The Stresses due to a non-uniform Change in the Temperature of a Truss

Die Spannungen in Fachwerken, verursacht durch ungleichmäßige Temperaturänderungen

Contraintes mises en jeu dans les treillis par les variations non uniformes de la température

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I. Introduction

The temperature of a truss or any other structure seldom remains constant. On the contrary it changes continually, following the change in the temperature of the surrounding atmosphere. Such a change, however, may be uniform as in the case of a through bridge without side walks, fig. 1a, where all members of the main girder trusses are exposed to the direct rays of the sun. Also in the case of a deck bridge, fig. 1b, where the floor projects beyond the main girders and thus covers all members from the direct sun beams.

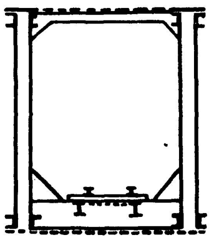

Fig. 1a

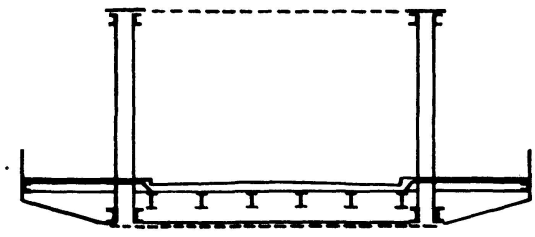

Fig. 1c

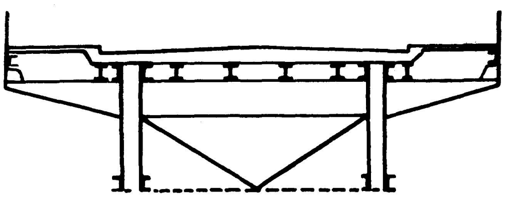

Fig. 1b

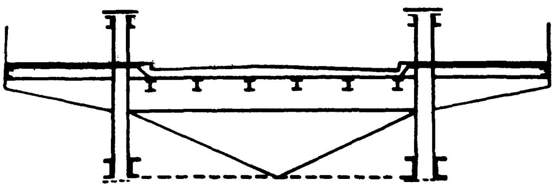

Fig. 1d

The change in the temperature of the truss may also be non-uniform. This is for example the case for a through bridge with side-walks, fig. 1c, which shelter the lower chord members from the direct heat of the sun or a direct cold wind. Also in the case of a deck bridge with upper chord members projecting above the floor, fig. 1d, the upper chord will be exposed to the direct sun beams, while the web members and the lower chord will be sheltered by the side walks and the floor.

Now, a change in temperature is accompanied by a corresponding change in length. A uniform change in temperature causes a uniform change in length, and a non-uniform change in temperature a non-uniform change in length. In the first case, every member of length s becomes $s + \Delta s = s \pm \alpha \cdot s \cdot \Delta t$, where: α = coefficient of linear expansion of the material of construction, and Δt = the uniform change of temperature. All members will expand or contract in the same ratio as their original lengths. Consequently the shape of the truss at $t + \Delta t$ degrees will be similar to its shape at t degrees. The straight members remain straight and all angles unaltered, fig. 2.

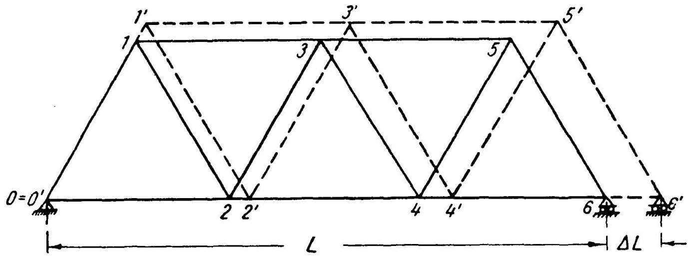

Fig. 2

In other words, the members are shifted parallel to themselves without suffering any change in direction. There are thus no rotations of the members at all. In short the effect of a uniform change of temperature is equivalent to a change of $\alpha \cdot \Delta t$ in the linear scale of the truss. If the truss is hinged at one end and freely supported at the other so that it can expand or contract without constraint, no stresses will be produced by a uniform change in temperature. This is true for statically determinate or indeterminate trusses, with pinned or with rigid joints.

On the other hand, if the change in the temperature of the truss is not uniform, for example, if there is a relative difference of temperature Δt between the lower chord and the rest of the members, then the geometrical shape of the truss will be altered. The members of the lower chord, in this case, will expand or contract relatively more than the other members. Even if the truss is simply supported, and is thus free to expand and to contract along the span, the angles between the system lines will be changed and the joints themselves deflected, fig. 3a.

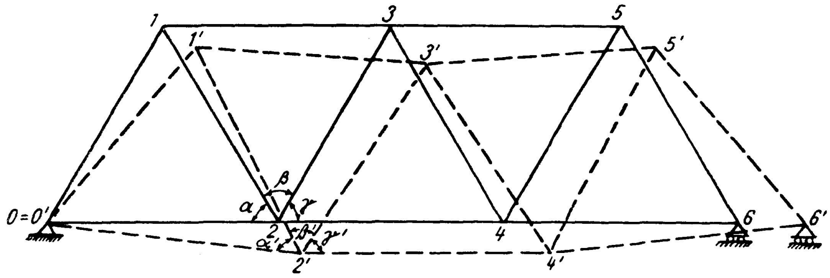

Fig. 3a

The members are not shifted parallel to themselves as in the case of a uniform change of temperature. On the contrary, they will be rotated and their slope angles changed. However, no stresses are produced in the members as long as the truss is provided with pinned connections and is statically determinate. The pinned connections allow every member to rotate freely about its end hinges, without any constraint.

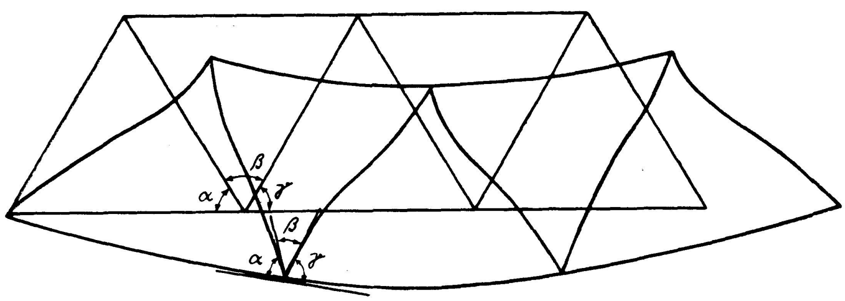

Fig. 3b

On the other hand, if the members are riveted or welded at their end connections, the change in the geometrical shape of the truss is followed by a distortion of the members, fig. 3b. Owing to the rigidity of the connections, the angles enclosed between the end tangents of the members are not altered. Every joint rotates as a rigid whole. But the angles between the straight centre lines joining the hinges are changed. Consequently the tangents at the ends of a certain member $m-n$ will not coincide with the straight line $m-n$ between the two joints m and n. The shape of the member is thus distorted, and the member is subject to moments M_{mn} and M_{nm} at its end points. This applies of course to all other members. The rigid truss may be considered similar to an equivalent truss with pinned connections, if the corresponding moments M_{mn} and M_{nm} are applied to the hinged ends of the different members. These end moments produce axial as well as shearing forces in the members of the truss.

The stresses, produced in the members of a truss by a non-uniform change of temperature, are thus a combination of the effects of the bending moments at the ends of the members and the corresponding axial and shearing forces in these members. It should be remembered that these stresses are due to the rigidity of the connections, so that the problem of their determination can be referred to the general problem of determining the bending stresses (often called secondary stresses!) in a truss with rigid joints. This general problem has been treated by many authors and analysed for different types of trusses. The chief case of loading, which has been generally discussed, is the case of panel-point loading. Only few authors have treated direct loading between the panel points. However, the effect of a non-uniform change of temperature seems not yet to have been treated.

II. The different Methods of Computation

The common methods adopted for determining the bending moments, and consequently the stresses due to the rigidity of the connections, are:

a) Method of Strain-Energy or Virtual Work

The rigid truss is considered to be a statically indeterminate frame. Its degree of redundancy is obviously $3\,n$, where n is the number of closed polygons in the figure. Consequently the truss shown in fig. 3, for example, is $3 \cdot 5 = 15$ times indeterminate. The problem involves in this case the solution of 15 equations of work. However, if the truss is symmetrical and the loading is also symmetrical, or opposite symmetrical, the number of equations can be reduced to 8 or almost half the degree of redundancy.

Generally the statically indeterminate truss with rigid connections is reduced to a statically determinate main system by introducing $3\,n$ hinges at the ends of the members, fig. 4. At a joint where m members are connected, $(m-1)$

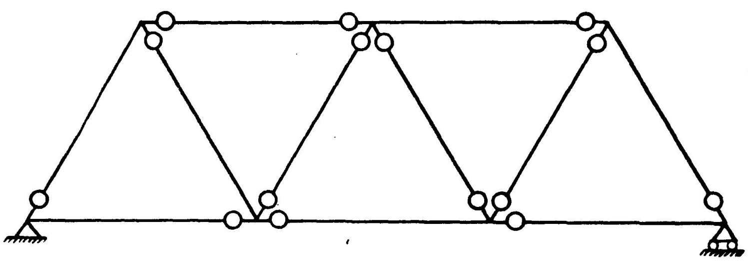

Fig. 4

hinges are introduced leaving the moment in the remaining member to be determined from the condition of equilibrium $\Sigma M = 0$ at the joint. The moments at the introduced hinges are the unknown indeterminates in the equations of work.

This method is rather tedious as it requires a good deal of mathematical computation. The effect of the shearing forces on the deformations is generally neglected. Also the effect of the axial forces is seldom considered in the calculations. The accuracy of the method will thus depend on how far the shearing and axial forces bear upon the results.

b) *Method of Joint Rotations (Mohr's Method)*

The deformations of the truss are considered to be partly due to the displacements and partly due to the rotations of the joints. The joint displacements are caused by the changes in the lengths of the members, which change the geometrical shape of the system and the directions of the straight centre lines between the panel points. The joint rotations, on the other hand, are due to the rigidity of the connections, which keeps the angles between the end tangents of the elastic lines of the members unaltered.

The end moments are then expressed in terms of joint and member, or system line, rotations. The latter are generally calculated from the joint displacements of an equivalent truss with pinned connections. In this way the effect of the end moments on the joint displacements is neglected and the displacements of the rigid truss are simply replaced by the corresponding displacements of the hinged truss. Accordingly the member rotations are calculated in advance so that the joint rotations are the only unknown values in the problem.

There are thus as many unknowns as there are joints in the truss. In fig. 3, for example, the total number of unknowns is 7 although the truss is 15 times indeterminate. This number is further reduced to about a half, if the truss is symmetrical and the loading is also symmetrical, or opposite symmetrical. In such a case the number of unknowns will be only 4 instead of 8 as in the method of work. The reduced number of unknowns naturally simplifies the computation.

The joint rotations are obtained by solving the equations of equilibrium, $\Sigma M = 0$, of the different joints. The end moments are finally calculated from the corresponding joint and member rotations. The degree of accuracy of this method depends on how far the displacements of the rigid truss coincide with those of a similar truss with pinned connections.

c) *Method of Moment Distribution*

This method assumes also that the displacements of the joints are known in advance. The common practice is to determine these displacements for an equivalent truss with pinned connections in the same way as in the method of joint rotations. The effect of the end moments on the joint displacements is thus neglected.

All joints are then considered to be locked and therefore prevented from rotation. In other words all members are considered to be rigidly fixed at their ends. For this condition, the moments at the fixed ends of every member are computed according to the relative displacement of its end joints.

The fixed-end moments so obtained, will not be in equilibrium at every joint of the truss. They are balanced out by releasing one locked joint after the other successively. The unbalanced moment at a certain joint is distributed among the members of the connection according to their relative stiffness. The distributed moments call into play further new moments at the other ends of the members, which are distributed in a similar way and so on. The process is repeated until all joints are balanced out.

Similar to the previous method of joint rotations, the results of the method of moment distribution depend for their accuracy on how far the displacements, found for the equivalent truss with pinned connections, coincide with the real displacements of a truss with rigid joints.

d) Comparison between the different Methods

The first classical method of work claims undoubtedly a higher degree of accuracy above the results of all other methods. This is the case only if the effect of the axial forces, and possibly that of the shearing forces, on the displacement of the joints, are included in the computation. Even if the effect of the shearing forces is neglected, the axial forces arising from the virtual couples $M = 1$ at the introduced hinges in the main system have to be considered. Otherwise the degree of accuracy attained by the method of work is reduced to the standard of the other methods.

This is obvious, since neglecting the effect of the axial forces due to $M = 1$ means simply that the end moments bend the truss members only without displacing the joints. The same assumption is made by the other methods, which take the displacements of the rigid truss to be identical with those of an equivalent truss with pinned connections. The error involved in this proposition is merely the effect of the end moments on the displacement of the joints. Consequently the results obtained by the method of work by neglecting the effect of the axial forces, coincide with those obtained by the methods of joint rotations or moment distribution, which are based on the displacements of the pinned truss.

In the case of the method of work, the results obtained are improved by including the effect of the axial forces in the calculations. Similarly, the results of the two other methods can be improved, if the displacements of the pinned truss are corrected in order to give a better coincidence with the displacements of the rigid truss. This is achieved by introducing the effect of the end moments on the joint displacements into the calculations.

However, the effect of the end moments on the displacement of the joints is not so important in the case of a simple triangular system fig. 3. For such a truss, the results obtained by the ordinary methods of computation, which are based on the displacements of an equivalent truss with pinned connections, show a reasonable degree of accuracy. On the other hand, if the truss is provided with posts, or what is still more conspicuous, in a truss with sub-panelling or a multiple web system, e. g. rhombic diagonals, the results of the ordinary method of computation are unreliable. This is due to the greater discrepency between the displacements of the rigid and the pinned systems. The results based on the displacements of the pinned truss are much bigger than the real values.

If the classical method of work is used in the computation, the effect of the axial forces must be involved in the calculations. Practical accuracy is also attained if the truss is referred to a simple rigid one without posts nor underpanelling as main system. This main system is of course statically indeterminate. It can be calculated in the ordinary way from the displacements of an equivalent truss with pinned connections. Finally the effect of the posts or underpanelling on the rigid main system is determined.

III. The Effect of a non-uniform Change of Temperature on a simple triangular Frame

The object of this preliminary treatment is to illustrate the different methods used in determining the bending stresses due to the rigidity of the connections. It is also intended to show how far the effect of the axial forces on the joint displacements bears on the results. Later a similar investigation is carried out for a triangular frame with a central vertical member.

a) Method of Virtual Work

The closed frame, fig. 5, is thrice statically indeterminate. The main system is obtained by introducing hinges at a, b, and c respectively. In this way, the three unknown values are the end moments M_a, M_b, and M_c at the corresponding joints of the frame.

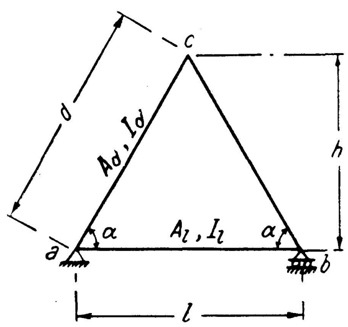

Fig. 5

At every hinge two equal and opposite moments $M = 1$ are then applied. These give the loading cases a, b, and c respectively. The corresponding bending moments, as well as axial and shearing forces, are shown in fig. 6.

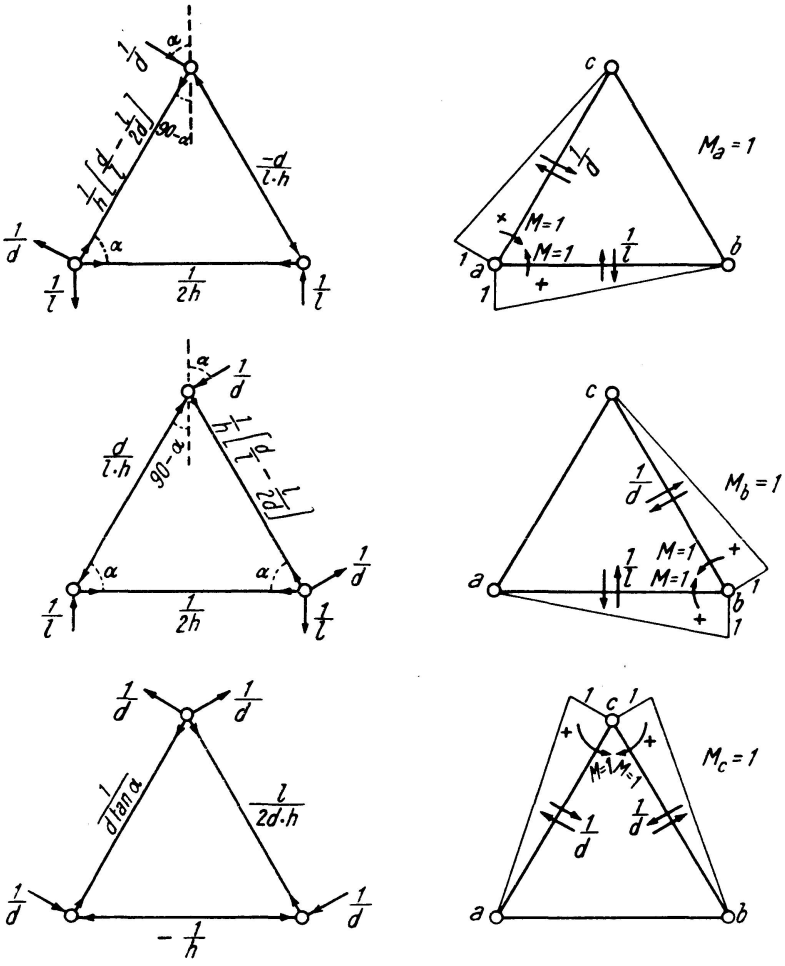

Fig. 6

Consider now the effect of a non-uniform change in the temperature of the main system. Let the side ab expand a distance $\Delta = \alpha \cdot t \cdot l$, due to a rise of t^0 in its temperature relative to the other two members. These latter experience no change in length.

