + 8 r \frac{\partial^3 v_r}{\partial r^3} + 12 \frac{\partial^2 v_r}{\partial r^2} = r \frac{\partial^3 v_s}{\partial r^3} + 4 \frac{\partial^2 v_s}{\partial r^2} + \frac{\partial^4 v_s}{\partial r^2 \partial \varphi^2},$$

$$\frac{\partial^2 (14)}{\partial \varphi^2} \ldots r^2 \frac{\partial^4 v_r}{\partial r^2 \partial \varphi^2} + 4 r \frac{\partial^3 v_r}{\partial r \partial \varphi^2} + 2 \frac{\partial^2 v_r}{\partial \varphi^2} = r \frac{\partial^3 v_r}{\partial r \partial \varphi^2} + 2 \frac{\partial^2 v_s}{\partial \varphi^2} + \frac{\partial^4 v_s}{\partial \varphi^4}.$$

For the elimination of v_s it is necessary to multiply the foregoing equations successively by

$$\alpha_1 = \frac{2}{r^2}, \quad \alpha_2 = \frac{1}{r}, \quad \alpha_3 = \frac{1}{r^2}, \quad \alpha_4 = \frac{1}{r}, \quad \alpha_5 = 1, \quad \alpha_6 = \frac{1}{r^2}$$

and to add them; the values of α can be determined analogically as above. In this manner we obtain for v_r the equation

$$r^2 \frac{\partial^4 v_r}{\partial r^4} + 2 \frac{\partial^4 v_r}{\partial r^2 \partial \varphi^2} + \frac{1}{r^2} \frac{\partial^4 v_r}{\partial \varphi^4} + 10 r \frac{\partial^3 v_r}{\partial r^3} + \frac{6}{r} \frac{\partial^3 v_r}{\partial r \partial \varphi^2} + 23 \frac{\partial^2 v_r}{\partial r^2}$$
$$+ \frac{4}{r^2} \frac{\partial^2 v_r}{\partial \varphi^2} + \frac{9}{r} \frac{\partial v_r}{\partial r} = 0. \tag{15}$$

Eliminating in the same manner v_r from the equations (13) and (14), we obtain for v_s an equation identical with the equation (15).

An independent equation for the tangential stress τ can be obtained by eliminating v_r, v_s from the equations (6), (7) and (10). The equation (6) gives

$$\nu_s = \frac{\partial (r\nu_r)}{\partial r} + \frac{\partial \tau}{\partial \varphi} = r\frac{\partial \nu_r}{\partial r} + \nu_r + \frac{\partial \tau}{\partial \varphi};$$

differentiating we obtain

$$\frac{\partial \nu_s}{\partial \varphi} = r\frac{\partial^2 \nu_r}{\partial r\partial \varphi} + \frac{\partial \nu_r}{\partial \varphi} + \frac{\partial^2 \tau}{\partial \varphi^2}, \qquad \frac{\partial^2 \nu_s}{\partial \varphi^2} = r\frac{\partial^3 \nu_r}{\partial r\partial \varphi^2} + \frac{\partial^2 \nu_r}{\partial \varphi^2} + \frac{\partial^3 \tau}{\partial \varphi^3},$$

$$\frac{\partial \nu_s}{\partial r} = r\frac{\partial^2 \nu_r}{\partial r^2} + 2\frac{\partial \nu_r}{\partial r} + \frac{\partial^2 \tau}{\partial r\partial \varphi}, \qquad \frac{\partial^2 \nu_s}{\partial r^2} = r\frac{\partial^3 \nu_r}{\partial r^2} + 3\frac{\partial^2 \nu_r}{\partial r^2} + \frac{\partial^3 \tau}{\partial r^2\partial \varphi}.$$

Substituting these derivatives, we find

$$\frac{\partial^2 (r\nu_s)}{\partial r^2} = r\frac{\partial^2 \nu_s}{\partial r^2} + 2\frac{\partial \nu_s}{\partial r} = r^2\frac{\partial^3 \nu_r}{\partial r^3} + 5r\frac{\partial^2 \nu_r}{\partial r^2} + 4\frac{\partial \nu_r}{\partial r} + 2\frac{\partial^2 \tau}{\partial r\partial \varphi} + r\frac{\partial^3 \tau}{\partial r^2\partial \varphi}.$$

