Approximations following from the maximum condition applied in theories of plasticity and earth pressure

Approximations Following from the Maximum Condition Applied in Theories of Plasticity and Earth Pressure

Aus der Maximalbedingung abgeleitete Näherungen in der Plastizitätstheorie und bei Erddruckproblemen

Approximations résultant de la condition du maximum appliquée dans les théories de la plasticité et de la pression des terres

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Statically indeterminate problems cannot be solved by equilibrium only, but a further condition is to be added. In the theory of elasticity Hooke's law is introduced but it is well known that, for most materials, it no more holds good in the higher ranges of stress, i. e. near failure. For the theoretical investigation of this state of a structure, therefore, the theory of plasticity has been created which, in its simplest form, is based on the assumption of a so-called ideally-plastic behaviour, i. e. independence of stress from strain after a maximum value, the yield stress σ_y, has been reached. The plastic theory has been applied to beams and frames and, recently, to slabs.

A quite similar yield condition has been in use in the theory of earth-pressure for more than a century, assuming that in the ultimate state of the elements concerned the angle φ be tween the resultant stress and the normal to the element reaches a maximum, i. e. the frictional angle or angle of repose, ρ, which is independent of the strain in the element.

Thus, since the yield conditions are similar and further analogies will appear later, beams, frames, plates and soil problems can be investigated along corresponding lines as far as the failure range is concerned, and approximations devised for one field may be transfered to the other.

1. *Beams and frames.* A cross-section of a beam or frame is in its limiting state when all of its elements are stressed with σ_y. Then, if there is no normal force, the bending moment is equal to the full plastic moment

$$M_y = \sigma_y S, \tag{1}$$

whereby S is the term replacing the section modulus occuring in the theory of elasticity, e. g. $S = b h^2/4$ for a rectangle. The strains at the edges then are infinitely great and the cross-section acts as a plastic hinge, i. e. a part without bending stiffness submitted to a constant moment. The structure, for its part, enters the limiting state when so many hinges have formed that the system has obtained at least one degree of kinematic freedom.

If the position of the plastic hinges is known, M_y for *a* given load is readily determined by equilibrium; this is the case under single loads and under distributed loads if there is symmetry. Thus, in fig. 1, if S is assumed to be a constant, the hinges can only be at the righthand support and under the load; equilibrium therefore yields $R_a - R_b = P$, $R_a l/2 = M_y$, $R_b l/2 = 2\,M_y$, thus

$$M_y = \frac{Pl}{6}. \qquad (2)$$

This result could have been got easier by the principle of virtual work, which states

$$L = L_e + L_i = 0, \qquad (3)$$

L_e and L_i being the external and internal work for any assumed deformation. By imposing at the load a deflection equal to unity, we make the two halves of the bar rotate by an angle $2/l$ and, as the two parts remain straight-lined, only the plastic moments do internal work; it amounts to $-M_y \cdot 2/l$ for the left hand part and $-2\,M_y \cdot 2/l$ for the right hand one, thus $-6\,M_y/l$ in all. As the external work is $P \cdot l$, we immediately obtain eq. (2).

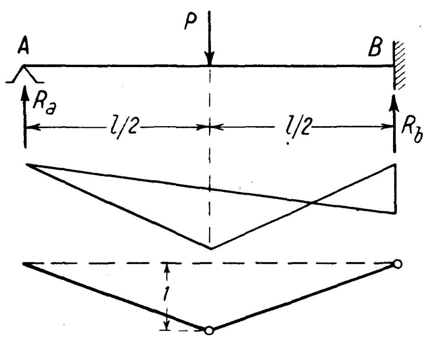
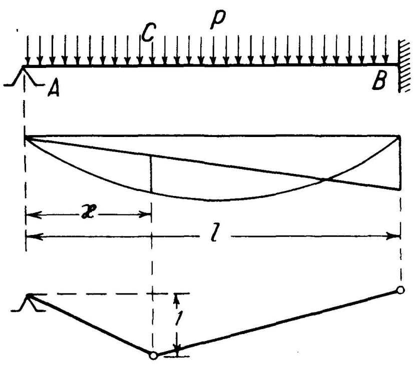