The change Δ in the length of the member ab causes a displacement of the joints b and c relative to the joint a. Taking a to be fixed and b to move along the direction of ab, the members bc and ca will rotate relative to their original directions. This rotation is due to the change in the angles of the $\Delta\, abc$. The distorted shape of the hinged main system is shown in fig. 7. The corresponding changes in the angles of the frame are:

$$\psi_{ao} = \psi_{bo} = \frac{\rho}{d} = \frac{\Delta}{2 \cdot \sin \alpha \cdot d} = \frac{\Delta}{2\,h}$$

and

$$\psi_{co} = -\,(\psi_{ao} + \psi_{bo}) = -\,\frac{\Delta}{h}.$$

Further, neglecting the effect of the shearing forces, the changes in the angles due to the loading cases a, b, and c will be partly due to the bending moments and partly due to the corresponding axial forces. Thus: $\epsilon = \tau + \psi$, where ϵ = total change in a certain angle, τ = change due to the bending of the members, and ψ = change due to the elongations and contractions of the sides of the frame.

The corresponding values are:

$$\begin{aligned}
\epsilon_{aa} &= \epsilon_{bb} = \tau_{aa} + \psi_{aa} = \tau_{bb} + \psi_{bb} \\
&= \frac{1}{3E}\left[\frac{d}{I_d} + \frac{l}{I_l}\right] + \frac{1}{E\cdot h^2}\left[\frac{l}{4A_l} + \frac{d}{A_d}\left\{\frac{d^2}{l^2} + \left(\frac{d}{l} - \frac{l}{2d}\right)^2\right\}\right],
\end{aligned}$$

$$\epsilon_{cc} = \tau_{cc} + \psi_{cc} = \frac{2\cdot d}{3\cdot E\cdot I_d} + \frac{l}{E\cdot h^2}\left[\frac{1}{A_l} + \frac{l}{2\cdot d\cdot A_d}\right],$$

$$\epsilon_{ab} = \epsilon_{ba} = \tau_{ab} + \psi_{ab} = \frac{l}{6EI_l} + \frac{1}{Eh^2}\left[\frac{l}{4A_l} - \frac{d}{A_d}\left(\frac{2\cdot d^2}{l^2} - 1\right)\right],$$

$$\epsilon_{bc} = \epsilon_{cb} = \tau_{bc} + \psi_{bc} = \frac{l}{6EI_d} - \frac{l}{2Eh^2}\left[\frac{1}{A_l} + \frac{l}{2dA_d}\right], \quad \text{and}$$

$$\epsilon_{ac} = \epsilon_{ca} = \tau_{ac} + \psi_{ac} = \tau_{bc} + \psi_{bc} = \epsilon_{bc} \quad \text{since the frame is symmetrical.}$$

However, owing to the rigidity of the connections of the frame, there will be no change in the angles measured between the end tangents of the members. Hence the equations of elasticity are:

$$\begin{aligned}
M_a\cdot\epsilon_{aa} + M_b\cdot\epsilon_{ab} + M_c\cdot\epsilon_{ac} + \psi_{ao} &= 0\\
M_a\cdot\epsilon_{ba} + M_b\cdot\epsilon_{bb} + M_c\cdot\epsilon_{bc} + \psi_{bo} &= 0\\
\text{and } M_a\cdot\epsilon_{ca} + M_b\cdot\epsilon_{cb} + M_c\cdot\epsilon_{cc} + \psi_{co} &= 0.
\end{aligned}$$

The solution of these equations is relatively simple because the frame is symmetrical. The end moments are given by the following expressions:

$$M_a = M_b = \frac{(\psi_{co}\cdot\epsilon_{ac} - \psi_{ao}\cdot\epsilon_{cc})}{[\epsilon_{cc}(\epsilon_{aa} + \epsilon_{ab}) - 2\epsilon_{ac}^2]},$$

and
$$M_c = \frac{1}{\epsilon_{ac}}\cdot\left[\frac{(\psi_{ao}\cdot\epsilon_{cc} - \psi_{co}\cdot\epsilon_{ac})\cdot(\epsilon_{aa} + \epsilon_{ab})}{\epsilon_{cc}\cdot(\epsilon_{aa} + \epsilon_{ab}) - 2\epsilon_{ac}^2} - \psi_{ao}\right].$$

Further, if the effect of the axial forces is neglected, the equations of elasticity become:

$$\begin{aligned}
M_a\cdot\tau_{aa} + M_b\cdot\tau_{ab} + M_c\cdot\tau_{ac} + \psi_{ao} &= 0\\
M_a\cdot\tau_{ba} + M_b\cdot\tau_{bb} + M_c\cdot\tau_{bc} + \psi_{bo} &= 0\\
\text{and}\quad M_a\cdot\tau_{ca} + M_b\cdot\tau_{cb} + M_c\cdot\tau_{cc} + \psi_{co} &= 0
\end{aligned}$$

Accordingly:

$$M_a = M_b = \frac{(\psi_{co}\cdot\tau_{ac} - \psi_{ao}\cdot\tau_{cc})}{[\tau_{cc}(\tau_{aa} + \tau_{ab}) - 2\tau_{ac}^2]},$$

and
$$M_c = \frac{1}{\tau_{ac}}\cdot\left[\frac{(\psi_{ao}\cdot\tau_{cc} - \psi_{co}\cdot\tau_{ac})\cdot(\tau_{aa} + \tau_{ab})}{\tau_{cc}\cdot(\tau_{aa} + \tau_{ab}) - 2\cdot\tau_{ac}^2} - \psi_{ao}\right].$$

Introducing the values previously obtained for the changes in the angles of the frame in the above expressions of M_a, M_b, and M_c, the end moments of the members are obtained. A numerical example is solved hereafter, in order to show analytically the effect of the axial forces on the results.

Numerical Example: Let $l = d = 5$ ms, $h = 4{,}33$ ms, $t = 20^0$, $\alpha = 0{,}000012$, $E = 2150$ t/cm^2,

$$\left.\begin{aligned} A_d &= 81{,}2 \text{ cm}^2 \\ I_d &= 2022 \text{ cm}^4 \end{aligned}\right\} \begin{aligned} &4 \text{ Angles } 80 \cdot 120 \cdot 8 \\ &1 \text{ Plate } 240 \cdot 8 \end{aligned}$$

$$\left.\begin{aligned} A_l &= 364{,}8 \text{ cm}^2 \\ I_l &= 56980 \text{ cm}^4 \end{aligned}\right\} \begin{aligned} &4 \text{ Angles } 100 \cdot 100 \cdot 10 \\ &6 \text{ Plates } 400 \cdot 10 \\ &2 \text{ Plates } 200 \cdot 12 \end{aligned}$$

These dimensions correspond to the condition encountered in a central panel of a main girder truss of about 40 ms span.

Here: $\Delta = \alpha \cdot t \cdot l = 0{,}12$ cms.

Further: $$E \cdot \psi_{ao} = E \cdot \psi_{bo} = \frac{E \cdot \Delta}{2\,h} = 0{,}297921$$

$$E \cdot \psi_{co} = -\frac{E \cdot \Delta}{h} = -\,0{,}595842\,.$$

And
$$\begin{aligned} E \cdot \epsilon_{aa} &= E \cdot \epsilon_{bb} = 0{,}085352 + 0{,}000043 = 0{,}085395 \\ E \cdot \epsilon_{cc} &= 0{,}164853 + 0{,}000024 = 0{,}164877 \\ E \cdot \epsilon_{ab} &= E \cdot \epsilon_{ba} = 0{,}001462 - 0{,}000031 = 0{,}001431 \\ E \cdot \epsilon_{bc} &= E \cdot \epsilon_{cb} = E \cdot \epsilon_{ac} = E \cdot \epsilon_{ca} = 0{,}041213 - 0{,}000012 = 0{,}041201 \end{aligned}$$

In the above values of ϵ, the first number gives the effect τ of the bending moments, while the second number gives the effect ψ of the axial forces.

Therefore: $M_a = M_b = -\,\underline{6{,}746}$ t · cm and $M_c = +\,\underline{6{,}985}$ t · cm.

Further, by neglecting the effect of the axial forces:

$$M_a = M_b = -\,\underline{6{,}750} \text{ t} \cdot \text{cm} \quad \text{and} \quad M_c = +\,\underline{6{,}989} \text{ t} \cdot \text{cm}\,.$$

It follows that the results obtained by neglecting the effect of the axial forces on the deformations, are almost equal to those obtained by taking this effect into consideration. For a simple triangular system, it is thus permissible to neglect the effect of the axial forces altogether and to consider only the effect of the bending moments.

b) Method of Joint Rotations

Assume the joint displacements of the rigid frame to be identical with those of an equivalent frame with pinned connections, as shown in fig. 7. The change Δ in the length of ab causes a displacement of the joints b and c relative to the joint a of the frame. Consequently the members bc and ca will be rotated relative to their original directions.

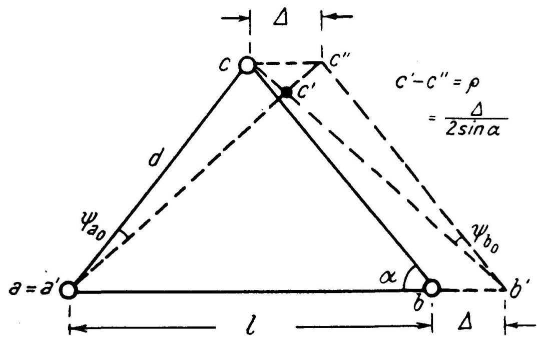

Fig. 7

Further, since the frame is symmetrical, the rotations ψ_{ac} and ψ_{bc} of the system lines ac and bc must be equal and opposite. On the other hand, the system line ab will not rotate at all so that $\psi_{ab} = 0$.

Now if the members were really hinged to one another at their ends, such a deformation in the geometrical shape of the frame would not produce any stresses in the members. The sides of the frame remain straight in spite of the joint displacements. This is not possible if the joints of the frame are rigid.

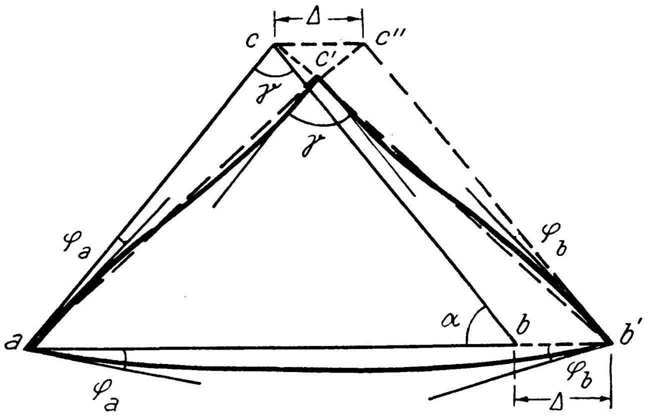

Fig. 8a

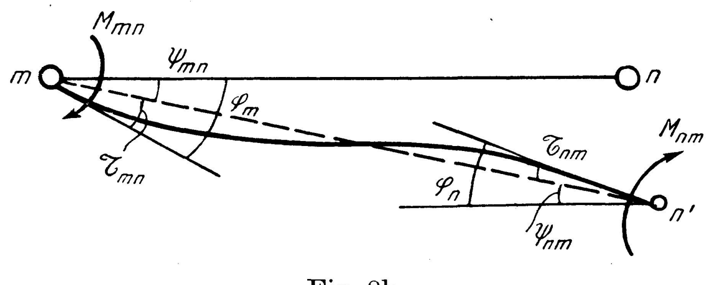

Fig. 8b

Since the angles between the system lines are changed, the members of the rigid frame must bend so as to keep the angles between the end tangents at the connections unaltered, fig. 8a. In this way the joints will prescribe angular rotations. From the symmetry of the frame abc, the corresponding angular rotations φ_a and φ_b of the joints a and b must be equal and opposite. For the same reason, $\varphi_c = 0$, since the joint c will not rotate.

Refer now to fig. 8b, where the end point n of the member mn is displaced to n', causing a rotation ψ_{mn} of the straight line mn. If φ_m and φ_n are the angular rotations of the end joints m and n respectively, the slope angles τ_{mn} and τ_{nm} of the end tangents to the elastic line of the member can be written in the form:

$$\tau_{mn} = \varphi_m - \psi_{mn} \quad \text{and} \quad \tau_{nm} = \varphi_n - \psi_{nm}.$$

But the end moments are functions of these slope angles. Therefore:

$$M_{mn} = \frac{2\,EI_{mn}}{l_{mn}}[2\,\tau_{mn} + \tau_{nm}] = N_{mn}\,[2\,\varphi_m + \varphi_n - 3\,\psi_{mn}]$$

and $$M_{nm} = \frac{2\,EI_{mn}}{l_{mn}}(2\,\tau_{nm} + \tau_{mn}) = N_{mn}\cdot(2\,\varphi_n + \varphi_m - 3\,\psi_{mn})$$

where: I_{mn} = Moment of inertia, l_{mn} = Length, and

$$N_{mn} = \frac{2\,EI_{mn}}{l_{mn}} = \text{Stiffness of the member } mn.$$

For the triangular frame abc: $\varphi_a = -\varphi_b = \varphi$ and $\varphi_c = 0$.

Also: $\psi_{ac} = -\psi_{cb} = \psi = \dfrac{\Delta}{2\,h}$ and $\psi_{ab} = 0$.

Therefore:
$$\begin{cases} M_{ab} = N_l\,(2\,\varphi_a + \varphi_b - 3\,\psi_{ab}) = N_l\cdot\varphi \\ M_{ac} = N_d\,(2\,\varphi_a + \varphi_c - 3\,\psi_{ac}) = N_d\,(2\,\varphi - 3\,\psi) \end{cases}$$
$$\begin{cases} M_{ca} = N_d\,(2\,\varphi_c + \varphi_a - 3\,\psi_{ac}) = N_d\,(\varphi - 3\,\psi) \\ M_{cb} = N_d\,(2\,\varphi_c + \varphi_b - 3\,\psi_{cb}) = N_d\,(-\varphi + 3\,\psi) \end{cases}$$
$$\begin{cases} M_{bc} = N_d\,(2\,\varphi_b + \varphi_c - 3\,\psi_{bc}) = N_d\,(-2\,\varphi + 3\,\psi) \\ M_{ba} = N_l\,(2\,\varphi_b + \varphi_a - 3\,\psi_{ab}) = -N_l\cdot\varphi \end{cases}$$

where: $$N_l = \frac{2\,EI_l}{l} \quad \text{and} \quad N_d = \frac{2\,EI_d}{d}.$$

But the sum of all moments acting at every joint of the frame must be zero. In other words: $\sum_a M = 0$, $\sum_b M = 0$, and $\sum_c M = 0$.

Therefore: $M_{ab} + M_{ac} = M_{ba} + M_{bc} = M_{ca} + M_{cb} = 0$.

Or $N_l\cdot\varphi + N_d\cdot(2\,\varphi - 3\,\psi) = 0$. Hence $\varphi = \dfrac{3\,N_d}{N_l + 2\,N_d}\cdot\psi$

but $\psi = \dfrac{\Delta}{2\,h}$. Therefore: $\varphi = \dfrac{1{,}5\,\Delta}{h\left[\dfrac{I_l}{I_d}\cdot\dfrac{d}{l} + 2\right]}$

And:

$$M_{ab} = N_l \cdot \varphi = \frac{3\,E\,\Delta}{h\left[\frac{d}{I_d} + \frac{2\,l}{I_l}\right]}$$

$$M_{ac} = -\,M_{ab} = \frac{-\,3\,E\,\Delta}{h\left[\frac{d}{I_d} + \frac{2\,l}{I_l}\right]}$$

$$M_{ca} = N_d \cdot (\varphi - 3\,\psi) = \frac{-\,3\,E\,\Delta}{h}\left[\frac{\frac{I_d}{d} + \frac{I_l}{l}}{2 + \frac{I_l}{I_d} \cdot \frac{d}{l}}\right]$$

$$M_{cb} = -\,M_{ca} = \frac{3\,E\,\Delta}{h}\left[\frac{\frac{I_d}{d} + \frac{I_l}{l}}{2 + \frac{I_l}{I_d} \cdot \frac{d}{l}}\right]$$

$$M_{bc} = -\,M_{ba} = M_{ab} = \frac{3\,E\,\Delta}{h\left[\frac{d}{I_d} + \frac{2\,l}{I_l}\right]}$$

Numerical Example: As before $l = d = 5$ ms, $h = 4{,}33$ ms, $I_l = 56980$ cm^4, $I_d = 2022$ cm^4, $\alpha = 0{,}000012$, $t = 20^0$.

Hence $\quad \Delta = \alpha \cdot t \cdot l = 0{,}12$ cms.

And $\quad M_{ab} = M_{bc} = +\,\underline{6{,}750}$ t · cm $\qquad M_{ca} = -\,\underline{6{,}989}$ t · cm

$\quad M_{ac} = M_{ba} = -\,\underline{6{,}750}$ t · cm $\qquad M_{cb} = +\,\underline{6{,}989}$ t · cm.