The equation (7) gives, when substituting for $\frac{\partial \nu_s}{\partial \varphi}$:

$$r\frac{\partial^2 \nu_r}{\partial r\partial \varphi} + \frac{\partial \nu_r}{\partial \varphi} + \frac{\partial^2 \tau}{\partial \varphi^2} + r\frac{\partial \tau}{\partial r} + 2\tau = 0 \qquad (16)$$

and from the equation (10) we find with

$$\frac{\partial^2 (r\tau)}{\partial r\partial \varphi} = r\frac{\partial^2 \tau}{\partial r\partial \varphi} + \frac{\partial \tau}{\partial \varphi}$$

and substituting for

$$\frac{\partial^2 (r\nu_s)}{\partial r^2}, \ \frac{\partial^2 \nu_s}{\partial \varphi^2} \text{ and } \frac{\partial \nu_s}{\partial r}:$$

$$(m-1)\left(r^3\frac{\partial^3 \nu_r}{\partial r^3} + 5r^2\frac{\partial^2 \nu_r}{\partial r^2} + 3r\frac{\partial \nu_r}{\partial r}\right) + (m-2)\frac{\partial^2 \nu_r}{\partial \varphi^2} - r\frac{\partial^3 \nu_r}{\partial r\partial \varphi^2} + (m-1)r^2\frac{\partial^3 \tau}{\partial r^2\partial \varphi}$$
$$-\frac{\partial^3 \tau}{\partial \varphi^3} - r\frac{\partial^2 \tau}{\partial r\partial \varphi} - 2m\frac{\partial \tau}{\partial \varphi} = 0. \qquad (17)$$

From the equations (16), (17) we can now eliminate ν_r as before ν_s; making use for this purpose of the equations

$$\frac{\partial (16)}{\partial r}, \ \frac{\partial^2 (16)}{\partial r^2}, \ \frac{\partial^3 (16)}{\partial r^3}, \ \frac{\partial^2 (16)}{\partial \varphi^2}, \ \frac{\partial^3 (16)}{\partial r\partial \varphi^2}, \ \frac{\partial (17)}{\partial \varphi}, \ \frac{\partial^2 (17)}{\partial r\partial \varphi}.$$

We obtain for τ an independent equation identical with (15).

All the three stresses ν_r, ν_s, τ have also an identical differential equation; integrating it, we find for ν_r, ν_s, τ formulas which differ only in constants of integration.

III.

The third method of solution is based on stress function F. Putting in polar coordinates [3])

$$\nu_r = \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \varphi^2}, \qquad \nu_s = \frac{\partial^2 F}{\partial r^2}, \qquad (18)$$

we obtain

[3]) See S. Timoshenko, "Theory of elasticity", p. 53.

$$\frac{\partial \nu_r}{\partial r} = \frac{1}{r}\frac{\partial^2 F}{\partial r^2} - \frac{1}{r^2}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^3 F}{\partial r \partial \varphi^2} - \frac{2}{r^3}\frac{\partial^2 F}{\partial \varphi^2},$$

$$\frac{\partial (r \nu_r)}{\partial r} = r\frac{\partial \nu_r}{\partial r} + \nu_r = \frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial^3 F}{\partial r \partial \varphi^2} - \frac{1}{r^2}\frac{\partial^2 F}{\partial \varphi^2}.$$

The equation (6) gives

$$\frac{\partial \tau}{\partial \varphi} = \nu_s - \frac{\partial (r \nu_r)}{\partial r} = \frac{1}{r^2}\frac{\partial^2 F}{\partial \varphi^2} - \frac{1}{r}\frac{\partial^3 F}{\partial r \partial \varphi^2};$$

integrating, we find

$$\tau = \frac{1}{r^2}\frac{\partial F}{\partial \varphi} - \frac{1}{r}\frac{\partial^2 F}{\partial r \partial \varphi}. \tag{19}$$

