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Autor(en): **Oravas, Gunhard**

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A Method of Successive Approximations for the Solution of Continuous Shells

Calcul des parois minces continues par approximations successives

Die Berechnung von durchlaufenden Schalen durch stufenweise Annäherungen

Dr. GUNHARD ORAVAS, Structural Engineer
Giffels & Vallet, Inc., Detroit, Michigan, U.S.A.

Introduction

This work has been written to present a method of successive approximations for the solution of continuous spherical shells of rotational symmetry and with constant thickness.

The present study is worked out for the design of reinforced concrete shells and therefore permits a number of simplifications in the "exact" elastic analysis. The results obtained that way are sufficiently accurate for practical design purpose when the shells are within the specified limits. The increased accuracy of the exact equations is in itself uncertain for non-homogeneous materials and therefore such rigorous mathematical treatments are hardly warranted for the design of modern reinforced concrete shells. Furthermore these mathematically exact solutions are not feasible for the design office use because of the excessive length of time required to achieve the necessary results. For most of the modern shells the chances are that an exact solution is quite hopeless and therefore the designer is compelled to search for approximate solutions which would satisfy the requirements imposed by the nature of the problem.

Procedure of Analysis

This analysis of spherical shells is based on the first term approximation of asymptotic integration. The Poisson's ratio has been taken equal to zero for reinforced concrete. This theory of spherical shells has been covered by various authors[1]) and therefore for lack of space only the final results are given.

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[1]) The most general discussion is given by Dr. F. B. HILDEBRAND: "On Asymptotic Integration in Shell Theory", Proc. Symp. in Appl. Math., vol. III, 1950, McGraw-Hill, New York.

The analysis is carried out in three step procedure:

1. Balancing of moments at the intersection of two shells by rotation of the joint from the fixed-end condition. At the same time the joint is kept stationary by applying arbitrary reactions.
2. The common boundary of the two shells is given an arbitrary radial sway (displacement) "Δ_F" and again the moments are balanced by joint rotation while maintaining the intersection of the shells at the specified radial displacement "Δ_F" by arbitrary reaction.
3. Superimposing the solutions (1) and (2) in proper proportions to satisfy the reaction requirements the solution of the problem can be obtained.

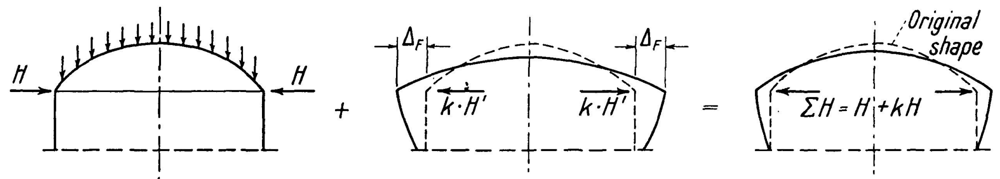


Fig. 1

See the Figure 1 above. The basic assumptions and the theory of shells have been omitted as it has been covered by several writers in detail[2]).

The results for shells under hydrostatic loading are only indicative as the simplified asymptotic theory is not quite applicable in this case. Also for shallow shells (with central angles approaching 20 degrees) this analysis is not quite accurate and care should be taken to guard against severe errors in design. Toward that end the paper by Dr. Eric Reissner in Journal of Mathematics and Physics, v. 25, 1946 and v. 27, 1948 should be studied as it treats the problem of shallow spherical shells.

The stresses in the shells are obtained as a sum of membrane and bending stresses[2]).

Membrane Forces

Dead Load

$$N_\phi = -\frac{g\,a}{1+\cos\phi} \qquad N_\theta = -g\,a\left(\cos\phi - \frac{1}{1+\cos\phi}\right)$$

Live Load

$$N_\phi = -\frac{w\,a}{2} \qquad N_\theta = -\frac{w\,a}{2}\cos 2\phi$$

Normal Load

$$N_\phi = N_\theta = -\frac{p\,a}{2}$$

[2]) "Theory of Plates and Shells" by Dr. S. Timoshenko, McGraw-Hill, New York; „Statik und Dynamik der Schalen" by Dr. W. Flügge, Springer, Berlin.