The results obtained by the ordinary method of joint rotations are thus identical with those obtained by the method of work, if the effect of the axial forces in the latter is neglected. In this connection it is reminded that the effect of the axial forces on the angles by the method of work is equivalent to the effect of the end moments in a rigid truss on the joint displacements. Neglecting the effect of the axial forces by the method of work is therefore equivalent to assuming the joint displacements of the rigid truss to be the same as those of a similar truss with pinned connections.

c) *Method of Moment Distribution*

It is assumed here also that the joint displacements of the rigid frame are the same as those of an equivalent frame with hinged connections. In this way they can be determined in advance as shown in fig. 7. Assuming further that the joints are locked against rotation, every member of the frame becomes a beam which is rigidly fixed at both ends. The directions of the end tangents are parallel to the original directions of the members, fig. 9a. The relative displacements of the ends produce bending moments in the members, which are calculated in a similar way to those produced by a settlement of the supports.

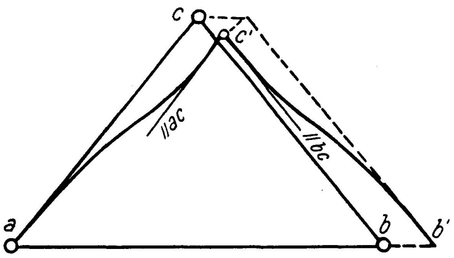

Fig. 9a

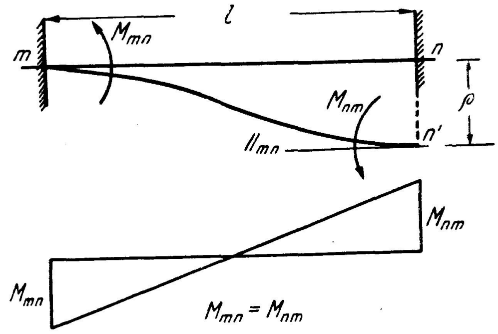

Fig. 9b

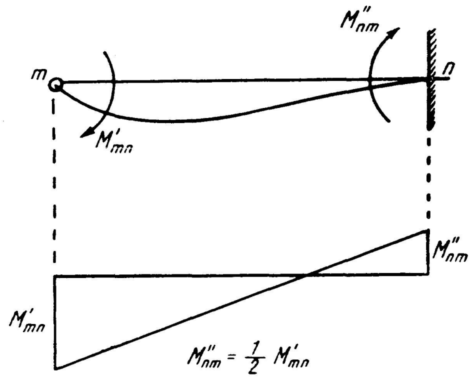

Fig. 9c

Let the end n, for example, of the fixed member mn, fig. 9b, deflect a distance ρ normal to the original direction of the member. The corresponding end moments are: $M_{mn} = M_{nm} = -3\,N \cdot \psi = -\frac{6\,E\,I\,\rho}{l^2}$ where I = moment of inertia, l = length, and N = stiffness of the member mn. The negative sign denotes that, for a clockwise rotation of the member mn, both M_{mn} and M_{nm} will be counter-clockwise.

Referring to fig. 7, the values of ρ can be determined from the change Δ in the length of ab.

Thus: $$\rho_{ab} = 0; \quad \rho_{ac} = \frac{\Delta \cdot d}{2\,h}; \quad \rho_{bc} = -\frac{\Delta \cdot d}{2\,h}$$

And the corresponding moments are simply: $M_{ab} = M_{ba} = 0$,

$$M_{ac} = M_{ca} = \frac{-3\,E \cdot I_d \cdot \Delta}{h \cdot d}, \qquad \text{and} \qquad M_{cb} = M_{bc} = \frac{3\,E\,I_d \cdot \Delta}{h \cdot d}.$$

These values refer, of course, to the case where the joints are locked against rotation. This is the reason why the moments at the joints a and b of the frame are not in equilibrium. If the joints are released, the sum of the moments at every joint must be zero.

Let the joint a, for example, be now released. The fixed end moments at this joint will be changed until the condition $\sum\limits_{a} M = 0$ is satisfied. The joint a

will of course rotate to this effect. In other words, since the sum of the fixed end moments at joint a:

$$\underset{a}{\Sigma M} = M_{ab} + M_{ac} = -\frac{3\,E\,I_d \cdot \Delta}{h \cdot d}$$

and not zero, it is necessary to apply a balancing moment $\frac{3\,E\,I_d \cdot \Delta}{h \cdot d}$ in order to bring about equilibrium. This moment is distributed between the members ab and ac in the ratio of their stiffness, i.e. in the ratio $\frac{I_l}{l} : \frac{I_d}{d}$. The introduction of these balancing moments into the system produces new moments at the joints b and c. These latter are termed the carried over moments.

Referring to fig. 9c, if the joint n at the end of the member mn is still locked and a moment M'_{mn}, for example, is applied at the released end m, an end moment $M''_{nm} = \frac{1}{2} M'_{mn}$ is produced at joint n. In the case of the triangular frame abc, this applies to the unbalanced joints a and b. The carried over moments M''_{nm} have still to be balanced out in a similar way as the fixed end moments. The process should be repeated until all joints are eased down.

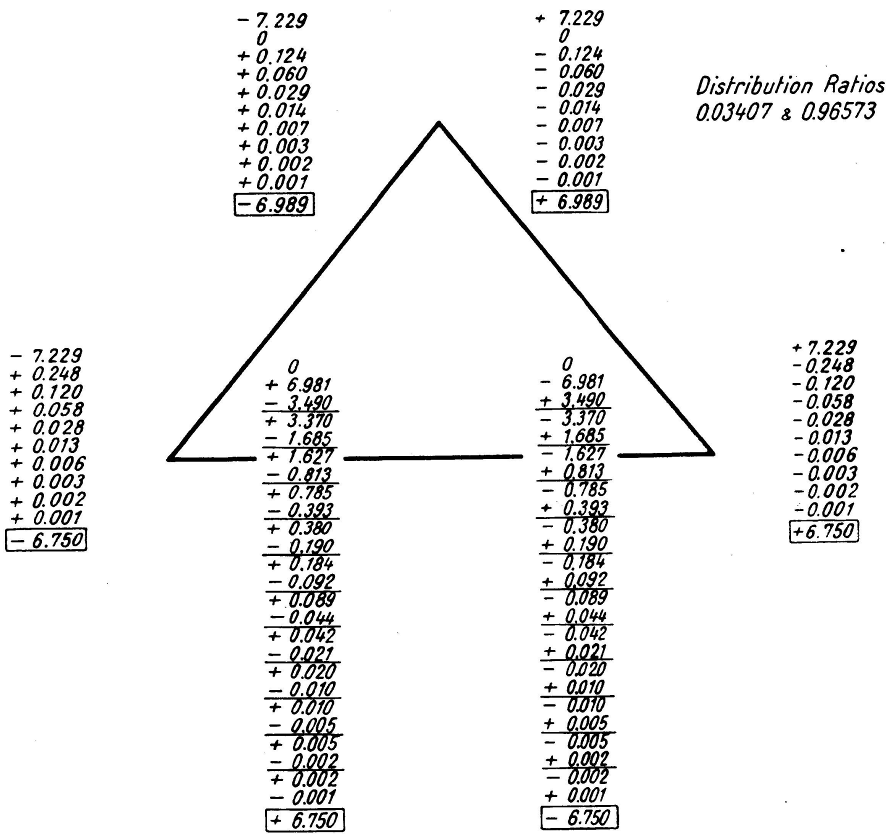

Fig. 10

Fig. 10 shows the method of obtaining the end moments of the numerical example by the method of moment distribution. The results so obtained coincide with those obtained before by the method of joint rotations. They coincide as well with the results of the method of work, if the effect of the axial forces is neglected.

IV. The Effect of a non-uniform Change in Temperature on a triangular Frame with a central Hanger

The triangular frame abc, which has been treated in the preceding paragraph, is now provided with a central hanger cd, see fig. 11a. An expansion Δ of the side adb due to a non-uniform change in the temperature of the frame causes a relative displacement of the joints and consequently a distortion of the members.

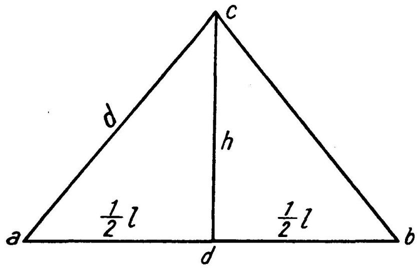

Fig. 11a

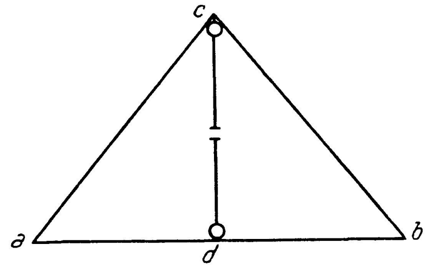

Fig. 11b

The bending moments produced at the rigid connections are calculated first by the method of work, and then by the method of joint rotations. In the latter case, the joint displacements of the rigid frame are taken from the displacement diagram of an equivalent frame with pinned connections, corresponding to what is generally done in practice.

As will be seen later, the results of the two methods are not the same. This is explained by the fact that, in the case of a central hanger, the displacement of the joints due to the end moments at the rigid connections cannot be altogether neglected. A direct solution based on the displacements of a hinged frame fails to give the same degree of accuracy.

a) *Method of Work*

The introduction of the central hanger cd renders the problem 6 times indeterminate. However, since the frame is considered symmetrical about the line cd, a change in the length of adb on either side does not change the direction of the central hanger, which remains straight after the deformation. Consequently the central hanger will not be subject to any bending, as if it were hinged at c and d respectively, fig. 11b. The moments produced at the ends of the other members are the same in both cases.

The frame shown in fig. 11b is only 4 times indeterminate. By cutting the central hanger cd, the problem is transformed into a simple triangular frame with rigid connections. This serves as a statically indeterminate main system to the frame with the central hanger.

1. Effect of non-uniform Temperature on Main System

This problem has been solved already. The results of a direct application of the method of joint rotations are here almost identical with those of the method of work. For this reason, the end moments are calculated directly from the joint displacements of the hinged system, fig. 12.

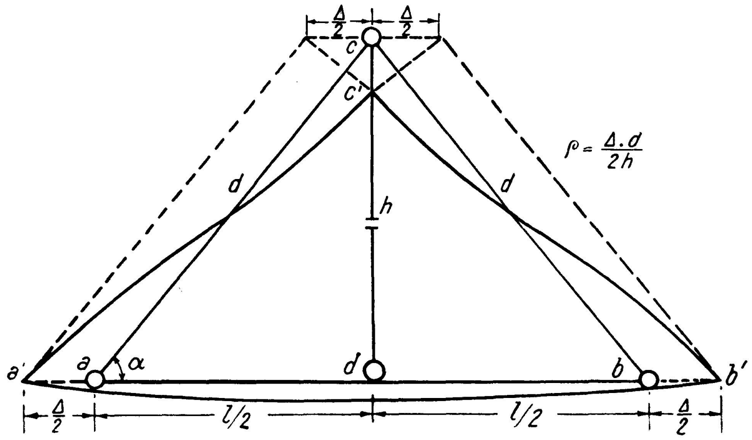

Fig. 12

Therefore:

$$\left.\begin{array}{l} M_{ab} = M_{bc} = \\ M_{ba} = M_{ac} = \end{array}\right\rangle \pm \frac{3\,E\cdot\Delta}{h\left[\dfrac{d}{I_d}+\dfrac{2\,l}{I_l}\right]}$$

and:

$$\left.\begin{array}{l} M_{cb} = \\ M_{ca} = \end{array}\right\rangle \pm \frac{3\,E\cdot\Delta}{h}\cdot\left[\frac{\dfrac{I_d}{d}+\dfrac{I_l}{l}}{2+\dfrac{I_l}{I_d}\cdot\dfrac{d}{l}}\right]$$

Further, the corresponding displacement of the severed ends of the central hanger:

$$\Delta_{10} = \overline{c-c'} - \overline{d-d'} = \rho\cos\alpha - \frac{M_{ab}-M_{ba}}{16\,E I_l}\cdot l^2$$

The first term is due to the displacement of the joint c, while the second is due to the flexure of the side ab. But $\rho = \dfrac{\Delta\cdot d}{2h}$ and

$$M_{ab} = -\,M_{ba} = \frac{3\,E\Delta}{h\cdot\left[\dfrac{d}{I_d}+\dfrac{2\,l}{I_l}\right]} \qquad \text{Hence: } \Delta_{10} = \frac{\Delta\cdot l}{4h}\left[1-\frac{\dfrac{3}{2}\cdot\dfrac{l}{I_l}}{\left[\dfrac{d}{I_d}+\dfrac{2\,l}{I_l}\right]}\right]$$

2. Effect of a force $P = \pm 1$ on Main System

The statically indeterminate main system is further reduced to a statically determinate problem by introducing 3 hinges at a, b, and c respectively. The forces $P = \pm 1\,t$ in the central hanger, fig. 13a, produce axial forces in the members. The diagonals receive a tensile force: $D = \dfrac{d}{2h}$ which causes a change in length: $\Delta_d = \dfrac{d^2}{2E \cdot h \cdot A_d}$.

On the other hand, the chord ab receives a compression: $L = -\dfrac{l}{4h}$ and contracts a distance: $\dfrac{\Delta}{2} = \dfrac{l^2}{8EhA_l}$ on either side. The diagram of joint displacements is given in fig. 13b.

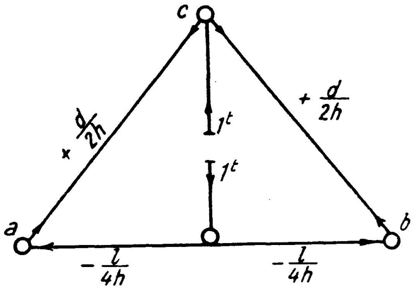

Fig. 13a

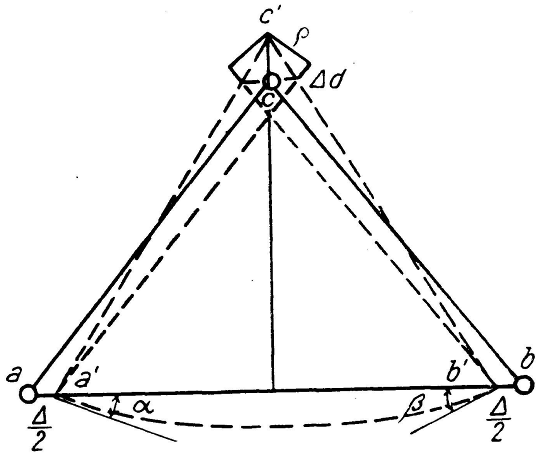

Fig. 13b

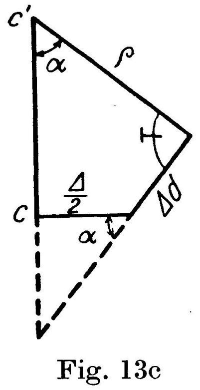

Fig. 13c

The change in the direction of the sides ac and bc is calculated from the normal displacement:

$$\rho = \frac{\Delta_d}{\tan\alpha} + \frac{\Delta}{2\sin\alpha} = \frac{d \cdot l}{4Eh^2}\left[\frac{d}{A_d} + \frac{l}{2A_l}\right].$$

The corresponding member rotations are:

$$\left.\begin{array}{l}\psi_{bc} = \\ \psi_{ac} = \end{array}\right\rangle \pm \frac{\rho}{d} = \pm \frac{l}{4Eh^2}\left[\frac{d}{A_d} + \frac{l}{2A_l}\right].$$

Besides, the direct load at d causes the side ab to deflect. The end slope angles at a and b in the hinged system are:

$$\left.\begin{array}{l}\alpha = \\ \beta = \end{array}\right\rangle \pm \frac{l^2}{16\,EI_l}\,.$$

Further, since the frame is symmetrical about cd, the joint rotations φ_c and φ_d are both equal to zero. For the same reason: $\varphi_b = -\varphi_a$.

In order now to maintain equilibrium at joint a: $\sum\limits_a M = 0$, or $M_{ab} + M_{ac} = 0$.

But $\quad M_{ab} = N_l[2(\varphi_a - \alpha) + (\varphi_b - \beta)] = N_l[\varphi_a - \alpha]$

and $\quad M_{ac} = N_d[2\,\varphi_a + \varphi_c - 3\,\psi_{ac}] = N_d[2\,\varphi_a - 3\,\psi_{ac}]$.

Thus: $\quad N_d[2\,\varphi_a - 3\,\psi_{ac}] + N_l[\varphi_a - \alpha] = 0$

and
$$\varphi_a = \frac{3\,N_d\cdot\psi_{ac} + N_l\cdot\alpha}{2\,N_d + N_l}\,.$$

Substituting $\quad \varphi_a' = E\,\varphi_a\,, \qquad \psi_{ac}' = E\cdot\psi_{ac} = -\frac{l}{4\,h^2}\left[\frac{d}{A_d} + \frac{l}{2\,A_l}\right]$

$$\alpha' = E\,\alpha = \frac{l^2}{16\,I_l}\,, \qquad N_d' = \frac{2\,I_d}{d} \quad \text{and} \quad N_l' = \frac{2\,I_l}{l}$$

$$\varphi_a' = \frac{\frac{l}{16} + 3\,\frac{I_d}{d}\cdot\psi_{ac}'}{\frac{I_l}{l} + 2\,\frac{I_d}{d}}$$

Further: $\quad M_{ab} = M_{bc} = \frac{2\,I_l}{l}\cdot\varphi_a' - \frac{l}{8}\,, \qquad M_{ba} = M_{ac} = -M_{ab}$

$$M_{ca} = N_d[2\,\varphi_c + \varphi_a - 3\,\psi_{ac}] = \frac{2\,I_d}{d}[\varphi_a' - 3\,\psi_{ac}']\,, \qquad M_{cb} = -M_{ca}\,.$$

Finally, the corresponding relative displacement Δ_{11} of the two severed ends of the central hanger is given by: $\Delta_{11} = \Delta_v + \overline{c - c'} + \delta_P + \delta_M$ where

$\Delta_v = \frac{h}{E\cdot A_v}$ = linear contraction of the hanger, and

A_v = its cross-sectional area.