From the stress function we can determine also the components of deformation ϱ and σ. Substituting from (18) into (8), we obtain

$$\frac{\partial \varrho}{\partial r} = \frac{1}{Em^2}\left[(m^2-1)\left(\frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \varphi^2}\right) - (m+1)\frac{\partial^2 F}{\partial r^2}\right]. \tag{20}$$

In the equation (9) we substitute

$$\frac{\partial (r \nu_r)}{\partial r} = r\frac{\partial \nu_r}{\partial r} + \nu_r, \quad \frac{\partial (r \nu_s)}{\partial r} = r\frac{\partial \nu_s}{\partial r} + \nu_s, \quad \frac{\partial \nu_s}{\partial r} = \frac{\partial^3 F}{\partial r^3},$$

$\frac{\partial \nu_r}{\partial r}$ as above and for ν_r, ν_s from the equations (18); whence

$$\frac{\partial^2 \sigma}{\partial r \partial \varphi} = \frac{1}{Em^2}\left[(m^2-1)\left(r\frac{\partial^3 F}{\partial r^3} + \frac{\partial^2 F}{\partial r^2} - \frac{1}{r}\frac{\partial F}{\partial r} - \frac{1}{r^2}\frac{\partial^2 F}{\partial \varphi^2}\right) - (m+1)\left(\frac{1}{r}\frac{\partial^3 F}{\partial r \partial \varphi^2} - \frac{1}{r^2}\frac{\partial^2 F}{\partial \varphi^2}\right)\right]. \tag{21}$$

The condition for F follows from the equation (4) which gives

$$r\tau = G\left(\frac{\partial \varrho}{\partial \varphi} + r\frac{\partial \sigma}{\partial r} - \sigma\right).$$

Differentiating with respect to r, we obtain

$$r\frac{\partial \tau}{\partial r} + \tau = G\left(\frac{\partial^2 \varrho}{\partial r \partial \varphi} + r\frac{\partial^2 \sigma}{\partial r^2}\right)$$

and differentiating again with respect to φ:

$$r\frac{\partial^2 \tau}{\partial r \partial \varphi} + \frac{\partial \tau}{\partial \varphi} = G\left(\frac{\partial^3 \varrho}{\partial r \partial \varphi^2} + r\frac{\partial^3 \sigma}{\partial r^2 \partial \varphi}\right). \tag{22}$$

From the equation (19) we find

$$\frac{\partial \tau}{\partial \varphi} = \frac{1}{r^2}\frac{\partial^2 F}{\partial \varphi^2} - \frac{1}{r}\frac{\partial^3 F}{\partial r \partial \varphi^2}, \quad \frac{\partial^2 \tau}{\partial r \partial \varphi} = \frac{1}{r^2}\frac{\partial^3 F}{\partial r \partial \varphi^2} - \frac{2}{r^3}\frac{\partial^2 F}{\partial \varphi^2} - \frac{1}{r}\frac{\partial^4 F}{\partial r^2 \partial \varphi^2} + \frac{1}{r^2}\frac{\partial^3 F}{\partial r \partial \varphi^2};$$

from the equation (20) it follows that

$$\frac{\partial^3 \varrho}{\partial r \partial \varphi^2} = \frac{1}{Em^2}\left[(m^2-1)\left(\frac{1}{r}\frac{\partial^3 F}{\partial r \partial \varphi^2} + \frac{1}{r^2}\frac{\partial^4 F}{\partial \varphi^4}\right) - (m+1)\frac{\partial^4 F}{\partial r^3 \partial \varphi^2}\right]$$

and from (21)

$$\frac{\partial^3 \sigma}{\partial r^2 \partial \varphi} = \frac{1}{Em^2}\left[(m^2-1)\left(r\frac{\partial^4 F}{\partial r^4} + 2\frac{\partial^3 F}{\partial r^3} - \frac{1}{r}\frac{\partial^2 F}{\partial r^2} + \frac{1}{r^2}\frac{\partial F}{\partial r} - \frac{1}{r^2}\frac{\partial^3 F}{\partial r \partial \varphi^2} + \frac{2}{r^3}\frac{\partial^2 F}{\partial \varphi^2}\right)\right.$$
$$\left. -(m+1)\left(\frac{1}{r}\frac{\partial^4 F}{\partial r^2 \partial \varphi^2} - \frac{2}{r^2}\frac{\partial^3 F}{\partial r \partial \varphi^2} + \frac{2}{r^3}\frac{\partial^2 F}{\partial \varphi^2}\right)\right].$$