Hydrostatic Load on Erect Shells

$$N_\phi = -\frac{\rho a^2}{2}\left[1-\frac{2}{3}\cos\phi - \frac{2}{3}\frac{1}{1+\cos\phi}\right]$$

$$N_\theta = -\frac{\rho a^2}{2}\left[1-\frac{4}{3}\cos\phi + \frac{2}{3}\frac{1}{1+\cos\phi}\right]$$

Hydrostatic Load on Inverted Shells

$$N_\phi = \frac{\rho a^2}{2}\left[1-\cos\alpha - \frac{1}{3}\frac{1+\cos\phi-2\cos^2\phi}{1+\cos\phi}\right]$$

$$N_\theta = -\frac{\rho a^2}{2}\left[1-\cos\alpha - \frac{1}{3}\frac{5-\cos\phi-4\cos^2\phi}{1+\cos\phi}\right]$$

Here ρ = unit volume weight of liquid.

Bending Forces

$$N_\phi = -M_\alpha \frac{2\kappa\sqrt{\sin\alpha}}{a k_\alpha \sqrt{\sin\phi}} e^{-\kappa\omega}\cot\phi\sin\kappa\omega + H_\alpha \frac{\sin\alpha\sqrt{\sin\alpha}}{\sqrt{\sin\phi}}\cot\phi \cdot e^{-\kappa\omega}\left[\cos\kappa\omega - \frac{1}{k_\alpha}\sin\kappa\omega\right]$$

$$N_\theta = M_\alpha \frac{2\kappa^2\sqrt{\sin\alpha}}{a k_\alpha \sqrt{\sin\phi}} e^{-\kappa\omega}\left[\cos\kappa\omega - k_\phi\sin\kappa\omega\right] + H_\alpha \frac{\kappa\sin\alpha\sqrt{\sin\alpha}}{\sqrt{\sin\phi}} \cdot e^{-\kappa\omega}\left[\frac{1+k_\alpha k_\phi}{k_\alpha}\cos\kappa\omega + \frac{k_\alpha - k_\phi}{k_\alpha}\sin\kappa\omega\right]$$

$$M_\phi = M_\alpha \frac{\sqrt{\sin\alpha}}{\sqrt{\sin\phi}} e^{-\kappa\omega}\left[\frac{1}{k_\alpha}\sin\kappa\omega + \frac{k_\phi}{k_\alpha}\cos\kappa\omega\right] + H_\alpha \frac{a\sin\alpha\sqrt{\sin\alpha}}{2\sqrt{\sin\phi}} \cdot e^{-\kappa\omega}\left[\frac{k_\phi - k_\alpha}{k_\alpha}\cos\kappa\omega + \frac{1+k_\alpha k_\phi}{k_\alpha}\sin\kappa\omega\right]$$

where $\kappa = \sqrt[4]{3}\sqrt{\frac{a}{h}}.$

Note that the membrane solution does not contribute to the sectional moments in the shell.

The horizontal force "H_α" and the moment "M_α" act at the boundary of the shell to satisfy the continuity of shells.

Fixed-End Moments and Reactions

Dead Load

$$M_F = g a^2 \left\{ -\frac{\frac{2\kappa\cos\alpha}{1+k_\alpha^2} - \frac{2\kappa}{(1+k_\alpha^2)(1+\cos\alpha)} + \sin\alpha + \frac{\sin\alpha}{(1+\cos\alpha)^2} - \cos\alpha\cot\alpha + \frac{2\cot\alpha}{1+\cos\alpha}}{4\kappa^3\left[\frac{1}{k_\alpha} - \frac{1}{k_\alpha(1+k_\alpha^2)}\right]} \right\}$$

$$H_F = g a\left[-\left(\frac{k_\alpha}{\kappa\sin\alpha(1+k_\alpha^2)}\right)\left(\frac{1}{1+\cos\alpha} - \cos\alpha\right)\right] - \left[\frac{2\kappa}{a\sin\alpha(1+k_\alpha^2)}\right] M_F$$

Live Load

$$M_F = w a^2 \left[- \frac{\sin 2\alpha - \frac{1}{2}(\cos 2\alpha - 1)\cot\alpha + \frac{\kappa \cos 2\alpha}{1+k_\alpha^2}}{4\kappa^3 \left(\frac{k_\alpha}{1+k_\alpha^2}\right)} \right]$$

$$H_F = w a \left[\frac{\cos 2\alpha\, k_\alpha}{\kappa \sin\alpha\,(1+k_\alpha^2)} \right] - \left[\frac{2\kappa}{a \sin\alpha\,(1+k_\alpha^2)} \right] M_F$$