$\overline{c - c'} = \rho\cos\alpha + \Delta_d\cdot\sin\alpha = \frac{1}{E}\left[\frac{d}{2\,A_d} - \frac{l}{2}\cdot\psi_{ac}'\right]$ = displacement of joint c

$\delta_P = \frac{l^3}{48\,EI_l}$ = deflection of point d due to $P = 1$ t

and $\delta_M = \frac{M_{ab} - M_{ba}}{16\,EI_l}\cdot l^2 = \frac{l}{4\,E}\cdot\varphi_a' - \frac{l^3}{64\,EI_l}$ = deflection at d due to end moments of side ab.

Hence: $$\Delta_{11} = \frac{1}{E}\left[\frac{h}{A_v} + \frac{d}{2\,A_d} + \frac{l}{4}\,(\varphi_a' - 2\,\psi_{ac}') + \frac{l^3}{192\,I_l}\right]$$

The corresponding compressive force in the central hanger is thus: $P = \frac{\Delta_{10}}{\Delta_{11}}$.
The corresponding end moments are calculated from the relation:

$M = M_0 + P \,.\, M_P$, where
M = Moment in frame with central hanger due to a non-uniform change of temperature t^0,
M_0 = Moment in simple triangular frame due to a non-uniform change of temperature t^0, and
M_P = Moment in simple triangular frame due to $P = \pm\, 1\, t$.

b) Method of Joint Rotations

The joint displacements are taken directly from the displacement diagram of an equivalent frame with pinned connections, fig. 14a. A rise of t^0 in the temperature of the lower chord causes an expansion of the chord members ad and db on either side of the symmetry axis cd. The lengths of the other members remain unaltered.

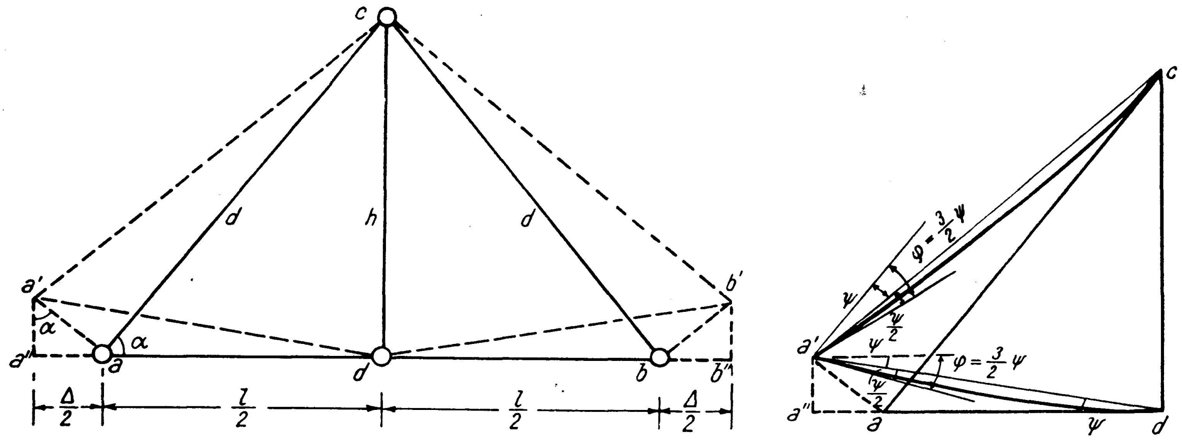

Fig. 14a Fig. 14b

Keeping the direction of the central hanger unchanged, the change in the temperature of the lower chord causes a relative displacement of the two joints a and b only. $a - a'' = b - b'' = \frac{\Delta}{2} = \frac{a \cdot t \cdot l}{2}$

The corresponding member rotations are thus: $\psi_{ad} = \psi_{ac} = \psi = \frac{\Delta}{2h}$.

Further, since the frame is symmetrical about cd: $\psi_{bd} = \psi_{bc} = -\,\psi = -\frac{\Delta}{2h}$.

For the same reason: $\psi_{cd} = 0$ and $\varphi_c = \varphi_d = 0$, i.e. there is no rotation of the joints c and d. Also $\varphi_b = -\,\varphi_a$.

In order to maintain equilibrium at the joint a, $\sum\limits_a M = M_{ad} + M_{ac} = 0$.

But $M_{ad} = N_l \,.\, [2\,\varphi_a + \varphi_d - 3\,\psi_{ad}] = N_l\,[2\,\varphi_a - 3\,\psi]$
and $M_{ac} = N_d \,.\, [2\,\varphi_a + \varphi_c - 3\,\psi_{ac}] = N_d\,[2\,\varphi_a - 3\,\psi]$.

Therefore: $(N_l + N_d) \cdot (2\,\varphi_a - 3\,\psi) = 0$ or $\varphi_a = \frac{3}{2}\psi = \frac{3}{4}\frac{\Delta}{h}$

Hence $M_{ad} = M_{ac} = 0$ and similarly $M_{bd} = M_{bc} = 0$, as if the joints a and b were really hinged. This result is explained in fig. 14b, where the elastic lines and end tangents of the sides ad and ac are drawn.

It is easily seen that: $\tau_{ad} = \tau_{ac} = \varphi_a - \psi = \frac{\psi}{2}$, and $\tau_{da} = \tau_{ca} = -\psi$.

Hence: $2\,\tau_{ad} + \tau_{da} = 2\,\tau_{ac} + \tau_{ca} = 0$, or $M_{ad} = M_{ac} = 0$.

On the other hand $M_{ca} = N_d\,[2\,\varphi_c + \varphi_a - 3\,\psi_{ac}] = -\frac{3}{2}\cdot\frac{E I_d}{d \cdot h}\cdot\Delta$

and $M_{da} = N_l\,[2\,\varphi_d + \varphi_a - 3\,\psi_{ad}] = -\frac{3\,E I_l}{l \cdot h}\cdot\Delta$

Further $M_{cb} = -M_{ca} = \frac{3}{2}\cdot\frac{E I_d}{d \cdot h}\cdot\Delta$ and $M_{db} = -M_{da} = \frac{3\,E I_l}{l h}\cdot\Delta$

Numerical Example:

Let $l = 8$ ms, $h = 5$ ms, $d = 6{,}403$ ms, $t = 20^0$, $\alpha = 0{,}000012$

$A_d = A_v = 81{,}2\ \mathrm{cm}^2$, $I_d = I_v = 2022\ \mathrm{cm}^4$ } 4 Angles 80 · 120 · 8 + Plate 240 · 8

$A_l = 364{,}8\ \mathrm{cm}^2$, $I_l = 56980\ \mathrm{cm}^4$ } 4 Angles 100 · 100 · 10, 6 Plates 400 · 10, 2 Plates 200 · 12.

These dimensions correspond to the two central panels of a main girder truss of about 40 ms span.

a) Method of Work

$\Delta = \alpha \cdot t \cdot l = 0{,}192$ cms

1. Moments due to Δ in Main System

$M_{ab} = M_{bc} = +\,\underline{7{,}184}\ t \cdot \mathrm{cm}$, $M_{ba} = M_{ac} = -\,\underline{7{,}184}\ t \cdot \mathrm{cm}$

$M_{ca} = -\,\underline{7{,}503\ t \cdot \mathrm{cm}}$, and $M_{cb} = +\,\underline{7{,}503}\ t \cdot \mathrm{cm}$

The corresponding displacement: $\Delta_{10} = \underline{0{,}0721084\ \mathrm{cm}}$.

2. Moments due to $P = \pm\,1$ in Main System

$\psi'_{ac} = -\,0{,}007185567$ and $\varphi'_a = 0{,}643944$

Hence: $M_{ab} = M_{bc} = -\,\underline{8{,}270}\ t \cdot \mathrm{cm}$, $M_{ba} = M_{ac} = +\,\underline{8{,}270}\ t \cdot \mathrm{cm}$

$M_{ca} = +\,\underline{4{,}203}\ t \cdot \mathrm{cm}$, $M_{cb} = -\,\underline{4{,}203}\ t \cdot \mathrm{cm}$

And the corresponding displacement: $\Delta_{11} = \underline{0{,}0877039}$

The compressive force in the hanger is thus: $P = \underline{0{,}822180\ t}$

3. Moments in the rigid Frame with Central Hanger

The corresponding end moments in the frame are:

$M_{ad} = M_{bc} = +\ \underline{0{,}385}\ t \cdot \text{cm}$, $M_{bd} = M_{ac} = -\ \underline{0{,}385}\ t \cdot \text{cm}$

Further: $M_{ca} = \underline{-\ 4{,}047}\ t \cdot \text{cm}$, and $M_{cb} = +\ \underline{4{,}047}\ t \cdot \text{cm}$

Finally $M_{db} = \frac{P \cdot l}{4} + \frac{M_{ad} - M_{bd}}{2} = +\ \underline{164{,}821}\ t \cdot \text{cm}$

and $M_{da} = -\ \underline{164{,}821}\ t \cdot \text{cm}$.

b) Method of Joint Rotations

$\Delta = a \cdot t \cdot l = \underline{0{,}192}$ cms

$M_{ad} = M_{ac} = M_{bd} = M_{bc} = 0$. However,

$M_{ca} = -\ \underline{3{,}911}\ t \cdot \text{cm}$; $M_{cb} = +\ \underline{3{,}911}\ t \cdot \text{cm}$

$M_{da} = -\ \underline{176{,}410}\ t \cdot \text{cm}$, and $M_{db} = +\ \underline{176{,}410}\ t \cdot \text{cm}$.

The results obtained by the method of work and the method of joint rotations are not identical. There is a difference of about 7 % between the values given by the two methods for the bending moment at point d. As already explained, this discrepency arises from neglecting the effect of the end moments on the joint displacements. If this effect is included in the calculations, the results obtained will be identical with those of the method of work.

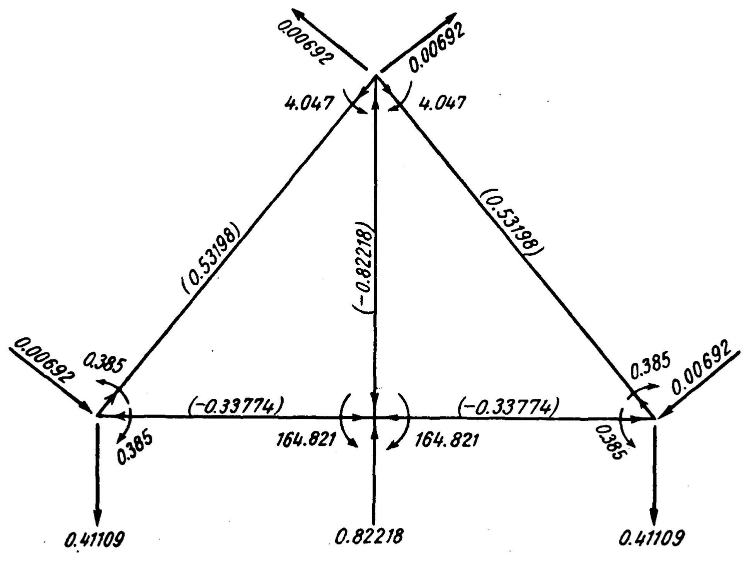

Fig. 15

Fig. 15 shows the end moments of the frame together with the corresponding panel loads. The axial forces produced in the different members are written in brackets. The changes in lengths due to these axial forces are:

$$E \,.\, \Delta_{ac} = 4{,}194903,\ E\psi_{ad} = -\ 0{,}370325;\ E\Delta_v = -\ 5{,}062685.$$

The corresponding displacement diagram supplies:

$$E \cdot \rho_{ac} = \frac{E \cdot \Delta_{ac}}{\tan \alpha} + \frac{E \cdot \Delta_{ad}}{\sin \alpha} = +\, 3{,}830161;$$

$$E \cdot \rho_{ad} = E \cdot \Delta_v + E \Delta_{ac} \sin \alpha + E \rho_{ac} \cos \alpha = 10{,}731146$$

Hence: $E \cdot \psi_{ac} = -\, 0{,}0059818$; and $E \cdot \psi_{ad} = -\, 0{,}026828$.

But the member rotation of the hinged frame is: $E \psi = \frac{E \Delta}{2h} = +\, 0{,}4128$.

Therefore the corrected member rotations are:

$$E \cdot \psi_{ac} = +\, 0{,}406818 \text{ and } E \psi_{ad} = 0{,}385972.$$

Further, for the equilibrium of the joint $a \cdot \sum_a M = 0$.

Or $N_{ac} \cdot (2\,\varphi_a + \varphi_c - 3\,\psi_{ac}) + N_{ad}\,(2\,\varphi_a + \varphi_d - 3\,\psi_{ad}) = 0$.

Substituting $\varphi_c = \varphi_d = 0$ for symmetry, $N_{ac}\,(2\,\varphi_a - 3\,\psi_{ac}) + N_{ad}\,(2\,\varphi_a - 3\,\psi_{ad}) = 0$.

Hence $$\varphi_a = \frac{3}{2} \cdot \frac{[N_{ac} \cdot \psi_{ac} + N_{ad} \cdot \psi_{ad}]}{[N_{ac} + N_{ad}]}$$

The corresponding value is thus: $E \cdot \varphi_a = +\, 0{,}5796363$.

Therefore $M_{ad} = N_{ad}\,(2\,\varphi_a - 3\,\psi_{ad}) = +\, \underline{0{,}386}\; t\,.\,\text{cm}$

$M_{da} = N_{ad}\,(\varphi_a - 3\,\psi_{ad}) = -\,\underline{164{,}752}\; t \cdot \text{cm}$, $M_{ca} = N_{ac}\,(\varphi_a - 3\,\psi_{ac}) = -\,\underline{4{,}047}\; t \cdot \text{cm}$.

As already anticipated, the results obtained by the method of joint rotations for the corrected slope angles almost coincide with those obtained by the method of work. The slight difference of about 0,04 % in the value of M_{da} is negligible. It is reminded, however, that the effect of the axial forces on the main system has been neglected in the method of work. The corresponding end moments have been calculated by the ordinary method of joints. For absolute coincidence, this effect has also to be included in the computation.

V. Effect of a non-uniform Change in Temperature on Warren Girder with Verticals

The object of investigation is a Warren truss with hangers and posts, which is designed as main girder of a single track railway bridge. The truss has a span of 40 ms divided into 10 panels. The height of girder is 5 ms. The cross sections

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The same girder has been investigated by the author for upper as well as lower panel point loading. The effect of direct loading between the joints has also been calculated. Those interested are referred to the following books by the same author:

1. „Beitrag zur Berechnung der Biegespannungen in Dreieck-Fachwerken mit und ohne Hilfssystem", 1931, Verlag AG. Gebr. Leemann & Co., Zürich und Leipzig.

2. "The Bending Stresses in the Members of a Truss due to the rigid Connections at the Joints", 1937, AG. Gebr. Leemann & Co., Zurich.

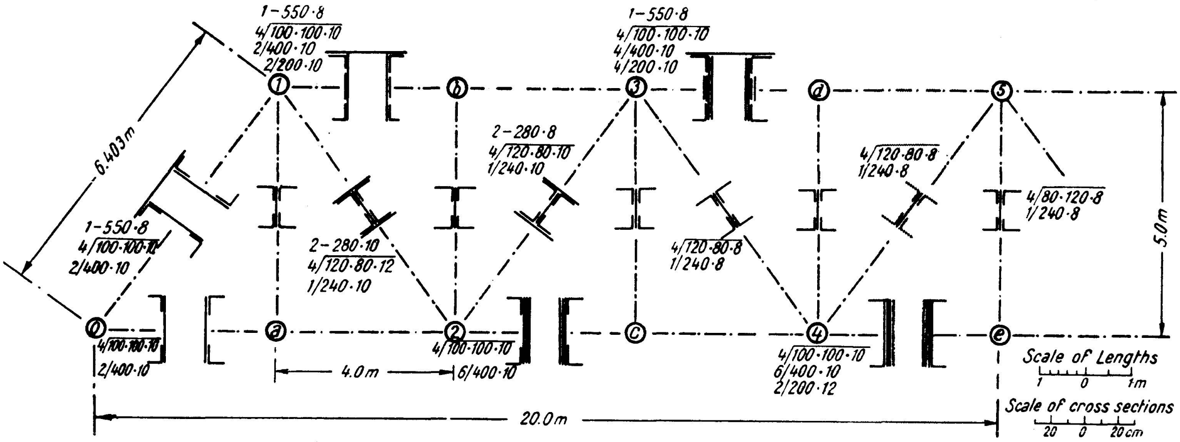

Fig. 16. Main Girder of a Single Track Railway Bridge designed according to Swiss Specifications

of the different members are given in fig. 16. The adjoining table I contains the necessary data of lengths, areas, moments of inertia and moduli of section for the computation. All dimensions are in accordance with ordinary design practice.