Substituting $G = \frac{Em}{2(m+1)}$ and the foregoing values of derivatives in the equation (22), we find the condition for F in the form

$$\frac{\partial^4 F}{\partial r^4} + \frac{2}{r^2}\frac{\partial^4 F}{\partial r^2 \partial \varphi^2} + \frac{1}{r^4}\frac{\partial^4 F}{\partial \varphi^4} + \frac{2}{r}\frac{\partial^3 F}{\partial r^3} - \frac{2}{r^3}\frac{\partial^3 F}{\partial r \partial \varphi^2} - \frac{1}{r^2}\frac{\partial^2 F}{\partial r^2} + \frac{4}{r^4}\frac{\partial^2 F}{\partial \varphi^2} + \frac{1}{r^3}\frac{\partial F}{\partial r} = 0. \tag{23}$$

Using for polar coordinates the symbol

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$$

and

$$\nabla^4 = \nabla^2(\nabla^2) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right)(\nabla^2) = \frac{\partial^4}{\partial r^4} + \frac{2}{r}\frac{\partial^3}{\partial r^3} - \frac{1}{r^2}\frac{\partial^2}{\partial r^2} + \frac{1}{r^3}\frac{\partial}{\partial r}$$
$$+ \frac{2}{r^2}\frac{\partial^4}{\partial r^2 \partial \varphi^2} - \frac{2}{r^2}\frac{\partial^3}{\partial r \partial \varphi^2} + \frac{4}{r^4}\frac{\partial^2}{\partial \varphi^2} + \frac{1}{r^4}\frac{\partial^4}{\partial \varphi^4},$$

we can write the equation (23) in the form

$$\nabla^4 F = 0. \tag{23'}$$

IV.

The general expression for the stress function F follows from the equation (23). Taking this function in the form

$$F = R\Phi,$$

where R is function of r and Φ function of φ only, we obtain

$$\frac{\partial^n F}{\partial r^n} = R^{(n)}\Phi, \quad \frac{\partial^n F}{\partial \varphi^n} = R\Phi^{(n)}, \quad \frac{\partial^{m+n} F}{\partial r^m \partial \varphi^n} = R^{(m)}\Phi^{(n)}.$$

Substituting in the equation (23) we find

$$\left(R^{(4)} + \frac{2}{r}R''' - \frac{1}{r^2}R'' + \frac{1}{r^3}R'\right)\Phi + \left(\frac{2}{r^2}R'' - \frac{2}{r^3}R' + \frac{4}{r^4}R\right)\Phi'' + \frac{1}{r^4}R\Phi^{(4)} = 0. \tag{24}$$

We can satisfy this equation by putting

$$\Phi'' = 0 \text{ (whence } \Phi^{(4)} = 0) \text{ and } R^{(4)} + \frac{2}{r}R''' - \frac{1}{r^2}R'' + \frac{1}{r^3}R' = 0.$$

From the first equation it follows that

$$\Phi = a_1\varphi + a_2.$$

The integral of the second equation is

$$R = c_1 r^2 \ln r + c_2 \ln r + c_3 r^2 + c_4.$$

The stress function is therefore

$$F_1 = R\Phi = (c_1 r^2 \ln r + c_2 \ln r + c_3 r^2 + c_4)(a_1\varphi + a_2). \tag{25 a}$$

The equation (24) can also be satisfied with

$$R = r \quad \text{or} \quad R' = 1,\ R'' = R''' = R^{(4)} = 0.$$

Substituting in (24) we find

$$\Phi^{(4)} + 2\,\Phi'' + \Phi = 0;$$

and the integral of this equation is

$$\Phi = c'_1 \sin\varphi + c'_2 \cos\varphi + c'_3 \varphi \sin\varphi + c'_4 \varphi \cos\varphi.$$

The second form of the stress function is

$$F_2 = R\Phi = r(c'_1 \sin\varphi + c'_2 \cos\varphi + c'_3 \varphi \sin\varphi + c'_4 \varphi \cos\varphi). \qquad (25\,\text{b})$$