Fig. 1

Fig. 2



For a load of constant intensity, p, fig. 2, the position of the second plastic hinge is not known. We therefore first consider a state of equilibrium in which at an *arbitrary* section x the bending moment M is equal to that at the support. A virtual displacement similar to that previously used then yields an internal work $-M \cdot l/x$ at the left and $-2\,M\,\frac{l}{l-x}$ at the right, thus $L_i = -\frac{l+x}{x(l-x)}M$; the shear forces do not contribute. The two resultants of the load acting left and right of the hinge both deflect by $1/2$, thus $L_e = pl/2$ or, by eq. (3)

$$M = \frac{pl}{2}\,\frac{x(l-x)}{l+x}. \qquad (4)$$

Now, in order to make the above state a limiting state, the position x must be fixed so as to make the moment at x a maximum with respect to the adjoining cross-sections. By deriving for x we find

$$x \quad = (\sqrt{2}-1)\,l = 0.414\,l \quad \text{and, by eq. (4)} \tag{5}$$

$$M_y = \tfrac{1}{2}(\sqrt{2}-1)^2 p\,l^2 = 0.0858\,p\,l^2. \tag{6}$$

It is obvious that the above-used "*maximum condition*" can be generalised if the position of more than one plastic hinge is unknown; so we obtain

$$\frac{\partial M}{\partial x_i} = 0, \; M = M_y, \; i = 1, 2, \dots n. \tag{7}$$

If the virtual deformation is assessed in such a way that only the bending moments do internal work, the total virtual work appears in the form

$$L = f(M, x_1, x_2 \dots) = 0 \text{ and from this we can get}$$

$$0 = \frac{\partial L}{\partial M} d M + \frac{\partial L}{\partial x_1} d x_1 + \frac{\partial L}{\partial x_2} d x_2 \dots + \frac{\partial L}{\partial x_n} d x_n .$$

Now, comparing this with the n conditions given by eq. (7) we find that the latter can only be fulfilled if also

$$\frac{\partial (L_e + L_i)}{\partial x_i} = 0, \; i = 1, 2, \dots n. \tag{8}$$

This modified form of the maximum condition leads sometimes to shorter arithmetic.

Now, since in the vicinity of the correct position of a plastic hinge determined according to eq. (7) or (8), the value of $\partial M/\partial x$ is still small and thus the bending moment is only slightly diminished with respect to the plastic moment, we can frequently obtain a fair approximation by using an *estimated value for the position of the plastic hinge.*

If, for instance, in the problem of fig. 2, instead of using eq. (5), we had just estimated $x = 0.5\,l$, and inserted this into eq. (4), we should have got $M_y = 0.0833\,p\,l^2$ which is only 3% less than the correct value after eq. (6).

This reasoning, however, does not hold for the plastic hinge at the support, since the moment at this place is not a maximum in the sense of eq. (7), $\partial M/\partial x$ being far from zero. It is also not valid for the hinge under a single load — but there is no necessity for an estimation in this case anyhow.

Let us now consider the frame shown in fig. 3a and let it be loaded by 12 kip/ft. run

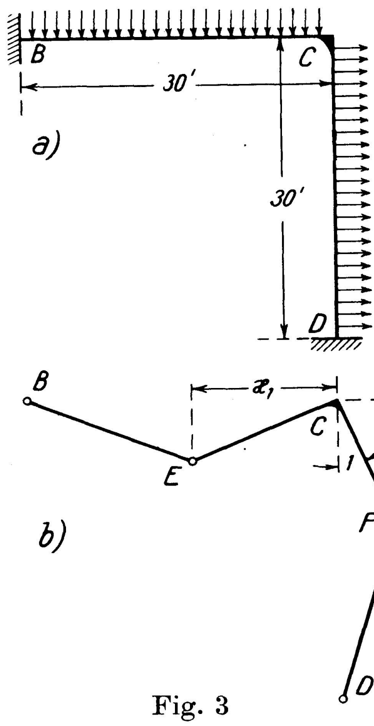


Fig. 3

on the beam and by 9 kip/ft. wind suction on the strut; S is assumed a constant for all cross-sections. Since at the corner C the moments due to the vertical and horizontal loads partly cancel each other, the plastic hinges will occur anywhere at E and F.