It is assumed that the temperature of the lower chord is raised 20^0 above that of the other members. This relative change in temperature causes joint displacements, which are followed by member distortions due to the rigid connections. The corresponding end moments are calculated a) by the method of work, b) by the method of joint rotations, similar to the procedure explained before for the triangular frame with central hanger. As will be seen later, the results of the two methods vary rather considerably.

Table I

	Member	Length l cm	A gross cm^2	I gross cm^4	Z gross cm^3
Lower Chord	0−2	800	156,8	34047	1702
	2−4	800	316,8	55380	2769
	4−4′	800	364,8	56980	2849
Upper Chord	1−3	800	240,8	50351	*U.* 2942 I. 2122
	3−5	800	360,8	63455	*U.* 3470 I. 2821
Diagonals	0−1	640	200,8	48343	*U.* 2967 I. 1975
	2−3	640	145,2	5528	394,9
	4−5	640	81,2	2022	163,1
	1−2	640	170,8	6800	485,7
	3−4	640	95,6	2522	203,4
	Posts	500	81,2	2022	163,1

Finally a method is developed by which the joint rotations may be calculated by successive correction without solving the equations of equilibrium. This method leads to almost exact results. It reduces the mathematical computation considerably.

a) Method of Work

Since all members of the truss are not hinged, but are rigidly connected at their ends, the Warren girder shown in fig. 17a becomes a redundant frame, which is $3 \cdot 18 = 54$ times statically indeterminate. The problem, however, is simplified by assuming the verticals to be hinged at their end connections to the chord members, fig. 17b. This assumption is justified because the chords are much stiffer than the vertical posts and hangers. The moments produced at the ends of the verticals are therefore very much smaller than those produced in the chords. A verification of this statement will be given later.

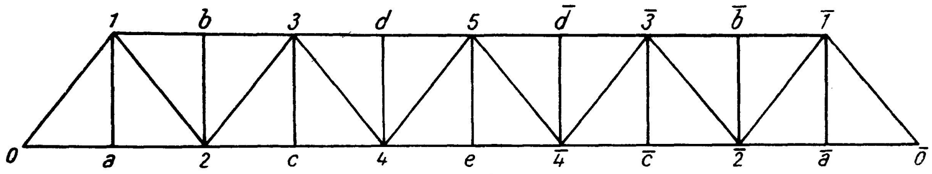

Fig. 17a. System with Rigid Connections of Verticals

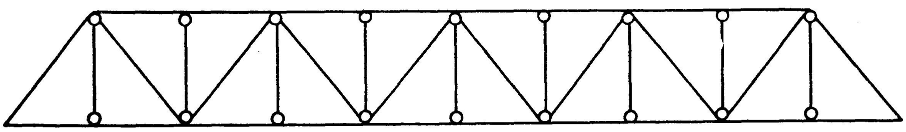

Fig. 17b. System with Hinged Verticals

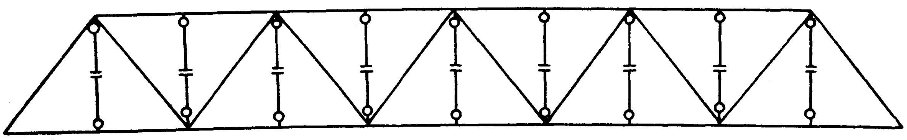

Fig. 17c. Simple Warren with Rigid Connections

Referring now to the system with hinged verticals, fig. 17b, and assuming all these verticals to be cut, the problem is reduced to a simple Warren girder with rigid connections, fig. 17c. This is still statically indeterminate and involves $3 \cdot 9 = 27$ redundant values, where 9 is the number of closed triangular frames in the system.

Consequently, the system with hinged verticals, as shown in fig. 17b, is $27 + 9 = 36$ times indeterminate, 9 being the surplus of unknown axial forces in the posts and hangers. But since the girder is symmetrical about the central hanger 5e, the deformations, due to a change in the temperature of the lower chord, will also be symmetrical about the central hanger. It is therefore sufficient to determine the moments and axial forces on one side only of the symmetry axis. The corresponding values on the other side are symmetrical. In this way the number of unknown values in the problem is reduced almost to a half.

The simple Warren girder with rigid connections is taken as a statically indeterminate main system for the girder with hinged verticals. The solution of the problem involves the determination of 14 end moments for the main system, and 5 axial forces in the verticals. There are thus $14 + 5 = 19$ unknown values from which all others can be derived. The steps followed in the solution are:

1. Determination of the end moments of the main system.
2. Calculation of the axial forces in the posts, and finally
3. Determination of the end moments of the Warren girder with hinged posts and hangers.

1. Determination of the End Moments of the Simple Warren Girder due to a Rise of 20° in the temperature of the Lower Chord

The simple Warren girder with rigid joints, fig. 17c, is 27 times statically indeterminate. A change in the temperature of its lower chord causes relative displacements of the joints. Further, since all joints are rigid, the angles enclosed at these joints remain unaltered. Consequently the members of the truss will be deflected, and bending moments are produced at their ends.

Owing to the symmetry of the girder and its deformations about the vertical line through joint 5, the determination of the end moments by the method of virtual work necessitates the solution of 14 equations of elasticity. It has been shown, however, that the results obtained by the method of work for a simple triangular frame are almost identical with those obtained by the ordinary method of joint rotations. In other words, the displacements of the rigid system are hardly affected by the end moments. There is thus no necessity to determine the end moments of the simple Warren girder, fig. 17c, by the method of work.

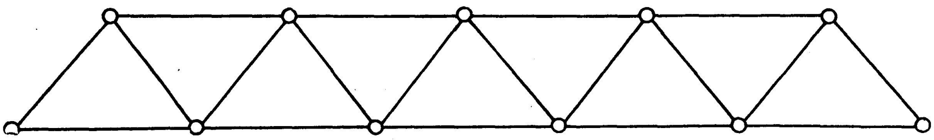

Fig. 17d. Simple Warren with Hinged Connections

The end moments of the statically indeterminate main system are calculated by the ordinary method of joint rotations. The displacements of the different joints are found for an equivalent truss with pinned connections, as shown in fig. 17d. Every member of the lower chord elongates a distance: $\Delta = \alpha \cdot t \cdot l = 0{,}192$ cms, where α = coefficient of linear expansion = 0,000012, $t = 20^0$, and $l = 800$ cms. The value Δ is multiplied by the modulus of elasticity $E = 2150$ t/cm² for the sake of convenience, so that: $\Delta' = E\Delta = 412{,}8$ t & cm units. All other members remain unaltered.

Table II

Member	$N' = \frac{N}{E} = \frac{2I}{l}$ cm³	$\Delta' = E\Delta = Eatl$	$\rho' = E\rho$	$\psi' = E\psi = \frac{E\rho}{l}$	$N'\psi' = N\cdot\psi$ t·cm
1 – 3	125,88	0	990,72	1,2384	155,89
3 – 5	158,64	0	330,24	0,4128	65,49
0 – 2	85,12	412,8	1320,96	1,6512	140,55
2 – 4	138,45	412,8	660,48	0,8256	114,30
4 – 4′	142,45	412,8	0	–	–
0 – 1	151,0	0	1321,58	2,0640	311,66
1 – 2	21,24	0	792,95	1,2384	26,304
2 – 3	17,267	0	792,95	1,2384	21,383
3 – 4	7,878	0	264,32	0,4128	3,252
4 – 5	6,316	0	264,32	0,4128	2,607

Table II gives the elongations Δ' and the relative normal displacements ρ' of the end points of every member. It contains further the member rotations ψ' and the products $N \cdot \psi$. All values are in t & cm units.

For the equilibrium of the joints, $\Sigma M = 0$ at every joint of the frame. Introducing the joint rotations, $\varphi' = E\varphi$, the end moments can be expressed as functions of these joint rotations, the member rotations ψ', and the stiffness N' of the corresponding members. For example:

$$M_{12} = N_{12}(2\,\varphi_1 + \varphi_2 - 3\,\psi_{12}) = N'_{12}(2\,\varphi_1' + \varphi_2' - 3\,\psi'_{12})$$

and ΣM at joint 1 becomes:

$$\Sigma N'_{12}(2\varphi'_1 + \varphi'_2 - 3\psi'_{12}) = 2\varphi'_1\,\Sigma N'_{12} + \Sigma N'_{12}\cdot\varphi'_2 - 3\,\Sigma N'_{12}\cdot\psi'_{12} = 0.$$

Owing to the symmetry of the girder about the vertical line through joint 5, this joint will experience no rotation, so that $\varphi_5 = 0$. It is thus sufficient to write down the equations of equilibrium for the other joints 0, 1, 2, 3, and 4, ommitting φ_5 altogether. These are:

$$\begin{aligned}
&472{,}24\,\varphi'_0 + 151\,\varphi'_1 + 85{,}12\,\varphi'_2 &&= 1356{,}63\\
&151\,\varphi'_0 + 596{,}24\,\varphi'_1 + 21{,}24\,\varphi'_2 + 125{,}88\,\varphi'_3 &&= 1481{,}56\\
&85{,}12\,\varphi'_0 + 21{,}24\,\varphi'_1 + 524{,}16\,\varphi'_2 + 17{,}27\,\varphi'_3 + 138{,}45\,\varphi'_4 &&= 907{,}61\\
&125{,}88\,\varphi'_1 + 17{,}27\,\varphi'_2 + 619{,}34\,\varphi'_3 + 7{,}88\,\varphi'_4 &&= 738{,}05\\
&138{,}45\,\varphi'_2 + 7{,}88\,\varphi'_3 + 447{,}74\,\varphi'_4 &&= 360{,}48
\end{aligned}$$

The solution of the above-mentioned equations supplies the unknown values of joint rotations. The solution itself is better done by Gauss method of elimination. The corresponding joint rotations are:

$$\varphi'_0 = +\,2{,}1031,\quad \varphi'_2 = +\,1{,}1806,\quad \varphi'_4 = +\,0{,}4260$$
$$\varphi'_1 = +\,1{,}7415,\quad \varphi'_3 = +\,0{,}7994,\quad \varphi'_5 = 0.$$

The next step is to determine the slope angles at the ends of every member from the joint rotations just obtained, and the member rotations ψ'. For example: $E\,\tau_{12} = \varphi'_1 - \psi'_{12}$ and $E\,\tau_{21} = \varphi'_2 - \psi'_{12}$ for the member $1-2$.

Hence, the corresponding end moments are:

$$M_{12} = N_{12}(2\,\tau_{12} + \tau_{21}) = N'_{12}(2\,E\,\tau_{12} + E\,\tau_{21}) \text{ and}$$
$$M_{21} = N_{12}(2\,\tau_{21} + \tau_{12}) = N'_{12}(2\,E\,\tau_{21} + E\,\tau_{12}).$$

The end moments of the different members are given in table III.

Table III

Moment at Joint	Opposite Joint	$\varphi' = E\varphi$	$\psi' = E\psi$	M t·cm
0	1	2,1031	2,0640	−36,89
	2		1,6512	+36,87
2	0	1,1806	1,6512	−41,65
	1		1,2384	+ 8,23
	3		1,2384	− 9,58
	4		0,8256	+42,97
4	2	0,4260	0,8256	−61,50
	3		0,4128	+ 3,25
	5		0,4128	− 2,44
	$\bar{4}$		0	+60,68
1	0	1,7415	2,0640	−91,49
	2		1,2384	+20,14
	3		1,2384	+71,40
3	1	0,7994	1,2384	−47,19
	2		1,2384	−16,16
	4		0,4128	+ 6,20
	5		0,4128	+57,17
5	3	0	0,4128	−69,64
	4		0,4128	− 5,13

2. Calculation of the Axial Forces in the Verticals

The axial forces in the vertical posts and hangers are determined in the usual way, using the simple Warren girder with rigid connections as a statically indeterminate main system. The number of unknowns corresponds to the number of verticals, so that the present problem involves the determination of 9 unknown values.

Remembering, however, that the girder is symmetrical about the central hanger, and that the deformations due to the assumed change in temperature is also symmetrical about the same axis, the number of unknowns is reduced to 5 only, namely the axial forces A, B, C, D, and E. Consequently there are five equations of elasticity of the form:

$$\delta'_{AA} \cdot A + \delta'_{AB} \cdot B + \delta'_{AC} \cdot C + \delta'_{AD} \cdot D + \delta'_{AE} \cdot E = \delta'_{A0}$$

The terms δ'_{A0} on the right hand side of the equations are the relative displacements of the severed ends of the verticals in the main system due to the rise of 20^0 in the temperature of the lower chord members. They are calculated from the displacements of the opposite joints in the upper and lower chords, taking into account the deflections due to the end moments of the main system. For example: $\delta'_{A0} = \delta'_{10} - (\delta'_{ao} + \delta'_{aM})$, where: $\delta'_{10} = 825{,}60$, and $\delta'_{ao} = \frac{\delta'_{00} + \delta'_{20}}{2} = 660{,}48$ are the displacements in t and cm units of the joint 1 and the mid-point a of the straight line $0-2$ of the hinged system, and $\delta'_{aM} = \frac{(M_{02} - M_{20}) \cdot l^2}{16\, I_{02}} = 92{,}25$ t and cm units $=$ deflection of point a of the member $0-2$ due to the end moments M_{02} and M_{20} of the indeterminate main system.

Hence: $\delta'_{A0} = 72{,}87$ t and cm units.

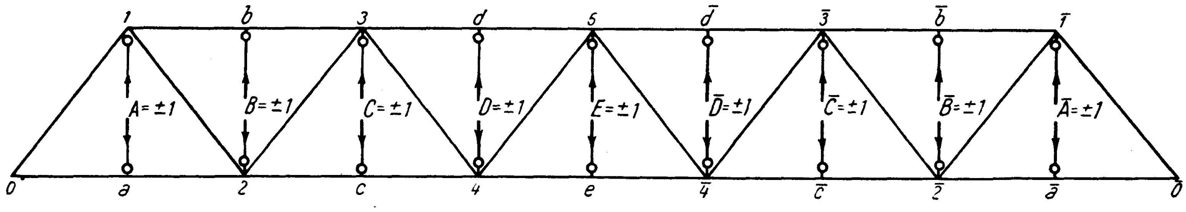

Fig. 18a

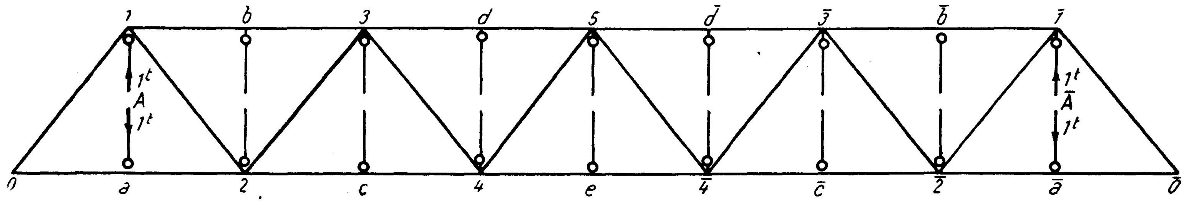

Fig. 18b

All values are multiplied by E for convenience. Downward deflections are considered to be positive. Further, δ'_{A0} is positive, when the distance between 1 and a is reduced.

Similarly the coefficients $\delta'_{AA}, \delta'_{AB}, \ldots$ etc. of the unknown values A, B, $C, \ldots$ on the left hand side of the equations are the relative displacements of the severed ends of the verticals due to the virtual forces $P = \pm 1$, fig. 18a. The effects of each pair of symmetrical posts is here combined. For example, the value δ'_{AA} is the relative displacement of the severed ends of the post A due to the forces $P = \pm 1$ applied to the symmetrical hangers A and $\bar{A}$ at the same time, fig. 18b. It is calculated from the difference of the deflections of the two opposite joints a and l. Besides, it contains the change in the length of the hanger A due to the axial forces $P = \pm 1$.

Thus: $$\delta'_{AA} = \{\delta'_{a(A+\bar{A})} - \delta'_{1(A+\bar{A})}\} + \Delta'_v\,, \qquad \text{where:}$$

$$\delta'_{a(A+\bar{A})} = \frac{\delta'_{0(A+\bar{A})} + \delta'_{2(A+\bar{A})}}{2} + \frac{(M_{02} - M_{20})\cdot l^2}{16\, I_{02}} + \frac{l^3}{48\, I_{02}}$$

= deflection of joint a due to $P = 1$ t acting simultaneously downwards at a and $\bar{a}$, and upwards at 1 and $\bar{1}$, fig. 18b. It is composed of:

1. the effect of the relative displacements of the joints 0 and 2,
2. the deflection due to the end moments M_{02} and M_{20}, and
3. the deflection due to the load $P = 1$ t applied directly at the mid-point a of the member $0-2$.

Further, $\delta'_{1(A+\bar{A})}$ = deflection of joint 1 due to the loading shown in fig. 18b, and $\Delta'_v = \dfrac{l}{A_v}$ = change in length of the vertical A due to $P = 1$ t.

Similarly all values are multiplied by E for convenience. Downward deflections are considered to be positive. Further, the value δ'_{AA} is positive, when the distance between the severed ends is increased.

The corresponding equations of elasticity are:

$$\begin{aligned}
139{,}770\,A + 11{,}096\,B - 33{,}202\,C - 2{,}730\,D + 7{,}659\,E &= 72{,}87\\
10{,}800\,A + 106{,}670\,B + 2{,}130\,C - 20{,}978\,D - 1{,}374\,E &= 94{,}21\\
-33{,}356\,A + 2{,}368\,B + 117{,}768\,C + 3{,}178\,D - 30{,}915\,E &= 89{,}66\\
-2{,}842\,A - 20{,}912\,B + 2{,}896\,C + 71{,}530\,D + 4{,}652\,E &= 79{,}94\\
15{,}446\,A - 2{,}696\,B - 61{,}682\,C + 9{,}534\,D + 108{,}209\,E &= 79{,}93
\end{aligned}$$

All coefficients are in t and cm units.