As a further possibility we take

$$\frac{\Phi''}{\Phi} = -1;$$

integrating we find

$$\Phi = a'_1 \sin\varphi + a'_2 \cos\varphi$$

and differentiating

$$\frac{\Phi^{(4)}}{\Phi} = 1.$$

From (24) it follows that

$$R^{(4)} + \frac{2}{r} R''' - \frac{3}{r^2} R'' + \frac{3}{r^3} R' - \frac{3}{r^4} R = 0;$$

and integrating we obtain

$$R = c''_1 r \ln r + c''_2 r^{-1} + c''_3 r^3 + c''_4 r.$$

The stress function may, therefore, have the form

$$F_3 = R\Phi = (c''_1 r \ln r + c''_2 r^{-1} + c''_3 r^3 + c''_4 r)(a'_1 \sin\varphi + a'_2 \cos\varphi). \qquad (25\,\text{c})$$

Finally we can take

$$\frac{\Phi''}{\Phi} = -n^2,$$

where n is an arbitrary integer. Then we have

$$\Phi = a_n \sin n\varphi + b_n \cos n\varphi, \qquad \frac{\Phi^{(4)}}{\Phi} = n^4.$$

Substituting in (24) we obtain

$$R^{(4)} + \frac{2}{r} R''' - \frac{1 + 2n^2}{r^2} R'' + \frac{1 + 2n^2}{r^3} R' - \frac{4n^2 - n^4}{r^4} R = 0.$$

Putting $R = r^\alpha$ we have

$$R' = \alpha r^{\alpha-1},\ R'' = \alpha(\alpha-1) r^{\alpha-2},\ R''' = \alpha(\alpha-1)(\alpha-2) r^{\alpha-3},\ R^{(4)} = \alpha(\alpha-1)(\alpha-2)(\alpha-3) r^{\alpha-4},$$

and the last equation reduces to the characteristic equation

$$(\alpha^2 - n^2)(\alpha^2 - 4\alpha - n^2 + 4) = 0.$$

The solutions of this equation are

$$\alpha_{1,2} = \pm n, \quad \alpha_{3,4} = 2 \pm n;$$

the general solution of the foregoing differential equation is, therefore,

$$R = c_n r^{\alpha_1} + d_n r^{\alpha_2} + e_n r^{\alpha_3} + f_n r^{\alpha_4} = c_n r^n + d_n r^{-n} + e_n r^{2+n} + f_n r^{2-n}.$$

This result is valid for each integer $n \geqq 2$, because only then the characteristic equation gives four different values α. For $n = 1$ we find $\alpha_1 = \alpha_4 = 1$, and it is necessary to include not only the terms with r, r^{-1}, r^3 but also a term with $r \ln r$, which gives the function F_3. For $n = 0$ we have $\alpha_{1,2} = 0$, $\alpha_{3,4} = 2$ and we obtain, besides the absolute term and the term containing r^2, also terms with $\ln r$ and $r^2 \ln r$. Hence the function F_1 with $a_1 = 0$.

The calculated values of R, Φ being valid for each integer $n \geqq 2$, the equation (24) is also satisfied by the series

$$F_4 = \Sigma R \Phi = \sum_{n=2}^{\infty} (c_n r^n + d_n r^{-n} + e_n r^{2+n} + f_n r^{2-n})(a_n \sin n\varphi + b_n \cos n\varphi). \quad (25\,\mathrm{d})$$

V.

The boundary conditions are satisfied only by the solution F_4 in the form of a series. Taking $F = F_4$ we obtain

$$\frac{\partial F}{\partial r} = \sum_2^{\infty} (n c_n r^{n-1} - n d_n r^{-n-1} + (2+n) e_n r^{1+n} + (2-n) f_n r^{1-n})(a_n \sin n\varphi + b_n \cos n\varphi),$$

$$\frac{\partial^2 F}{\partial r^2} = \sum_2^{\infty} [n(n-1) c_n r^{n-2} + n(n+1) d_n r^{-n-2} + (2+n)(1+n) e_n r^n + (2-n)(1-n) f_n r^{-n}](a_n \sin n\varphi + b_n \cos n\varphi),$$