It has been pointed out by other authors that the normal forces acting in frame members do not influence the position of the plastic hinges materially. We therefore examine a state of equilibrium for which the absolute value of the bending moment is the same at 2 arbitrary sections E and F and at the clamped-in sections B and D. The virtual deformation is chosen such as to make part ECF rotate by an angle equal to unity, see fig. 3b. The deflection then is x_1 at E, x_2 at F and parts BE and FD rotate by $x_1/(30-x_1)$ and $x_2/(30-x_2)$ respectively.

The external work thus follows to be

$$L_e = 12\cdot 30\cdot \tfrac{1}{2}x_1 + 9\cdot 30\cdot \tfrac{1}{2}x_2,$$

while the internal work, computed in the sequence BE, ECF, FD is

$$L_i = -2M\frac{x_1}{30-x_1} - 2M\cdot 1 - 2M\frac{x_2}{30-x_2}, \quad \text{hence}$$

$$180x_1 + 135x_2 - 2M\left(1 + \frac{x_1}{30-x_1} + \frac{x_2}{30-x_2}\right) = 0. \tag{9a}$$

By use of eq. (8) we find two more conditions:

$$180 - 2M\frac{30}{(30-x_1)^2} = 0 \quad \text{and} \tag{9b}$$

$$135 - 2M\frac{30}{(30-x_2)^2} = 0. \tag{9c}$$

The solution of the 3 equations is easily got by iteration, viz.

$$x_1 = 13.5 \text{ ft.}, \; x_2 = 11.0 \text{ ft.}, \; M = M_y = 815 \text{ kip-ft.} \tag{10}$$

Without the use of the maximum condition, by estimating $x_1 = x_2$ 15 ft., eq. (9a) would have rendered $M_y = 788$, thus only 3.5% less than the correct moment. This result is remarkable, as x_2 deviates rather much from the precise value.

We now shall briefly examine the case when S is a variable. Then, of course, not the moments but the stresses, $\sigma = M/S$ will equalise and, instead of eq. (7), the maximum condition runs

$$\frac{\partial \sigma}{\partial x_1} = 0, \; \sigma = \sigma_y, \; i = 1, 2, \ldots n, \tag{11}$$

while eq. (8) remains unchanged if the work is expressed in the form $L = f(\sigma, x_1, x_2 \ldots)$.

We apply this to fig. 2 and assume

$$S = S_a\left(1+\frac{x}{l}\right), \tag{12}$$

S_a and $2\,S_a$ being the corresponding values at the two supports. The moment at any place x then is

$$M_x = \frac{p\,x\,(l-x)}{2} - \frac{x}{l}M_b,$$

if M_b denotes the moment over the support B; the stress at x is

$\sigma_x = \dfrac{p\,l\,x\,(l-x) - 2\,x\,M_b}{2\,S_a\,(l+x)}$ and the same at the support

$\sigma_b = \dfrac{M_b}{2\,S_a}$. By inserting M_b from the last equation into the penultimate and equating $\sigma_x = \sigma_b = \sigma$, we get $\sigma = \dfrac{p\,l}{2\,S_a}\cdot\dfrac{l\,x - x^2}{l+3\,x}$. Finally, if the condition (11) is applied, we find x and, in connection with the other equations, M_x and M_b. The result is

$$x = \frac{l}{3}, \qquad M_x = \frac{2}{27}p\,l^2, \qquad |M_b| = \frac{1}{9}p\,l^2. \tag{13}$$

By use of $x = 0.5\,l$ we should have got $M_b = 0.100\,p\,l^2$, thus only 11% error in spite of the rough estimation.