The solution of the above equations supplies the axial forces in the verticals due to the rise of 20° in the temperature of the lower chord.

They are simply: $A = \underline{0{,}6855\text{ t}}$ $B = \underline{1{,}0661\text{ t}}$ $C = \underline{1{,}2263\text{ t}}$
$D = \underline{1{,}3256\text{ t}}$ $E = \underline{1{,}2496\text{ t}}$.

All are compression forces.

3. Determination of the End Moments of the Warren Girder with Verticals

Having found the axial forces in the verticals, the end moments at the rigid joints can be calculated. They are obtained from the corresponding moments in the simple Warren girder with rigid connections by adding up the end moments M_t due to the rise in the temperature of the lower chord to the end moments M_v due to the axial forces in the vertical members. In other words: $M = M_t + M_v$.

The end moments M_t are given in table III. They have been calculated in paragraph 1. The end moments M_v, however, which are due to the axial forces in the verticals, are calculated from the end moments of the simple Warren girder for panel point loads. These latter are given in table IV. The terms 2^a, a^a, . . . etc. refer to symmetrical cases of loading. Case 2^a, for example, corresponds to a load of $\frac{1}{2}$ t acting at joint 2 and another load of the same magnitude acting at joint $\bar{2}$ at the same time. The cases 5 and e, on the other hand, refer to a load of 1 t at joint 5 or joint e. Accordingly, the end moment at any joint is given by the sum: $M = M_t + \Sigma 2A \cdot (M_{m-a}{}^a - M_{m-1}{}^a)$.

Table IV

Moment at Joint	Opposite Joint	Moments in kg. cm for loading of Upper Chord					Moments in kg. cm for loading of Lower Chord				
		1^a	b^a	3^a	d^a	5	a^a	2^a	c^a	4^a	e
0	2	+ 36	− 6323	− 657	+ 428	− 289	− 37464	− 668	+ 7025	− 361	− **3947**
	1	− 36	+ 6323	+ 657	− 428	+ 289	+ 37464	+ 668	− 7025	+ 361	+ **3947**
2	0	− 704	− 3291	− 1091	− 672	− 400	+ 39433	− 1486	+ 20768	− 406	− **10665**
	1	+ 48	+ 2731	+ 108	− 377	+ 217	− 6153	− 83	+ 6108	+ 260	− **2568**
	3	+ 55	− 1546	+ 126	+ 1664	+ 476	− 4143	+ 181	+ 5090	+ 367	− **1945**
	4	+ 601	+ 2106	+ 858	− 615	− 293	− 29137	+ 1387	− 31966	− 222	+ **15178**
4	2	− 178	+ 76	− 1193	− 2314	− 2296	− 6937	− 325	+ 25224	− 2251	+ **56086**
	3	− 16	− 799	+ 50	+ 729	+ 89	+ 686	− 17	− 2453	+ 36	+ **3871**
	5	+ 1	+ 28	+ 47	+ 13	− 6	+ 494	− 3	− 1901	+ 119	+ **3201**
	$\bar{4}$	+ 193	+ 695	+ 1095	+ 1572	+ 2213	+ 5757	+ 344	− 20870	+ 2095	− **63158**
1	0	− 848	+ 28979	+ 570	− 4707	− 244	+ 11099	− 71	− 3264	+ 144	+ **1834**
	2	+ 119	+ 5161	+ 204	− 704	+ 170	− 4095	+ 14	+ 3207	+ 241	− **1189**
	3	+ 729	− 34140	− 774	+ 5411	+ 74	− 7004	+ 54	+ 57	− 385	− **645**
3	1	− 164	+ 37019	− 2304	+ 17965	+ 444	− 2316	− 1275	− 1505	− 1061	− **1410**
	2	− 11	− 3529	− 6	+ 3120	+ 488	− 1827	+ 80	+ 2518	+ 259	− **928**
	4	− 2	− 1589	+ 107	+ 1490	+ 208	+ 480	+ 35	− 1191	+ 102	+ **2006**
	5	+ 176	− 31901	+ 2202	− 22575	− 1140	+ 3664	+ 1160	+ 178	+ 700	+ **332**
5	3	− 332	− 16784	− 158	+ 60356	− 6013	+ 1412	− 260	− 1987	− 2962	− **3146**
	4	− 8	− 2	− 1	− 57	− 104	+ 239	− 18	− 976	+ 26	+ **1567**

Table V

Moment at point	Other end of member	Post A		Post B		Post C		Post D		Post E		M_t	Total M
		-1,3710 1^a	+1,3710 a^a	2.1322 2^a	-2,1322 b^a	-2,4526 3^a	+2.4526 c^a	2,6512 4^a	-2,6512 d^a	-1,2496 5	1,2496 e		t·cm
0	2	−0,05	−51,36	−1,42	+13,48	+1,61	+17,23	−0,96	− 1,13	+0,36	− 4,93	+36,89	+ 9,76
	1	+0,05	+51,36	+1,42	−13,48	−1,61	−17,23	+0,96	+ 1,13	−0,36	+ 4,93	−36,87	− 9,74
2	0	+0,97	+54,06	−3,17	+ 7,02	+2,68	+50,94	−1,08	+ 1,78	+0,50	−13,33	−41,65	+ 58,72
	1	−0,07	− 8,44	−0,18	− 5,82	−0,26	+14,98	+0,69	+ 1,00	−0,27	− 3,21	+ 8,23	+ 6,65
	3	−0,08	− 5,68	+0,39	+ 3,30	−0,31	+12,48	+0,97	− 4,41	−0,59	− 2,43	− 9,58	− 5,94
	4	−0,82	−39,95	+2,96	− 4,49	−2,10	−78,40	−0,59	+ 1,63	+0,37	+18,97	+42,97	− 59,46
4	2	+0,24	− 9,51	−0,69	− 0,16	+2,93	+61,87	−5,97	+ 6,13	+2,87	+70,09	−61,50	+ 66,29
	3	+0,02	+ 0,94	−0,04	+ 1,71	−0,12	− 6,02	+0,10	− 1,93	−0,11	+ 4,84	+ 3,25	+ 2,63
	5	—	+ 0,68	−0,01	− 0,06	−0,12	− 4,66	+0,32	− 0,04	0,01	+ 4,00	− 2,44	− 2,32
	$\bar{4}$	−0,27	+ 7,89	+0,73	− 1,48	−2,69	−51,19	+5,55	− 4,17	−2,77	−78,92	+60,68	− 66,61
1	0	+1,16	+15,22	−0,15	−61,79	−1,40	− 8,01	+0,38	+12,49	+0,31	+ 2,29	−91,49	−131,00
	2	−0,16	− 5,61	+0,04	−11,00	−0,50	+ 7,87	+0,64	+ 1,87	−0,21	− 1,44	+20,14	+ 11,57
	3	−1,00	− 9,60	+0,12	+72,79	+1,90	+ 0,14	−1,02	−14,35	−0,09	− 0,81	+71,40	+119,48
3	1	+0,23	− 3,18	−2,72	−78,93	+5,65	− 3,69	−2,81	−47,63	−0,56	− 1,76	−47,19	−182,59
	2	+0,02	− 2,51	+0,17	+ 7,53	+0,02	+ 6,18	+0,69	− 8,27	−0,61	− 1,16	−16,16	− 14,12
	4	—	+ 0,66	+0,07	+ 3,39	−0,26	− 2,92	+0,27	− 3,95	−0,26	+ 2,51	+ 6,20	+ 5,71
	5	−0,24	+ 5,02	+2,47	+68,02	−5,40	+ 0,44	+1,86	+59,85	+1,43	+ 0,42	+57,17	+191,02
5	3	+0,46	+ 1,94	−0,55	+35,79	+0,39	− 4,87	−7,85	−160,01	+7,51	− 3,93	−69,64	−200.79
	4	+0,01	+ 0,33	−0,04	+ 0,01	—	− 2,39	+0,07	+ 0,15	+0,13	+ 1,96	− 5,13	− 4,91

Finally, the moments in the mid-points of the chord members, where the verticals are connected, are calculated from the end moments at the next hinges, and the bending moments due to the direct loading from the axial force in the vertical member. For example:

$$\left.\begin{array}{l} M_{a2} = \\ M_{a0} = \end{array}\right\rangle \pm \frac{A \cdot l}{4} \pm \frac{M_{02} - M_{20}}{2}\,.$$

Moments turning clockwise are given a positive sign as usual. The end moments calculated for the Warren girder with hinged verticals are given in table V and V a.

Table Va

Moment at Joint	Other end of Member	$\frac{M_l + M_r}{2}$	$\frac{Pl}{4}$	Total M
a	0	− 24,48	− 137,10	− 112,62
	2	+ 24,48	+ 137,10	+ 112,62
c	2	− 62,88	− 245,26	− 182,38
	4	− 62,88	+ 245,26	+ 182,38
e	4	+ 66,61	− 249,92	− 183,31
	$\overline{4}$	− 66,61	+ 249,92	+ 183,31
b	1	− 151,04	+ 213,22	+ 62,19
	3	+ 151,04	− 213,22	− 62,19
d	3	− 195,90	+ 265,12	+ 69,22
	5	+ 195,90	− 265,12	− 69,22

b) Method of Joint Rotations

The end moments produced in the members of a truss with rigid connections can be expressed in terms of the corresponding member and joint rotations. The member rotations depend on the joint displacements of the truss, which depend in their turn on the end moments at the rigid joints. The general rule, however, is to neglect the effect of the end moments altogether, and to assume the joint displacements of the truss with rigid connections to be the same as those of an equivalent truss with pinned joints.

In this way, the member rotations can be calculated in advance from the joint displacements of the hinged truss. They are then introduced as known values in the equations of equilibrium, $\Sigma M = 0$, of the joints. The only unknowns in these equations remain thus the angular rotations of the joints.

Refer now to the Warren girder with posts and hangers, fig. 17a. The girder has 20 joints. Accordingly, there are 20 equations of equilibrium of the form: $\Sigma M = 0$, one for each joint. These equations contain 20 unknown joint rotations, corresponding to the total number of joints in the system.

However, the girder is symmetrical about the central hanger 5e. Also the deformations on either side are symmetrical about the same axis. Hence: $\varphi_5 = \varphi_e = 0$, which means that the joints 5 and e do not rotate at all. Further $\varphi_{\bar{m}} = -\varphi_m$, or the angular rotations of symmetrical joints are equal and opposite. In this way the number of unknowns is reduced to 9.

The problem is further simplified by assuming the verticals to be hinged at their ends, fig. 17b. This assumption allows the rotations of the intermediate joints, where the verticals are connected to the midpoints of the chords, to be expressed in terms of the main joint rotations.

For joint a, for example: $\Sigma M = M_{a0} + M_{a2} = 0$.

Hence: $N_{0a} \cdot (2\varphi_a + \varphi_0 - 3\psi_{0a}) + N_{a2} \cdot (2\varphi_a + \varphi_2 - 3\psi_{a2}) = 0$.

But $N_{0a} = N_{a2}$. Therefore: $4\varphi_a + \varphi_0 + \varphi_2 - 3(\psi_{0a} + \psi_{a2}) = 0$.

Or
$$\varphi_a = \frac{3(\psi_{0a} + \psi_{a2})}{4} - \frac{(\varphi_0 + \varphi_2)}{4}.$$

It is thus sufficient to write down the equations of equilibrium of the main joints only and to substitute the intermediate joint rotations by the corresponding terms just derived. In other words, there will be 5 equations of equilibrium with 5 unknowns only, namely the rotations of the main joints, 0, 1, 2, 3, and 4.

Table VI

Member	$N' = \frac{N}{E} = \frac{2I}{l}$ cm^3	$\Delta' = E\Delta = Eatl$	$\rho' = E\rho$	$\psi' = E\psi = \frac{E\varrho}{l}$	$N' \cdot \psi' = N \cdot \psi$ t·cm
1−b	251,76	–	495,36	1,2384	311,78
b−3	251,76	–	495,36	1,2384	311,78
3−d	317,28	–	165,12	0,4128	130,97
d−5	317,28	–	165,12	0,4128	130,97
0−a	170,24	206,4	825,60	2,0640	351,38
a−2	170,24	206,4	495,36	1,2384	210,83
2−c	276,90	206,4	495,36	1,2384	342,91
c−4	276,90	206,4	165,12	0,4128	114,30
4−e	284,90	206,4	165,12	0,4128	117,61
0−1	151,0	–	1321,58	2,0640	311,66
1−2	21,24	–	792,95	1,2384	26,306
2−3	17,267	–	792,95	1,2384	21,383
3−4	7,878	–	264,32	0,4128	3,252
4−5	6,316	–	264,32	0,4128	2,607

The elongations Δ' of the different members for an equivalent truss with hinged connections are given in table VI. The relative displacements ρ' of the two ends of every member, as well as the corresponding member rotations ψ', are also contained. All values are multiplied by E for convenience. They are expressed in t and cm units. The last column contains the products $N' \cdot \psi' = N \cdot \psi$ for the different members.

Table VII

Moment at Joint	Opposite Joint	$E\,\varphi$	$E\,\psi$	M t·cm	M work
0	a	2,3371	2,0640	+ 8,14	+ 9,76
	1		2,0640	− 8,14	− 9,74
a	0	1,5656	2,0640	− 123,20	− 112,62
	2		1,2384	+ 123,20	+ 112,62
2	a	1,3076	1,2384	+ 79,26	+ 58,72
	1		1,2384	+ 7,73	+ 6,65
	3		1,2384	− 5,72	− 5,94
	c		1,2384	− 81,27	− 59,46
c	2	0,8065	1,2384	− 220,02	− 182,38
	4		0,4128	+ 220,05	+ 182,38
4	c	0,4201	0,4128	+ 113,06	+ 66,29
	3		0,4128	+ 2,92	+ 2,63
	5		0,4128	− 2,52	− 2,32
	e		0,4128	− 113,45	− 66,61
e	4	0	0,4128	− 233,13	− 183,31
	$\bar{4}$		−0,4128	+ 233,13	+ 183,31
1	0	1,4639	2,0640	− 139,99	− 131,00
	2		1,2384	+ 11,05	+ 11,57
	b		1,2384	+ 128,90	+ 119,48
b	1	1,2994	1,2384	+ 87,49	+ 62,19
	3		1,2384	− 87,51	− 62,19
3	b	0,7688	1,2384	− 221,10	− 182,59
	2		1,2384	− 15,02	− 14,12
	4		0,4128	+ 5,67	+ 5,71
	d		0,4128	+ 230,41	+ 191,02
d	3	0,4270	0,4128	+ 121,96	+ 69,22
	5		0,4128	− 121,96	− 69,22
5	d	0	0,4128	− 257,44	− 200,79
	4		0,4128	− 5,17	4,91

The corresponding equations of equilibrium for the main joints of the truss are:

$$
\begin{aligned}
599{,}92\,\varphi_0' + 151\,\varphi_1' - 42{,}56\,\varphi_2' &= 1567{,}47 \\
151\varphi_0' + 785{,}06\,\varphi_1' + 21{,}24\,\varphi_2' - 62{,}94\,\varphi_3' &= 1481{,}56 \\
-42{,}56\,\varphi_0' + 21{,}24\,\varphi_1' + 859{,}51\,\varphi_2' + 17{,}267\,\varphi_3' - 69{,}23\,\varphi_4' &= 1039{,}72 \\
-62{,}94\,\varphi_1' + 17{,}267\,\varphi_2' + 1046{,}11\,\varphi_3' + 7{,}878\,\varphi_4' &= 738{,}03 \\
-69{,}23\,\varphi_2' + 7{,}878\,\varphi_3' + 1082{,}76\,\varphi_4' &= 370{,}40
\end{aligned}
$$

These equations have been solved by the Gauss method of elimination. They supply the following main joint rotations:

$$
\begin{aligned}
\varphi_0' &= 2{,}3371; & \varphi_2' &= 1{,}3076; & \varphi_4' &= 0{,}4201 \\
\varphi_1' &= 1{,}4639; & \varphi_3' &= 0{,}7688; & \varphi_5' &= 0.
\end{aligned}
$$

Thence the intermediate joint rotations can be calculated. They are:

$$
\begin{aligned}
\varphi_a' &= 1{,}5656; & \varphi_c' &= 0{,}8065; & \varphi_e' &= 0 \\
\varphi_b' &= 1{,}2994; & \varphi_d' &= 0{,}4270.
\end{aligned}
$$

Finally the end moments are calculated from the corresponding angles τ_m' and τ_n' of the end tangents of every member. They are given in table VII. The last column in table VII contains the values obtained before by the method of work. The results of the two methods vary considerably. The greatest differences are encountered in the moments of the chord members.

This shows that the effect of the end moments on the joint displacements of a truss with verticals cannot be neglected. In other words, the displacements of a truss with rigid joints cannot be replaced by those of an equivalent truss with pinned connections. The differences between the corresponding deflections affect the chord moments most of all, owing to the relatively higher stiffness of the chord members.

c) *The effect of the rigid Connections of the Verticals*

The foregoing investigations are limited to the calculation of the end moments in a Warren girder with hinged verticals. In a practical design, however, the verticals are rigidly connected to the chords, so that they receive bending moments as all the other members of the girder.