Normal Load

$$M_F = p a^2 \left(- \frac{1}{4\kappa^2 k_\alpha} \right) \qquad H_F = p a \left(\frac{1}{2\kappa \sin\alpha\, k_\alpha} \right)$$

Hydrostatic Load on Erect Shells

$$M_F = \rho a^3 \left\{ - \frac{\frac{\kappa}{1+k_\alpha^2}\left[1 - \frac{2}{3}\left(2\cos\alpha + \frac{1}{1+\cos\alpha}\right)\right] - \left[\frac{2}{3}\sin\alpha + \frac{1}{3}\frac{\sin\alpha}{(1+\cos\alpha)^2} - \left(\cos\alpha - \frac{2}{1+\cos\alpha}\right)\frac{\cot\alpha}{3}\right]}{4\kappa^3 \left(\frac{k_\alpha}{1+k_\alpha^2}\right)} \right\},$$

$$H_F = \rho a^2 \frac{k_\alpha}{2\kappa^2 \sin\alpha} \left[\frac{2}{3}\sin\alpha + \frac{1}{3}\frac{\sin\alpha}{(1+\cos\alpha)^2} - \left(\cos\alpha - \frac{2}{1+\cos\alpha}\right)\frac{\cot\alpha}{3} \right] - \left[\frac{2\kappa}{a\sin\alpha} \right] M_F.$$

Hydrostatic Load on Inverted Shells

$$M_F = \rho a^3 \left\{ \frac{\left(\frac{\kappa}{1+k_\alpha^2}\right)\frac{(1-\cos\alpha)(2+\cos\alpha)}{(1+\cos\alpha)} + \frac{(3+4\cos\alpha+2\cos^2\alpha)\sin\alpha}{(1+\cos\alpha)^2}}{12\kappa^3\left(\frac{k_\alpha}{1+k_\alpha^2}\right)} \right\},$$

$$H_F = \rho a^2 \left\{ \left(\frac{k_\alpha}{6\kappa^2 \sin\alpha}\right) \frac{(3+4\cos\alpha+2\cos^2\alpha)\sin\alpha}{(1+\cos\alpha)^2} \right\} - \left[\frac{2\kappa}{a\sin\alpha} \right] M_F.$$

Change of Reaction

In the moment balancing procedure the joint has to be kept stationary, but when a moment is applied to the edge of the shell it tends to undergo deflection. In order that it can be kept from deflecting a horizontal reaction has to be applied. This reaction is defined as a change of reaction

$$\Delta H = \left[- \frac{2\kappa}{a \sin\alpha\,(1+k_\alpha^2)} \right] \Delta M_\alpha.$$

Stiffness

By the stiffness of the spherical shell is meant the moment necessary to give a specified rotation in the meridian plane to the edge of the shell while allowing no deflection of the boundary

$$S_\alpha = M_\alpha = \left[- \frac{a h}{4\kappa^3} \left(\frac{1+k_\alpha^2}{k_\alpha} \right) \right] \Theta E.$$

The angle "α" has little influence on stiffness and therefore can be neglected in numerical computations.

Fixed-End Moment and Reaction Due to Horizontal Displacement "Δ_F"

In the second operation of the analysis this boundary condition is necessary

$$M_{\Delta_F} = \left(-\frac{h}{2\,\kappa^2 \sin\alpha\, k_\alpha}\right)\Delta_F E \qquad H_{\Delta_F} = \left(\frac{h}{a\,\kappa \sin^2\alpha\, k_\alpha}\right)\Delta_F E$$

where
$$k_\phi = 1 - \frac{\cot\phi}{2\,\kappa} \qquad k_\alpha = 1 - \frac{\cot\alpha}{2\,\kappa}.$$

Application

A spherical shell subjected to its own weight $g = 0.075$ kips/$\Box$. is continuous with a semi-infinite cylindrical shell shown in Fig. 2. Then the following formulas can be derived for the cylinder