It should be noted that, of course, an estimated solution will lead the more safely to success, the more and more care and consideration has been spent on the estimation. For instance, in the case of fig. 2 it is obvious from the bending moment diagram that the maximum moment occurs to the left of midspan, thus $x < 0.5\,l$. If, however, S diminishes to the left, the most stressed cross-section will be still more to the left than for a constant cross-section.

One of the few problems which, partly, are solved less simply by the theory of plasticity than by elasticity, is the *superposition of two* or more *cases of loading*. If, for instance, the loads shown in figs. 1 and 2 act simultaneously, the moment at failure is, at least strictly speaking, *not* equal to the sum of the terms in eqs. (2) and (6). This follows from the fact that at $x = 0.5\,l$ the moment due to the distributed load is less than $0.0858\,p\,l^2$, since this is a value occuring as a maximum at $x = 0.414\,l$ and correspondingly the value due to the single load after eq. (2) occurs at $x = 0.5\,l$ and its influence is smaller at $x = 0.414\,l$.

Thus, if we write nevertheless

$$M_y{}' = \frac{P\,l}{6} + \tfrac{1}{2}\,(\sqrt{2}-1)^2\,p\,l^2, \tag{14}$$

we obtain at any rate *more* than the correct value. This applies always *when the plastic hinges* for the superposed cases *are at different places*. Only if, exceptionally, the hinges occur at the same place — say, if a single load would stand at $x = 0.414\,l$ — the superposition will be rigidly correct.

Now, we have already pointed out that near a plastic hinge the moments decrease but slowly; the error due to the superposition therefore will be the smaller, the smaller the distance between the two hinges for the considered cases. Moreover, the error will decrease if one of the cases dominates; eq. (14), for instance, is perfectly correct for both $p=0$ and $P=0$.

Let us now derive the strict solution for the above combined case of loading, assuming S = const. Then the hinge will be at a cross-section $x \leqq \frac{1}{2} l$ and, considering equal moments at this place and at the support, we have

$$M = \frac{Px}{2} + \frac{px(l-x)}{2} - \frac{x}{l} M \text{ and from eq. (7) we find}$$

$$x = \left(\sqrt{2 + \frac{P}{pl}} - 1\right) l \quad \text{and} \tag{15a}$$

$$M_y = \frac{pl^2}{2} \left(\sqrt{2 + \frac{P}{pl}} - 1\right)^2. \tag{15b}$$

As $x \leqq \frac{1}{2} l$, this holds good as long as

$$\frac{P}{pl} \leqq \frac{1}{4}. \tag{15c}$$

For $P=0$, i.e. distributed load only, this leads back to eqs. (5) and (6). If, however,

$$\frac{P}{p\,l} \geqq \frac{1}{4}, \text{ the hinge will always be at} \tag{16a}$$

$$x = \tfrac{1}{2} l, \text{ therefore} \tag{16b}$$

$$M_y = \frac{pl^2}{12} + \frac{Pl}{6}. \tag{16c}$$

The relative error in eq. (14) is $\frac{M_y' - M_y}{M_y}$ and depends only on the term $\frac{P}{pl}$. In the range of eqs. (15) it has a maximum of 2.5% corresponding to $\frac{P}{pl} = 0.21$; in the range of eqs. (16) the error is even less. Thus, in this example, the superposition is completely justified.

2. *Plates.* In the following we restrict ourselves to plates with constant thickness. The application of theory of plasticity to plates has been established by K. W. Johansen [1], see also a brief representation by the author [2]. The plastic hinges here are replaced by the "fracture lines" about which the adjoining parts rotate at failure; along these lines the moment then is equal to the full plastic moment which will be denoted by m_y per unit run. Mostly the fracture lines are straight-lined. If the "fracture pattern", i.e. the totality of the fracture lines, is known, m_y can be determined by equilibrium.

If this is not the case, we first assume a pattern with n geometrical unknowns x_i and consider a state of equilibrium for which in each of the assumed lines the moment has the same amount m per unit length. Now, this pattern will be the fracture pattern if the maximum condition

$$\frac{\partial m}{\partial x_i} = 0, \quad m = m_y, \quad i = 1, 2, \ldots n \tag{7a}$$

is fulfilled. Eqs. (3) and (8) can be applied as before.