However, owing to the relatively small stiffness of the verticals, the moments produced at their ends will be very small compared with the other end moments at the same joints. The rigid connections of the verticals have thus a negligible effect on the other end moments.

It follows that the member and joint rotations of a girder with rigidly connected verticals are practically the same as those of a similar girder with hinged verticals. They are, however, different from the member and joint rotations given in table VII. These latter correspond to a Warren girder with pinned connections in which no account is taken of the end moments.

It must be remembered, however, that the end moments at the rigid joints of the truss produce panel point loads and subsequently axial forces in the different members. The moments, M_{mn} and M_{nm}, at the ends of the member mn, for example, produce two equal and opposite parallel forces $\frac{M_{mn}+M_{nm}}{l_{mn}}$ acting normal to the member mn at the joints m and n respectively.

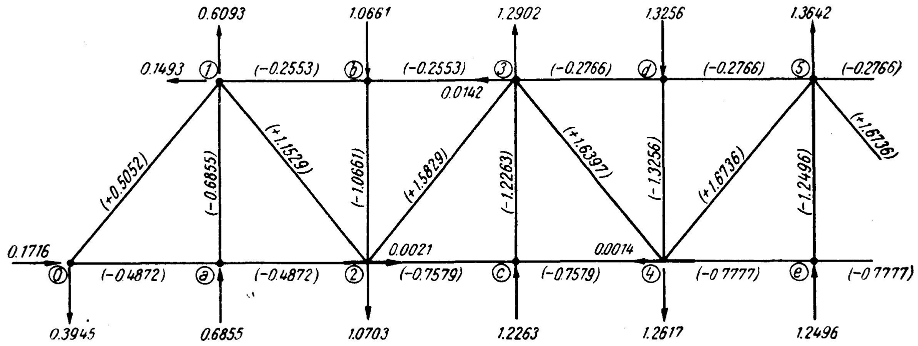

Fig. 19

Table VIII

Member	S kg	$\Delta' = \frac{Sl}{A}$	ρ'	$\Delta\psi'$	ψ' *hinged*	ψ' *rigid*
1 − b	− 255,3	− 0,42	+ 11,1	+ 0,028	1,238	1,266
b − 3	− 255,3	− 0,42	− 17,25	− 0,043	1,238	1,195
3 − d	− 276,6	− 0,31	+ 22,15	+ 0,055	0,413	0,468
d − 5	− 276,6	− 0,31	− 25,8	− 0,065	0,413	0,348
0 − a	− 487,2	− 1,24	− 9,4	− 0,023	2,064	2,040
a − 2	− 487,2	− 1,24	+ 8,8	+ 0,022	1,238	1,260
2 − c	− 757,9	− 0,96	− 18,2	− 0,045	1,238	1,193
c − 4	− 757,9	− 0,96	+ 21,5	+ 0,054	0,413	0,467
4 − c	− 775,7	− 0,85	− 25,2	− 0,063	0,413	0,350
0 − 1	+ 505,2	+ 1,61	− 6,25	− 0,010	2,064	2,054
1 − 2	+ 1152,9	+ 4,37	+ 1,8	+ 0,003	1,238	1,241
2 − 3	+ 1582,9	+ 6,98	− 8,45	− 0,013	1,238	1,225
3 − 4	+ 1639,7	+ 10,98	+ 8,55	+ 0,013	0,413	0,426
4 − 5	+ 1670,4	+ 13,17	− 11,65	− 0,018	0,413	0,395
1 − a	− 685,5	− 4,22	− 2,6	− 0,005	1,651	1,646
2 − b	− 1066,1	− 6,56	− 1,8	− 0,004	1,238	1,234
3 − c	− 1226,3	− 7,55	− 1,2	− 0,003	0,826	0,823
4 − c	− 1325,6	− 8,16	− 0,6	− 0,001	0,413	0,412
5 − e	− 1249,6	− 7,69	—	—	0	0

All these forces, summed up for all joints of the truss, are obviously in equilibrium, since $\Sigma M = 0$ at every joint. They produce axial forces in the members. The panel point loads due to the end moments of the Warren girder with hinged verticals, together with the corresponding axial forces are shown in fig. 19.

The corresponding displacement diagram supplies the necessary corrections of the member rotations. They have to be added to the member rotations of the hinged girder, in order to obtain the member rotations of the rigid system. Table VIII contains the member rotations of the hinged, as well as those of the rigid girder.

Once the member rotations of the rigid truss are known, the corresponding joint rotations are easily calculated. Expressing the end moments M_{mn} and M_{nm} of the member mn, for example, in terms of member and joint rotations:

$$M_{mn} = N_{mn} \cdot (2\,\varphi_m + \varphi_n - 3\,\psi_{mn})$$
$$\text{and } M_{nm} = N_{mn}\,(2\,\varphi_n + \varphi_m - 3\,\psi_{mn}).$$

Multiply M_{mn} by 2 and subtract M_{nm}:

$$2\,M_{mn} - M_{nm} = N_{mn} \cdot [3\,\varphi_m - 3\,\psi_{mn}] \qquad \text{or} \qquad \varphi_m = \frac{2\,M_{mn} - M_{nm}}{3\,N_{mn}} + \psi_{mn}\,.$$

Table IX

Moment at Joint	Opposite Joint	$E\,\varphi$	$E\,\psi$	$E\,\tau_m$	$E\,\tau_n$	Moments t. cm
a	1	1,583	1,646	−0,063	−0,146	−2,20
1	a	1,500	1,646	−0,146	−0,063	−2,88
2	b	1,269	1,234	+0,035	+0,038	+0,87
b	2	1,272	1,234	+0,038	+0,035	+0,90
c	3	0,826	0,823	+0,003	−0,028	−0,18
3	c	0,795	0,823	−0,028	+0,003	−0,43
4	d	0,408	0,412	−0,004	+0,002	−0,05
d	4	0,414	0,412	+0,002	−0,004	—

The values of the end moments are obtained from tables V and V a, while those of the member rotations are taken from the above table VIII. The joint rotations of the girder with hinged verticals, calculated from the above expression, are contained in table IX. This table contains, further, the end moments of the vertical members. They are calculated from the joint rotations just found and the corresponding member rotations of table VIII. For example:

$$M_{a1} = N_v \cdot (2\,\varphi_a + \varphi_1 - 3\,\psi_{a1})$$

= end moment at joint a of the vertical member $a-1$. No end moments are produced in the central vertical $5-e$. $N'_v =$ stiffness of vertical = 8,088.

It is clear from the results shown in table IX, that the end moments of the rigidly connected verticals are much less than the end moments of the other members. This justifies of course the simplification of the rigid system by assuming the verticals to be hinged.

The introduction of the end moments of the vertical members disturbs the equilibrium of the joints. This is merely due to the fact, that the other end moments at the joints have been calculated for the assumption of hinged verticals. In order to maintain equilibrium, the end moments of the verticals may be distributed among the members of the connection according to their stiffness, similar to what has been explained by the method of moment distribution. This, however, has not been done here as the end moments of the verticals are really very small.

d) The Stresses due to the non-uniform Change in the Temperature of the Truss

The max. bending moments produced in the members of the truss occur directly at the joints, where the members are connected. Consequently, the max. bending stresses are obtained by dividing the end moments of the different members through the corresponding moduli of section.

Further, the end moments produce panel point loads, and subsequently axial forces in the members. These axial forces produce normal stresses in the cross sections. There are thus bending, as well as direct stresses in the members of the truss. The total max. stresses are found by suitable addition of the corresponding values.

The bending, direct, and total stresses in the Warren girder with rigid connections due to a rise of 20^0 in the temperature of the lower chord members are given in table X. They are calculated for the gross sections, as they may be tension or compression according to an increase or a decrease in the temperature of the chord members, and vice versa.

If it is desired to take account of the reduction due to the rivet holes, the values contained in table X may be increased about 20 %. Accordingly, the absolute max. stress in the lower chord becomes $1{,}2 \cdot 69{,}3$ = about 83 kg/cm² or about 6 % of the allowable stress of 1,4 t/cm².

On the other hand, the max. total stress in the upper chord is 85 kg/cm² calculated for the gross section. Taking the allowable buckling stress to be say 1,2 t/cm² in the corresponding member, the total stress due to non-uniform temperature makes thus about 7 % of this value.

These results give an idea of the stresses that may be produced in the members of a common type of truss by a possible non-uniform change in its temperature. They are valuable for specifying, or revising the allowable stresses used in ordinary design practice.

Table X

Joint	Opposite Joint	M t·cm	Z cm³	f bending kg/cm²	N t	A cm²	f axial kg/cm²	f total kg/cm²
0	1	− 9,74	u 2967 l 1975	+ 3,3 − 4,9	+ 0,51	200,8	+ 2,5	+ 5,8 − 2,4
	a	+ 9,74	1702	− 5,7 + 5,7	− 0,49	156,8	− 3,1	− 8,8 + 2,6
a	0	− 112,62	1702	∓ 66,2	− 0,49	156,8	− 3,1	− 69,3 + 63,1
	1	− 2,20	163,1	± 13,5	− 0,69	81,2	− 8,5	+ 5,0 − 22,0
	2	+ 112,62	1702	∓ 66,2	− 0,49	156,8	− 3,1	− 69,3 + 63,1
2	a	+ 58,72	1702	± 34,5	− 0,49	156,8	− 3,1	+ 31,4 − 37,6
	1	+ 6,65	485,7	± 13,7	+ 1,15	170,8	+ 6,7	+ 20,4 − 7,0
	b	+ 0,87	163,1	∓ 5,3	− 1,07	81,2	− 13,2	− 18,5 + 7,9
	3	− 5,94	394,9	± 15,1	+ 1,58	145,2	+ 10,9	+ 26,0 − 4,2
	c	− 59,46	2769	± 21,5	− 0,76	316,8	− 2,4	+ 19,1 − 23,9
c	2	− 182,38	2769	∓ 65,8	− 0,76	316,8	− 2,4	− 68,2 + 63,4
	3	− 0,18	163,1	± 1,1	− 1,23	81,2	− 15,2	− 14,1 − 16,3
	4	+ 182,38	2769	∓ 65,8	− 0,76	316,8	− 2,4	− 68,2 + 63,4
4	c	+ 66,29	2769	± 24,0	− 0,76	316,8	− 2,4	+ 21,6 − 26,4
	3	+ 2,63	203,4	± 12,9	+ 1,64	95,6	+ 17,2	+ 30,1 + 4,3
	d	− 0,05	163,1	± 0,3	− 1,33	81,2	− 16,4	− 16,1 − 16,7
	5	− 2,32	163,1	± 14,2	+ 1,67	81,2	+ 20,6	+ 34,8 + 6,4
	e	− 66,61	2849	± 23,4	− 0,78	364,8	− 2,1	+ 21,3 − 25,5
e	4	− 182,31	2849	∓ 64,0	− 0,78	364,8	− 2,1	− 66,1 + 61,9
	5	—	163,1	—	− 1,25	81,2	− 15,4	− 15,4 − 15,4

(Continued page 145)

Table X (Continued)

Joint	Opposite Joint	M t·cm	Z cm³	f bending kg/cm²	N t	A cm²	f axial kg/cm²	f total kg/cm²
1	0	−131	u 2967 l 1975	−44,1 +66,4	+0,51	200,8	+2,5	−41,6 +68,9
	a	−2,88	163,1	∓17,7	−0,69	81,2	−8,5	−26,2 +9,2
	2	+11,57	485,7	∓23,8	+1,15	170,8	+6,7	−17,1 +30,5
	b	+119,48	u 2942 l 2122	−40,6 +56,3	−0,26	240,8	−1,1	−41,7 +55,2
b	1	+62,19	u 2942 l 2122	+20,8 −28,8	−0,26	240,8	−1,1	+19,7 −29,9
	2	+0,90	163,1	±5,5	−1,07	81,2	−13,2	−7,7 −18,7
	3	−62,19	u 2942 l 2122	+21,1 −29,3	−0,26	240,8	−1,1	+20,0 −30,4
3	b	−182,59	u 2942 l 2122	−62,1 +86,0	−0,26	240,8	−1,1	−63,2 +84,9
	2	−14,12	394,9	∓35,8	+1,58	145,2	+10,9	−14,9 +46,7
	c	−0,43	163,1	∓2,6	−1,23	81,2	−15,2	−17,8 +12,6
	4	+5,71	203,4	∓28,1	+1,64	95,6	+17,2	−10,9 +45,3
	d	+191,02	u 3470 l 2821	−55,1 +67,8	−0,28	360,8	−0,8	−55,9 +67,0
d	3	+69,22	u 3470 l 2821	+20,0 −24,5	−0,28	360,8	−0,8	+19,2 −25,3
	4	—	163,1	—	−1,33	81,2	−16,4	−16,4 −16,4
	5	−69,22	u 3470 l 2821	+20,0 −24,5	−0,28	360,8	−0,8	+19,2 −25,3
5	d	−200,79	u 3470 l 2821	−57,8 +71,2	−0,28	360,8	−0,8	−58,6 +70,4
	4	4,91	163,1	∓30,1	+1,67	81,2	+20,6	−9,5 +50,7
	e	—	163,1	—	−1,25	81,2	−15,4	−15,4 −15,4

VI. Method of Successive Correction of Joint Rotations

The solution of the equations of equilibrium, which supply the joint rotations, becomes more and more tedious as the number of joints in the system is increased. For this reason, it is intended here to give a method for obtaining the joint rotations by successive correction. The proposed method leads to fairly good results, which are practically the same as those obtained by a proper solution of the equations.

According to this method, the joint rotations are first found approximately, and then corrected successively. The first approximation must be fairly good, and the successive corrections should decrease continually. Otherwise, the method will diverge and no good results are obtained.

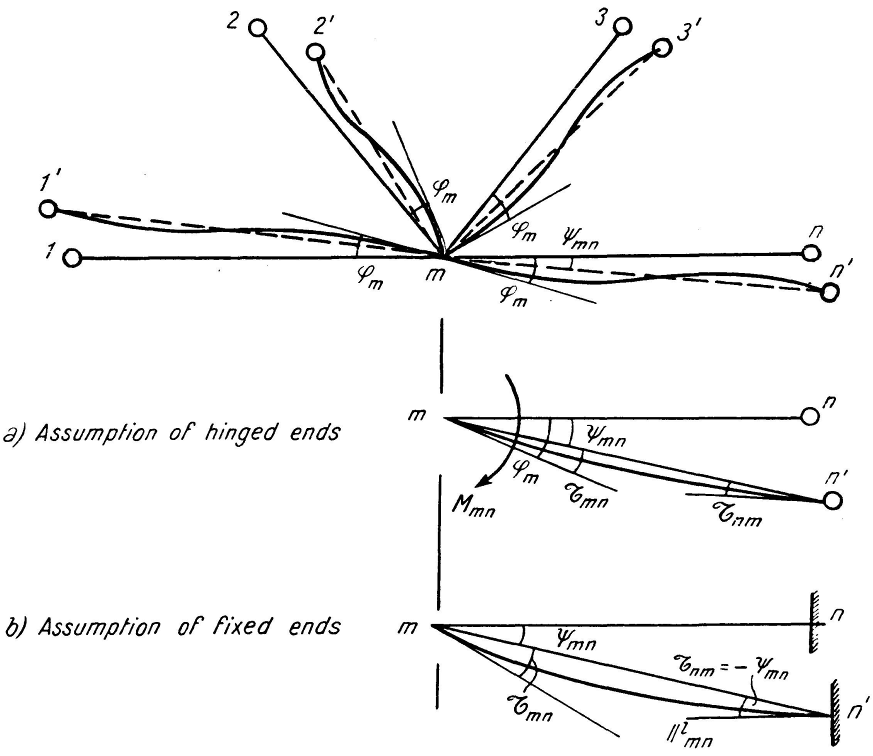

Fig. 20

Consider now a group of members to be rigidly connected together at a certain joint m, fig. 20. Assume further that the ends 1, 2, 3, . . and n of the members are displaced, relative to the common joint m. This case represents obviously the condition of a similar joint in a truss.

Owing to the rigidity of the connections at m, the members will be deformed, and moments are produced at their ends. For the sake of equilibrium ΣM at joint m must be zero. In other words:

$$\Sigma M = M_{m1} + M_{m2} + M_{m3} + \ldots . M_{mn} = 0 .$$

These moments, however, are functions of the rotations ψ and φ of the members and joints respectively. For example, the end moment

$$M_{mn} = N_{mn} \left[2 \varphi_m + \varphi_n - 3 \psi_{mn} \right], \quad \text{where} \quad N_{mn} = \frac{2 E I_{mn}}{l_{mn}} =$$

stiffness of the member mn. But the member rotations ψ are obtained from the relative displacements of the joints, and the joint rotations φ are obtained from the condition of fixation at the end joints. Therefore, if the relative displacements of the joints are given and the conditions of fixation at the end joints 1, 2, 3, ... and n are known or calculated, all ψ and φ angles except φ_m are known. φ_m is thus the only unknown value in the equation of equilibrium $\sum\limits_m M = 0$. The solution of this equation supplies φ_m, and consequently the end moments of the members.