$$\frac{\partial F}{\partial \varphi} = \sum_2^{\infty} (c_n r^n + d_n r^{-n} + e_n r^{2+n} + f_n r^{2-n})(n a_n \cos n\varphi - n b_n \sin n\varphi),$$

$$\frac{\partial^2 F}{\partial \varphi^2} = \sum_2^{\infty} (c_n r^n + d_n r^{-n} + e_n r^{2+n} + f_n r^{2-n})(-n^2 a_n \sin n\varphi - n^2 b_n \cos n\varphi),$$

$$\frac{\partial^2 F}{\partial r \partial \varphi} = \sum_2^{\infty} (n c_n r^{n-1} - n d_n r^{-n-1} + (2+n) e_n r^{1+n} + (2-n) f_n r^{1-n})(n a_n \cos n\varphi - n b_n \sin n\varphi).$$

The first formula (18) gives

$$\nu_r = \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \varphi^2} = \sum_2^{\infty} [n(1-n) c_n r^{n-2} - n(1+n) d_n r^{-n-2} + (1+n)(2-n) e_n r^n + (1-n)(2+n) f_n r^{-n}](a_n \sin n\varphi + b_n \cos n\varphi). \quad (26)$$

On the inner surface of the shell we have

$$r = r_1, \quad \nu_r = -p_1 = \gamma r_1 \cos \varphi. \quad (27)$$

We can develop $f(\varphi) = \cos\varphi$ in the interval $0 - \pi$ in a Fourier's series [4])

$$f(\varphi) = a + \sum_1^{\infty} a'_n \cos 2n\varphi + \sum_1^{\infty} b'_n \sin 2n\varphi,$$

where

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[4]) The function $\cos\varphi$ in the interval $0 - \pi$ signifies a periodical intermittent function having for the limits of the interval ($\varphi = 0, \pi, 2\pi, \ldots$) the values $\cos\varphi = \mp 1$. The series in which we develop $\cos\varphi$, gives for the limits of the interval the arithmetic average of both values of the function, therefore zero.

$$a=\frac{1}{\pi}\int_0^\pi f(\varphi)\,d\varphi=\frac{1}{\pi}\int_0^\pi \cos\varphi\,d\varphi=\frac{1}{\pi}[\sin\varphi]_0^\pi=0,$$

$$a'_n=\frac{2}{\pi}\int_0^\pi f(\varphi)\cos 2n\varphi\,d\varphi=\frac{2}{\pi}\int_0^\pi \cos\varphi\cos 2n\varphi\,d\varphi$$

$$=\frac{2}{\pi}\cdot\frac{1}{2}\int_0^\pi[\cos(2n+1)\varphi+\cos(2n-1)\varphi]\,d\varphi=\frac{1}{\pi}\left[\frac{\sin(2n+1)\varphi}{2n+1}+\frac{\sin(2n-1)\varphi}{2n-1}\right]_0^\pi=0,$$

$$b'_n=\frac{2}{\pi}\int_0^\pi f(\varphi)\sin 2n\varphi\,d\varphi=\frac{2}{\pi}\int_0^\pi \cos\varphi\sin 2n\varphi\,d\varphi$$

$$=\frac{2}{\pi}\cdot\frac{1}{2}\int_0^\pi[\sin(2n+1)\varphi+\sin(2n-1)\varphi]\,d\varphi=\frac{1}{\pi}\left[-\frac{\cos(2n+1)\varphi}{2n+1}-\frac{\cos(2n-1)\varphi}{2n-1}\right]_0^\pi$$

$$=\frac{1}{\pi}\left(\frac{2}{2n+1}+\frac{2}{2n-1}\right)=\frac{8}{\pi}\cdot\frac{n}{(2n-1)(2n+1)}.$$

We obtain

$$\cos\varphi=\sum_1^\infty b'_n\sin 2n\varphi=\frac{8}{\pi}\sum_1^\infty\frac{n}{(2n-1)(2n+1)}\sin 2n\varphi$$

$$=\frac{8}{\pi}\left(\frac{1}{1\cdot 3}\sin 2\varphi+\frac{2}{3\cdot 5}\sin 4\varphi+\frac{3}{5\cdot 7}\sin 6\varphi+\frac{4}{7\cdot 9}\sin 8\varphi+\ldots\right). \qquad (28)$$