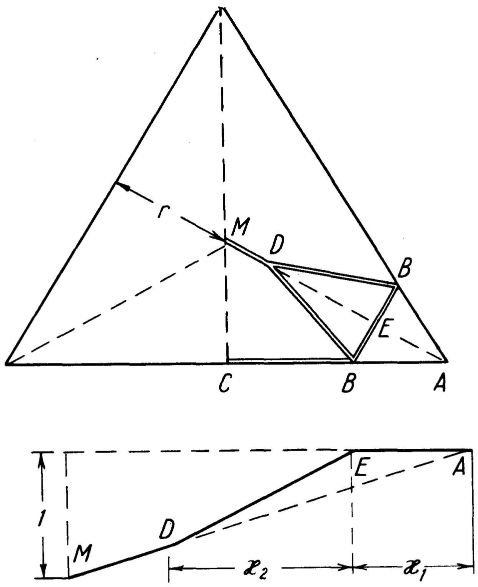


Fig. 4

In fig. 4 we examine an equilateral triangle equally loaded with intensity p and clamped in at the edges. The fracture lines are marked by double lines; hereby the moments along MD and DB are positive and along EB and CB negative, compare the cut shown in the figure; x_1 and x_2 are the geometrical unknowns. For symmetry it is sufficient to examine a sixth of the slab, viz. MCA, and from geometrical relations we readily get $MA = 2r$, $AC = 1.733\,r$, $AB = 1.155\,x_1$ and $EB = 0.578\,x_1$.

Now we impart to the system a virtual deflection such as to leave the pieces between the assumed lines plane and make point M deflect by unity. Then the deflection at D is $(x_1 + x_2)/2r$. In the piece $MCBD$ the bending moments have a component $2m$. CB turning round an axis parallel to CB and the piece rotates by an angle $1/r$; the internal work therefore is insofar $2m\,CB/r$. Similarly in the piece BDE the component is $2m\,EB$ and the angle of rotation is $(x_1 + x_2)/2r\,x_2$. Using this and inserting the above-stated geometrical relations, we obtain

$$L_i = -2m\left(1.733 - 1.555\frac{x_1}{r} + 0.578\,x_1\frac{x_1 + x_2}{2r\,x_2}\right).$$

To find the external work, we consider the parts MCA, DBE, ABE with the areas $\frac{1}{2}\,1.733\,r^2$, $\frac{1}{2}\,0.578\,x_1 x_2$ and $\frac{1}{2}\,0.578\,x_1{}^2$ respectively. In order to carry out the agreed virtual deformation, the centroid of the first part has to deflect by $\frac{1}{3}$ and those of the two others have to move upwards by $\frac{x_1}{6r}$ each with respect to the first. Thus the external work is

$$L_e = 0.289\,p\,r^2 - 0.0481\,p\,x_1{}^2\frac{x_1+x_2}{r}.$$

By inserting into eq. (3) and introducing the abbrevations:

$$\mu = \frac{12\,m}{p\,r^2}, \qquad \xi = \frac{x_1}{r}, \qquad \eta = \frac{x_2}{r} \tag{17}$$

we arrive at

$$\mu\left(1-\tfrac{1}{2}\xi+\frac{\xi^2}{6\,\eta}\right)-\left[1-\frac{1}{6}\xi^2(\xi+\eta)\right] = 0 \tag{18}$$

and, by deriving for ξ and η according to eq. (8),

$$\mu\left(-\tfrac{1}{2}+\frac{\xi}{3\,\eta}\right)+\left(\tfrac{1}{2}\xi^2+\frac{\xi\,\eta}{3}\right) = 0 \quad \text{and}$$

$$\mu\left(-\frac{\xi^2}{6\,\eta^2}\right)+\frac{\xi^2}{6} = 0.$$

From the last equation follows

$$\eta = \sqrt{\mu}; \tag{19a}$$

inserting this into the two other equations and transforming, we come to

$$\mu = \frac{1-\frac{1}{6}\xi^2(\xi+\sqrt{\mu})}{1-\frac{1}{2}\xi+\frac{1}{6}\xi^2/\sqrt{\mu}} \quad \text{and} \tag{20a}$$

$$\xi = \frac{3}{4}\sqrt{\mu}\left(1-\frac{\xi^2}{\mu}\right). \tag{20b}$$

In this form the last two equations are easily solved by iteration in the following way