In order to get a good first approximation of the joint rotation φ_m, different assumptions are made for the condition of fixation at the other joints. The corresponding value of φ_m is calculated for every assumption. Finally a comparison is drawn between the different results with a view of chosing the best first approximation of φ_m.

a) Assumption of hinged Ends

All members are assumed to be rigidly connected at joint m, and hinged at their other ends 1, 2, 3, ... and n, fig. 20a. There are thus no end moments at these joints. Consequently: $M_{nm} = 0$, and

$$\tau_{mn} = \frac{2}{3}\,\frac{M_{mn}}{E\,I_{mn}}\cdot\frac{l_{mn}}{2} = \frac{2}{3}\cdot\frac{M_{mn}}{N_{mn}}, \text{ where } N_{mn} = \frac{2\,E I_{mn}}{l_{mn}} = \text{stiffness of member } mn.$$

Therefore $M_{mn} = \frac{3}{2} N_{mn}\cdot\tau_{mn}$. But $\tau_{mn} = \varphi_m - \psi_{mn}$.

Hence $$M_{mn} = \frac{3}{2} N_{mn}\,[\varphi_m - \psi_{mn}]\,.$$

Further, for the equilibrium of the joint m: $\Sigma M_{mn} = 0$.

Therefore $\Sigma\,\frac{3}{2}\,N_{mn}\,(\varphi_m - \psi_{mn}) = 0$. Hence $\varphi_m\cdot\Sigma N_{mn} = \Sigma N_{mn}\cdot\psi_{mn}$.

Or $$\varphi_m = \frac{\Sigma N_{mn}\cdot\psi_{mn}}{\Sigma N_{mn}}.$$

b) Assumption of rigidly fixed Ends

Every member is assumed to be fixed at its other end, fig. 20b, so that $\varphi_1 = \varphi_2 = \varphi_3 = \ldots. = \varphi_n = 0$. Therefore: $\tau_{mn} = \varphi_m - \psi_{nm}$ and $\tau_{nm} = \varphi_n - \psi_{mn} = -\psi_{mn}$. Further: $M_{mn} = N_{mn}\cdot(2\,\varphi_m + \varphi_n - 3\,\psi_{mn}) = N_{mn}\cdot(2\,\varphi_m - 3\,\psi_{mn})$.

Introducing this value into the equation of equilibrium:

$\Sigma M_{mn} = \Sigma N_{mn}\cdot(2\,\varphi_m - 3\,\psi_{mn}) = 0$. Hence $2\,\varphi_m\,\Sigma N_{mn} = 3\,\Sigma N_{mn}\cdot\psi_{mn}$.

Or $$\varphi_m = \frac{3}{2}\cdot\frac{\Sigma N_{mn}\cdot\psi_{mn}}{\Sigma N_{mn}}.$$

c) Assumption of equal Joint Rotations

All members are assumed to be elastically fixed at their ends. It is further assumed that all joints experience equal angular rotations, so that: $\varphi_1 = \varphi_2 = \varphi_3 = \ldots\, \varphi_n = \varphi_m$.

In this way: $M_{mn} = N_{mn} \cdot [2\varphi_m + \varphi_n - 3\psi_{mn}] = 3 N_{mn} \cdot [\varphi_m - \psi_{mn}]$.

Hence $\Sigma M_{mn} = 3 \Sigma N_{mn} (\varphi_m - \psi_{mn}) = 3 \varphi_m \Sigma N_{mn} - 3 \Sigma N_{mn} \cdot \psi_{mn} = 0$.

Or
$$\varphi_m = \frac{\Sigma N_{mn} \cdot \psi_{mn}}{\Sigma N_{mn}}.$$

The above results show that the assumption a) of hinged ends gives the same value $\varphi_m = \frac{\Sigma N_{mn} \cdot \psi_{mn}}{\Sigma N_{mn}}$ as the assumption c) of equal joint rotations. Further, the assumption b) of rigidly fixed ends gives $1^1/_2$ times this value.

The members of a real truss, however, are neither hinged nor rigidly fixed at their ends. They are in fact elastically fixed due to the angular rotations of the joints. These joint rotations need not be equal to one another. Consequently the real value of φ_m in a practical case will be different from the above mentioned results.

However, the conditions of the joints in ordinary designs are far from being absolutely rigid. The joint rotations are by no means equal to zero. This is due to the fact that the members of a real truss are not subject to direct loading. They are chiefly designed for axial loads and are thus very flexible.

Consequently, it is more likely that the real value of φ_m will lie in the neighbourhood of the value $\frac{\Sigma N_{mn} \cdot \psi_{mn}}{\Sigma N_{mn}}$ found for assumptions a) and c). This value gives at least a good first approximation of the real joint rotation φ_m.

Consider now a practical joint m in a certain truss, which is surrounded by the joints 1, 2, 3, ... and n. The end moment M_{mn} at joint m of the member mn is given by the expression: $M_{mn} = N_{mn} \cdot [2\varphi_m + \varphi_n - 3\psi_{mn}]$. Hence the general condition of equilibrium for all end moments at joint m is:

$$\Sigma M_{mn} = \Sigma N_{mn} [2\varphi_m + \varphi_n - 3\psi_{mn}] = 2\varphi_m \cdot \Sigma N_{mn} + \Sigma N_{mn} \cdot \varphi_n - 3 \Sigma N_{mn} \cdot \psi_{mn} = 0$$

Now let $\varphi_{mn} = \varphi_m - \varphi_n$, so that $\varphi_n = \varphi_m - \varphi_{mn}$.

Introduce this value in the above equation of equilibrium. Thus

$$2\varphi_m \cdot \Sigma N_{mn} + \Sigma N_{mn} \cdot (\varphi_m - \varphi_{mn}) = 3 \Sigma N_{mn} \cdot \psi_{mn}.$$

Or
$$3\varphi_m \cdot \Sigma N_{mn} - \Sigma N_{mn} \cdot \varphi_{mn} = 3 \Sigma N_{mn} \cdot \psi_{mn}.$$

Hence
$$\varphi_m = \frac{\Sigma N_{mn} \cdot \psi_{mn}}{\Sigma N_{mn}} + \frac{\Sigma N_{mn} \cdot \varphi_{mn}}{3 \cdot \Sigma N_{mn}} = \varphi_m^1 + \Delta \varphi_m.$$

This is the general expression of the joint rotation φ_m, on which the method of successive correction is based. The first term φ_m^1 is simply the value obtained before under a) and c) for hinged ends and equal joint rotations respectively. It is the proposed first approximation. The second term, on the other hand, is the required correction $\Delta \varphi_m$, which is neglected in the first approximation.

In order now to determine the joint rotations corresponding to certain joint displacements of a given truss, the following steps are to be followed:

1. Calculate the N and ψ values for the different members, and work out consequently the first approximation $\varphi_m^1 = \frac{\Sigma N_{mn} \cdot \psi_{mn}}{\Sigma N_{mn}}$ for every joint. These are of course not the same for all joints.

2. Find the differences $\varphi_{mn}^1 = \varphi_m^1 - \varphi_n^1$ from the first approximations of the joint rotations at the ends of every member, and hence calculate the first corrections $\Delta \varphi_m^1 = \frac{\Sigma N_{mn} \cdot \varphi_{mn}^1}{3 \Sigma N_{mn}}$ for every joint.

These corrections are not final, since they are calculated from the approximate values φ_m^1. They have still to be corrected. By adding the corrections $\Delta \varphi_m^1$ to the first approximations of the joint rotations, the second approximations are obtained.

$\varphi_m^2 = \varphi_m^1 + \Delta \varphi_m^1$, and $\varphi_n^2 = \varphi_n^1 + \Delta \varphi_n^1$, etc.

The new differences are: $\varphi_{mn}^2 = \varphi_m^2 - \varphi_n^2 = (\varphi_m^1 + \Delta \varphi_m^1) - (\varphi_n^1 + \Delta \varphi_n^1) =$
$= (\varphi_m^1 - \varphi_n^1) + (\Delta \varphi_m^1 - \Delta \varphi_n^1) = \varphi_{mn}^1 + \Delta \varphi_{mn}^1$

where: $\Delta \varphi_{mn}^1 = \Delta \varphi_m^1 - \Delta \varphi_n^1 =$ difference arising from the first corrections.

3. Find the differences $\Delta \varphi_{mn}^1 = \Delta \varphi_m^1 - \Delta \varphi_n^1$ from the first corrections $\Delta \varphi_m^1$, and calculate the corresponding second corrections:

$$\Delta \varphi_m^2 = \frac{\Sigma N_{mn} \cdot \Delta \varphi'_{mn}}{3 \Sigma N_{mn}}, \quad \text{and so on.}$$

The values are really corrections of the first corrections.

After some further steps, the corrections become too small and thus negligible. This is of course due to the fact that a) the first approximation is not very far from the real values, and b) that the denominator of the corrections is 3 times bigger than the denominator of the first approximation, rendering the corrections relatively smaller, and c) that the successive corrections are in fact corrections of corrections, which become smaller and smaller. The computation is then brought to an end, and the corrected values are found by addition: $\varphi_m = \varphi_m^1 + \Delta \varphi_m^1 + \Delta \varphi_m^2 + \ldots$

The determination of the joint rotations of the simple Warren truss by the proposed method of successive correction is given in fig. 21. Every set of dashes corresponds to a step in the correction. The first number gives always the difference between the joint rotations or corrections on either side. The second number, on the other hand, gives the product of this difference and the corresponding member stiffness. All the necessary mathematical computations have been carried out by an ordinary slide rule of 50 cm length.

For the sake of comparison, the corresponding values obtained by solving the equations of equilibrium by the Gauss method of elimination are written in brackets. These latter have to be found by means of a calculating machine. The results obtained by the proposed method of successive correction are

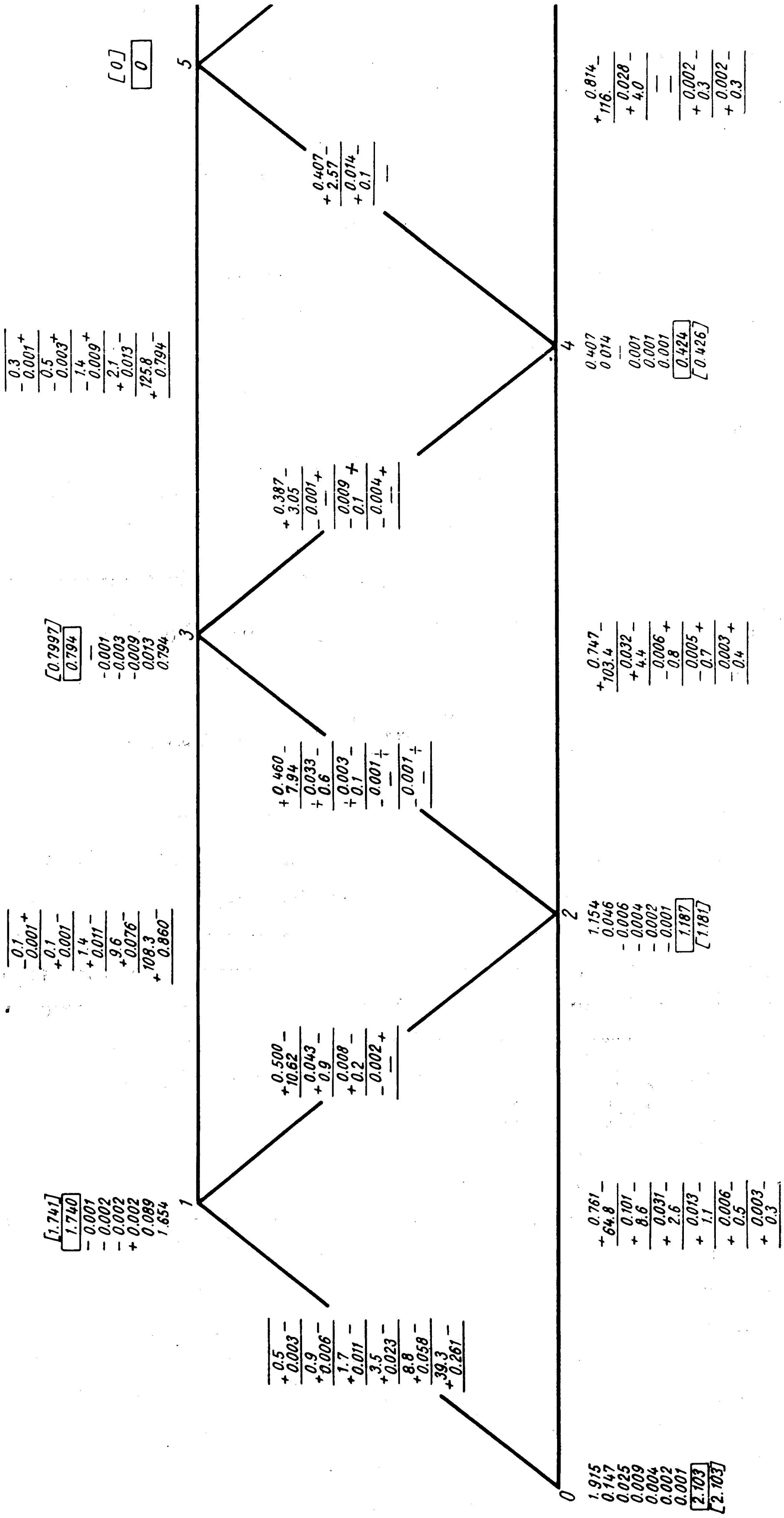

Fig. 21

almost identical with those obtained by solving the equations in the ordinary way. This justifies the use of the quick and simple method of successive correction.

Summary

Truss bridges in which a non-uniform change of temperature takes place due to the direct action of sun rays or cold wind are illustrated. Due to the presence of rigid connections, an increase or decrease in the temperature of one of the chords produces stresses in the different members. The computation of these stresses is referred to the general case of the so-called "secondary stresses".

A review of the following chief methods of calculation is given:

a) Virtual Work
b) Joint-Rotations (Mohr's Method)
c) Moment distribution.

A comparison between the given methods as to their degree of accuracy is drawn.

Further, a numerical example based on a common type of a Warren girder truss of 40 ms span with posts and hangers is worked out for an increase of 20° C in the temperature of the lower chord members. The results obtained by the ordinary method of Mohr are found to vary considerably from those of the more accurate method proposed by the author.

Finally, the author gives a method of determining the joint rotations by successive correction. Taking the angular rotation of every joint to be equal to the rotations of the surrounding joints, approximate values of the joint rotations are initially obtained. These are of course not equal to one another. They are then corrected successively for the differences and so on until ultimately the corrections become negligible. This method involves a great reduction in the mathematical computation, and leads quickly to almost accurate results. A similar procedure can be applied to related problems.

Zusammenfassung

Es werden Fachwerkbrücken behandelt, in denen ungleichmäßige Temperaturänderungen infolge der direkten Wirkung von Sonnenstrahlen oder kaltem Wind entstehen. Infolge der steifen Knotenpunkte verursacht die Zu- oder Abnahme der Temperatur in einem der Gurte Spannungen in den verschiedenen Gliedern. Die Berechnung dieser Spannungen wird zurückgeführt auf den allgemeineren Fall der Bestimmung der sog. Nebenspannungen.

Es wird eine Übersicht über die folgenden Berechnungsmethoden gegeben:

a) Virtuelle Arbeit
b) Drehwinkelmethode (Methode Mohr)
c) Momentenverteilung.

Diese Methoden werden miteinander verglichen in Bezug auf ihren Genauigkeitsgrad.

Ferner wird als numerisches Beispiel ein gewöhnlicher, 40 m langer Warren-Fachwerkträger mit Pfosten, dessen Temperatur im Untergurt um 20^0 zunimmt, durchgerechnet. Die Resultate der Theorie von Mohr weichen von denjenigen der genaueren Methode des Verfassers beträchtlich ab.

Abschließend gibt der Verfasser eine Methode zur Bestimmung der Knotendrehungen auf der Grundlage von sukzessiver Approximation. Indem die Drehung jedes Knotens gleich derjenigen der benachbarten Knoten angenommen wird, können sofort erste Näherungswerte gerechnet werden. Da die Winkeldrehungen der Knoten natürlich nicht in jedem Punkte gleich groß sind, werden sie sukzessive korrigiert, bis die Korrekturwerte vernachlässigbar klein werden. Diese Methode vereinfacht die mathematische Berechnung stark und führt schnell zu genügend genauen Resultaten.

Bei ähnlichen Problemen kann analog vorgegangen werden.

Résumé

L'auteur étudie la question des ponts en treillis dans lesquels se manifestent des variations non uniformes de la température sous l'action directe des rayons solaires ou du vent froid. Par suite de la rigidité des noeuds, l'élévation ou l'abaissement de la température dans l'une des membrures met en jeu des contraintes dans les différents éléments de l'ouvrage. L'auteur ramène le calcul de ces contraintes au cas plus général de la détermination des contraintes dites secondaires.

Il passe en revue les méthodes de calcul suivantes:

a) Travail virtuel
b) Rotations des noeuds (méthode de Mohr)
c) Répartition des moments.

Ces méthodes sont ensuite comparées entre elles du point de vue de leurs degrés respectifs de précision.

L'auteur calcule, à titre d'exemple numérique, un pont à poutres trapézoidales en treillis avec montants, de 40 m de longueur, dans lequel la membrure inférieure subit une élévation de température de 20^0C. Les résultats de la théorie de Mohr s'écartent notablement de ceux que donne la méthode plus précise de l'auteur.

L'auteur expose enfin une méthode pour la détermination des rotations dans les noeuds, méthode basée sur les approximations successives. En admettant tout d'abord que la rotation de chaque noeud est égale à celle des noeuds voisins, on arrive immédiatement à une première approximation. Les rotations des différents noeuds n'étant pas les mêmes, on procède à des corrections successives, jusqu'à ce que les termes de correction atteignent une valeur négligeable. Cette méthode simplifie considérablement le calcul mathématique et conduit rapidement à des résultats suffisamment précis.

Des problèmes semblables peuvent être traités dans des conditions analogues.