Substituting $r=r_1$ in (26) and developing $\cos\varphi$ in the series (28), we find from the condition (27)

$$\sum_2^\infty[n(1-n)c_n r_1^{n-2}-n(1+n)d_n r_1^{-n-2}+(1+n)(2-n)e_n r_1^n$$
$$+(1-n)(2+n)f_n r_1^{-n}](a_n\sin n\varphi+b_n\cos n\varphi)=\gamma r_1\frac{8}{\pi}\sum_1^\infty\frac{n}{(2n-1)(2n+1)}\sin 2n\varphi.$$

In order that this condition may be satisfied for each value of φ, it is necessary that $b_n=0$ and also that each a_n with an odd index must vanish. Joining the coefficient a_n with the coefficients c_n, d_n, e_n, f_n, we have, putting on the left hand side $2n$ instead of n, the condition

$$2n(1-2n)c_{2n}r_1^{2n-2}-2n(1+2n)d_{2n}r_1^{-2n-2}+(1+2n)(2-2n)e_{2n}r_1^{2n}$$
$$+(1-2n)(2+2n)f_{2n}r_1^{-2n}=\gamma r_1\cdot\frac{8}{\pi}\cdot\frac{n}{(2n-1)(2n+1)}. \qquad (29\text{a})$$

This condition is valid for each integer $n=1\div\infty$.

As all b_n and all a_n with odd indexes vanish and all other a_n terms can be joined with the constants c, d, e, f, the equation (26) becomes

$$\nu_r=\sum_1^\infty[2n(1-2n)c_{2n}r^{2n-2}-2n(1+2n)d_{2n}r^{-2n-2}+(1+2n)(2-2n)e_{2n}r^{2n}$$
$$+(1-2n)(2+2n)f_{2n}r^{-2n}]\sin 2n\varphi. \qquad (26\text{a})$$

The second formula (18) gives generally

$$v_s = \frac{\partial^2 F}{\partial r^2} = \sum_2^\infty [n(n-1)c_n r^{n-2} + n(n+1)d_n r^{-n-2} + (2+n)(1+n)e_n r^n + (2-n)(1-n)f_n r^{-n}](a_n \sin n\varphi + b_n \cos n\varphi);$$

omitting all b_n and all a_n with odd indexes and joining all a_n with even indexes to the constants c, d, e, f, we arrive at the formula

$$v_s = \sum_1^\infty [2n(2n-1)c_{2n} r^{2n-2} + 2n(2n+1)d_{2n} r^{-2n-2} + (2+2n)(1+2n)e_{2n} r^{2n} + (2-2n)(1-2n)f_{2n} r^{-2n}] \sin 2n\varphi. \quad (30)$$

Finally with respect to (19) the general equation emerges:

$$\tau = \frac{1}{r^2}\frac{\partial F}{\partial \varphi} - \frac{1}{r}\frac{\partial^2 F}{\partial r \partial \varphi} = \sum_2^\infty [n(1-n)c_n r^{n-2} + n(1+n)d_n r^{-n-2} - n(1+n)e_n r^n - n(1-n)f_n r^{-n}](a_n \cos n\varphi - b_n \sin n\varphi);$$

and by analogy with v_r, v_s we find

$$\tau = \sum_1^\infty [2n(1-2n)c_{2n} r^{2n-2} + 2n(1+2n)d_{2n} r^{-2n-2} - 2n(1+2n)e_{2n} r^{2n} - 2n(1-2n)f_{2n} r^{-2n}] \cos 2n\varphi. \quad (31)$$

For the constants c_{2n}, d_{2n}, e_{2n}, f_{2n} we can obtain, besides the equation (29 a), three further equations from boundary conditions. On the outer surface $r = r_2$ and $v_r = 0$, which gives by each $\sin 2n\varphi$ in (26 a) the coefficient zero:

$$2n(1-2n)c_{2n} r_2^{2n-2} - 2n(1+2n)d_{2n} r_2^{-2n-2} + (1+2n)(2-2n)e_{2n} r_2^{2n} + (1-2n)(2+2n)f_{2n} r_2^{-2n} = 0. \quad (29\,b)$$