$\mu = 1$ 1.165 1.175 1.180
$\xi = 0$ 0.75 0.42 0.69 0.48. We therefore have finally

$$m_y = 1.18\frac{p\,r^2}{12}, \qquad x_1 = 0.48\,r, \qquad x_2 = 1.085\,r. \tag{21}$$

In the above iteration it is seen that already its first step leads to to a very reasonable term for the moment, viz. $\mu=1.165$ whereas the first value for $x_1 = 0.75\,r$ is far from the final one. This is another proof of the possibility of getting a good approximation even from a rather inexact geometrical pattern. In practice therefore small deviations in the properties of the building material will often produce remarkable deviations in the geometrical aspect of the fracture pattern with respect to the theoretical one, without influencing the bearing capacity. As a consequence, in analyses of test results, the observed and the theoretically predicted failure loads coincided well, but the fracture often pattern did not.

The condition that the geometrical pattern is to make the moment a maximum cannot always be fulfilled by a derivation according to eqs. (7) or (8), since it may happen that principally different types of patterns must be compared.

Consider a quadratic slab after fig. 5, equally loaded by p and freely supported along two sides AB and AD, but completely free at CD and CB. We can first take into account a fracture line DB with negative moment. By imparting to point C a unit virtual deflection, we get $L_e = \frac{1}{2} p a^2 \cdot \frac{1}{3}$. The rotation about DB of the piece DBC is $\sqrt{2}/a$, thus

$$L_i = -m_y a \sqrt{2} \cdot \frac{\sqrt{2}}{a}, \qquad \text{hence } m_y = \frac{p a^2}{12}.$$

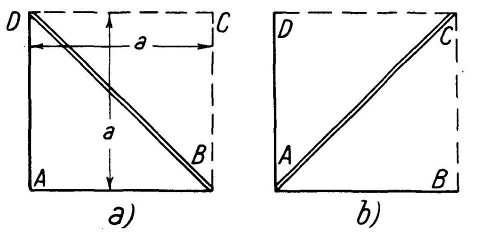


Fig. 5

The positive fracture line AC in fig. 5b, however, is more unfavorable. By deflecting C by unity we get $L_e = p a^2 \cdot \frac{1}{3}$; the rotation of the parts about AD and AB is $1/a$, the corresponding moments have a component of $m_y \cdot a$ each, therefore $L_i = -2 m_y \frac{a}{a}$ or $m_y = p a^2/6$. Since in this case two different types have to be distinguished, the decision cannot be made by using eqs. (7) or (8).

As a further example for the use of estimated fracture patterns, the equally loaded irregular pentagon with the dimensions inscribed in fig. 6 will be examined. The edges shall be freely supported, the total load be 3 kip. In a compact figure like this we may well assume that all fracture lines meet at the same point A. By imposing at this point a unit deflection, we get $L_e = 3 \cdot \frac{1}{3} = 1$. The angles of rotation about each side of the polygon are $1/h$ if by h the height of the corresponding triangle is denoted which may be picked out of a plan drawn at a sufficient scale. In this way we obtain

$$L_i = -m_y \left(\frac{7.4}{4.0} + \frac{3.2}{5.0} + \frac{6.3}{3.8} + \frac{5.7}{3.5} + \frac{6.9}{3.7} \right), \quad \text{hence}$$

$$m_y = 0.131 \text{ kip. ft./ft. run.}$$

Now, if a is the length of any side, the equilibrium yields $m_y a = \frac{1}{2} p a h \cdot \frac{h}{3}$ or $m_y = p h^2/6$. The fracture pattern therefore can only be strictly correct if all heights are equal. This, however, is only possible for a polygon which possesses an inscribed circle. Notwithstanding we can approach to this condition by chosing a shifted point B and thus more or less equalising the heights which, in the first trial, deviated between 3.5 and 5.0′. We thus get the heights 3.65, 4.40, 3.55, 4.00 and 4.15 and $m_y = 0.132$.

Of course we could have chosen the intersection point more carefully just from the beginning, but what we wanted to demonstrate was just the small influence of the position of this point. Although the solution is not and cannot be "rigid", it is of no practical use trying to improve the result by more complicated patterns.

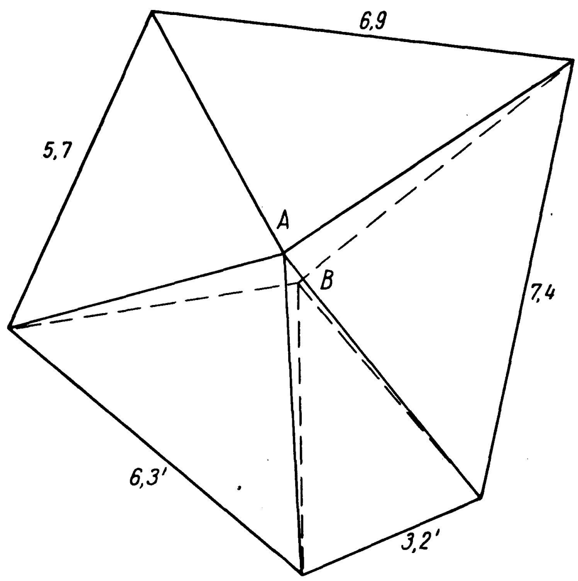


Fig. 6

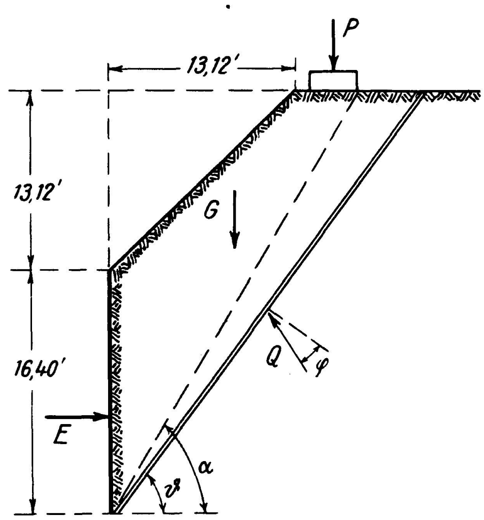


Fig. 7

3. *Earth pressure.* In earth and similar materials an element is in the limiting state if

$$\tau/\sigma = \tan \rho, \tag{22}$$

σ and τ being the normal and shear stresses and ρ the frictional angle; possible cohesion is not taken into account. For every other element, however, the angle between σ and τ is

$$\varphi < \rho. \tag{23}$$

In turn, the system as a whole reaches the limiting state when the above state has spread along certain continuous surfaces and the latter are arranged in such a way that the parts on both sides of such a "sliding surface" may glide on each other, the system thus obtaining at least one degree of kinematic freedom.

This occurence has its counter-part in beams when the yield stress is first reached at the edges and after this spreads over a whole cross-section; the sliding surfaces in soil mechanics hereby correspond to the plastic hinges in the plastic theory of beams and frames.

Coulomb's theory is based on plane sliding surfaces, thus only a special type of fracture pattern is considered; we also adopt this assumption. It follows that all stresses acting on such a sliding plane and, consequently, their resultant are parallel. The corresponding angle φ therefore is a maximum with respect to neighboring planes, or

$$\frac{\partial \varphi}{\partial \vartheta_i} = 0, \quad \varphi = \rho, \quad i = 1, 2, \ldots n. \tag{24}$$

This maximum has its analogy in eqs. (7) and (11). By ϑ_i the angles are denoted which fix the fracture pattern geometrically. In this general form the maximum condition can be used for a fuller investigation of several problems on earth pressure and bearing capacity of soils to which we shall refer at another occasion.