On the inner surface $r = r_1$ and $\tau = 0$; in the equation (31) the coefficient of $\cos 2n\varphi$ must vanish, therefore

$$2n(1-2n)c_{2n} r_1^{2n-2} + 2n(1+2n)d_{2n} r_1^{-2n-2} - 2n(1+2n)e_{2n} r_1^{2n} - 2n(1-2n)f_{2n} r_1^{-2n} = 0. \quad (29\,c)$$

Finally, also, on the outer surface ($r = r_2$) the stress $\tau = 0$, which gives analogically the condition

$$2n(1-2n)c_{2n} r_2^{2n-2} + 2n(1+2n)d_{2n} r_2^{-2n-2} - 2n(1+2n)e_{2n} r_2^{2n} - 2n(1-2n)f_{2n} r_2^{-2n} = 0. \quad (29\,d)$$

We have, therefore, four linear equations for c_{2n}, d_{2n}, e_{2n}, f_{2n} and can calculate the constants from them. It is necessary to compute values for $n = 1, 2, 3, \ldots$ and to include in the series for v_r, v_s, τ as many terms as are needed for exact calculation.

For the elongations ϱ and σ equations (20) and (21) can be used.

Summary.

In a hollow circular cylinder subjected to a variable radial pressure the stresses and strains are functions of the coordinates r and φ only. For two strains we obtain from general equations two partial simultaneous differential equations of the second order, which can be transformed into two independent

equations for each strain separately. From two conditions of equilibrium of an element and three relations of stresses and strains, which are partial differential equations of the first order, we can eliminate four unknown functions and obtain also for each of the three stresses an independent equation as a differential equation of the fourth order. The stresses and strains can be expressed by means of a stress function; this being a biharmonical function given by a differential equation of the fourth order. Integrating it we obtain stress functions of different forms. The boundary conditions are satisfied in the present case only by a function in the form of a series.

Zusammenfassung.

Ist eine zylindrische Schale durch einen veränderlichen radialen Druck innen beansprucht, so sind die Spannungen und die Verformungen allein Funktionen der Koordinaten r und φ. Für die Verformungen erhält man aus den allgemeinen Gleichungen zwei partielle simultane Differentialgleichungen 2. Ordnung, aus welchen zwei unabhängige Gleichungen für beide Verformungen abgeleitet werden können. Zwei Gleichgewichtsbedingungen eines Elements und drei Gleichungen, welche die Abhängigkeit der Spannungen und Verformungen ausdrücken, sind partielle simultane Differentialgleichungen 1. Ordnung; durch Elimination von vier unbekannten Funktionen kann man auch für die Spannungen unabhängige Gleichungen in der Form von Differentialgleichungen 4. Ordnung aufstellen. Die Spannungen und Verformungen können auch durch eine Spannungsfunktion ausgedrückt werden, die durch eine Differentialgleichung 4. Ordnung gegeben ist. Durch Integration findet man die Spannungsfunktion in verschiedenen Formen. Die Oberflächenbedingungen können in diesem Falle nur durch eine Spannungsfunktion in der Form einer Reihe befriedigt werden.

Résumé.

Dans une enveloppe cylindrique circulaire, sollicitée par une pression radiale variable, les composantes des tensions et des déformations sont des fonctions de deux coordonnées r, φ seulement. Pour deux composantes de déformation on tire des équations générales deux équations différentielles partielles simultanées que l'on peut transformer séparément en deux équations indépendantes pour chaque composante de déformation. De deux conditions d'équilibre d'un élément et de trois relations entre les tensions et les déformations qui sont des équations différentielles de premier ordre, on peut éliminer quatre fonctions inconnues et parvenir ainsi pour chacune des trois tensions à une équation différentielle indépendante de quatrième ordre. Les tensions et les déformations peuvent être exprimées par une seule fonction; c'est une fonction biharmonique donnée par une équation différentielle de quatrième ordre. En intégrant cette équation on obtient différentes formes de la fonction. Dans le cas présent, les conditions aux surfaces sont satisfaites seulement par une fonction en forme de série.