We examine the earth pressure against the wall shown in fig. 7; $\rho = 30°$ and a specific weight of the soil of 125 lb./cub. ft. is supposed. By P a load distributed along a line parallel to the wall is denoted; regarding its position it is only supposed that it stands between the wall and the sliding plane, $\alpha \geqq \vartheta$. The back of the wall is supposed to be perfectly frictionless.

Then, the weight per ft. run between wall and sliding plane is $G = 54500 \cot \vartheta - 10750 + P$ and from the equilibrium against shifting normally to the force Q acting in the sliding plane we get

$$E = (54500 \cot \vartheta - 10750 + P) \tan (\vartheta - \varphi). \tag{25}$$

Carrying out the operation after eq. (24) we get after arithmetic

$$\cot \vartheta = \frac{\sin 2 (\vartheta - 30°)}{2 \sin^2 \vartheta} + 0.1975 - \frac{P}{54500}, \tag{26}$$

which can be *easily* solved by iteration. If, for the moment, we leave P out of consideration, we arrive at

$$\vartheta = 52.5°, \quad E = 12850 \text{ lb./ft. run}, \quad P = 0. \tag{27}$$

It is seen that, by utilisation of the iteration method, the analytical way leads more rapidly to a solution than old-fashioned graphic methods. The arithmetic, however, can still be simplified by using an estimated value for ϑ, e.g. the value $\vartheta' = 60°$ which applies for horizontal ground level; we then get directly from eq. (25)

$$\vartheta' = 60°, \quad E' = 11950 \text{ lb./ft. run}, \quad P = 0. \tag{28}$$

and need not consider eqs. (24) and (26). The error is 7%; as the assumption of plane sliding surfaces is an approximation in itself and, moreover, the frictional angle is usually but roughly guessed, the above solution is allowable. It should be remembered that a decrease of ρ, say from 30 to 25°, produces a much bigger error, viz. 22%.

For various magnitudes of P the following strict results have been computed according to eq. (25) and (26):

$$P = 6720 \text{ lb./ft.}, \quad \vartheta = 57°, \quad E = 15900 \text{ lb./ft.} \tag{29}$$
$$P = 13440 \text{ lb./ft.}, \quad \vartheta = 62.5°, \quad E = 19700 \text{ lb./ft.}$$

If, instead, $\vartheta' = 60°$ is assumed and eq. (25) used directly, the results deviate less than 1%, due to the close coincidence of the $\vartheta' s$.

References

1. K. W. Johansen, Brudlinieteorier; Kopenhagen 1943; in Danish.
2. H. Craemer, Slabs spanning in two directions; Concrete and Constr. Eng., Oct. 1950.

Summary

In the critical sections of beams, frames and plates, the moments are a maximum with respect to neighboring sections; this fact can be used for either a rigorous computation or a fair approximation. Similarly, in the critical surfaces of soil, the angle between the stress and the normal to this surface is a maximum; hence, soil problems can be examined along similar lines.

Zusammenfassung

In den maßgebenden Schnitten von Balken, Rahmen und Platten sind die Momente Maxima in bezug auf die Nachbarschnitte. Diese Tatsache kann sowohl zur genauen Berechnung als auch zu guten Näherungen benützt werden. Auf ähnliche Weise können auch Erddruckprobleme untersucht werden, da in den Gleitflächen der Winkel zwischen der Spannung und der Normalen zu dieser Gleitfläche ein Maximum wird.

Résumé

Dans les sections critiques des poutres, cadres et dalles, les moments sont maxima par rapport aux sections voisines; ce fait peut être utilisé en vue d'un calcul rigoureux ou d'une approximation suffisante. De même, dans les sections critiques du sol, l'angle entre la contrainte et la normale à la surface présente une valeur maximum; les problèmes relatifs aux sols peuvent donc être étudiée sur des bases analogues.