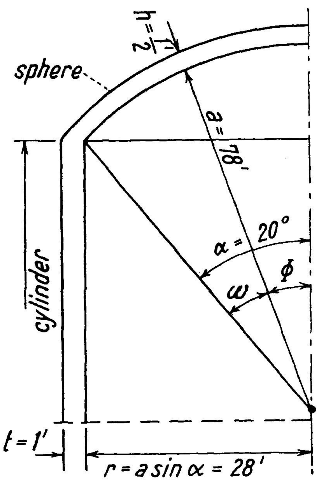


Fig. 2

$$S_0 = -\frac{\lambda t^3}{6}\Theta E \qquad \Delta H_0 = -\lambda(\Delta M)$$

$$M_{\Delta F_0} = \frac{\lambda^2 t^3}{6}\Delta_F E \qquad H_{\Delta F_0} = \frac{\lambda^3 t^3}{3}\Delta_F E$$

where
$$\lambda = \frac{\sqrt[4]{3}}{\sqrt{r\,t}}$$

Distribution Factors:

Sphere	$S_\alpha = 0.0044\,\Theta E$	0.097
Cylinder	$S_0 = 0.0415\,\Theta E$	0.903
	$\Sigma S = 0.0459\,\Theta E$	1.000

The First Step

$\Sigma H = 2.25 \rightarrow$

	0.097	Sphere
0.903	Cylinder	
		-0.435
0.393		0.042
$0.393'^k$		$-0.393'^k$

Reaction H		
Sphere	$F.E._\alpha$	$\overleftarrow{0.52}$
	ΔH_α	$\overrightarrow{0.03} = 0.042\,(0.675)$
Cylinder	$Memb._\alpha$	$\overrightarrow{2.84}$
	ΔH_0	$\overleftarrow{0.10} = 0.393\,(0.249)$
	ΣH	$= \overrightarrow{2.25}^k$

In this computation it should be noted that the cylinder has to provide the membrane horizontal reaction $N_\alpha \cos\alpha = \overrightarrow{2.84}^k$.

The Second Step:

Take $\Delta_F E = 1000$ k/ft.

$\Sigma H' = 7.72$ $\longleftarrow$

	Cylinder	Sphere
	0.903	0.097
	10.40	-3.01
	-6.67	-0.72
	$3.73'^{k}$	$-3.73'^{k}$

Reaction H'			
Sphere	$F.E.$	$\overleftarrow{3.74}$	
	ΔH_α	$\overleftarrow{0.49}$	$= 0.72\,(0.675)$
Cylinder	$F.E_0$	$\overleftarrow{5.15}$	
	ΔH_0	$\overrightarrow{1.66}$	$= 6.67\,(0.249)$
	$\Sigma H'$ =	$\overleftarrow{7.72^k}$	

The Third Step:

Exterior reaction acting at the joint has to be zero in this case or

$$2.25 - [(7.72)/1000]\Delta_F E = 0.$$

Then

$$\Delta_F E = \frac{(2.25)(1000)}{7.72} = 0.291\,(1000)$$

The final moment becomes

$$M_\alpha = -0.393 - (0.291)(3.73) = -1.478'^{k}$$

The horizontal reaction providing continuity between the sphere and cylinder is

$$H_s = \overleftarrow{0.49} + (\overleftarrow{4.23})(0.291) = \overleftarrow{1.723^k}$$

or

$$H_c = -\overrightarrow{2.74} + (\overleftarrow{3.49})(0.291) = \overrightarrow{1.725^k}$$

The actual reaction acting between the shells is

$$H_{act.} = \overleftarrow{2.84} - \overrightarrow{1.725} = \overleftarrow{1.115^k}$$

on the cylinder.

The little discrepancy between the values of horizontal reactions computed acting on each shell is due to slide rule inaccuracies. The sectional moment acting in the meridian plane can be obtained from

$$M_\phi = -1.478\frac{\sqrt{\sin\alpha}}{\sqrt{\sin\phi}}\,e^{-16.4\,\omega}\left[\frac{1}{0.915}\sin(16.4\,\omega) + \frac{k_\phi}{0.915}\cos(16.4\,\omega)\right]$$

$$+ 1.723\frac{78\cdot\sin\alpha\,\sqrt{\sin\alpha}}{2\,\sqrt{\sin\phi}}\,e^{-16.4\,\omega}\left[\frac{k_\phi - 0.915}{0.915}\cos(16.4\,\omega) + \frac{1+0.915\,k_\phi}{0.915}\sin(16.4\,\omega)\right]$$

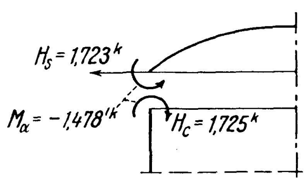
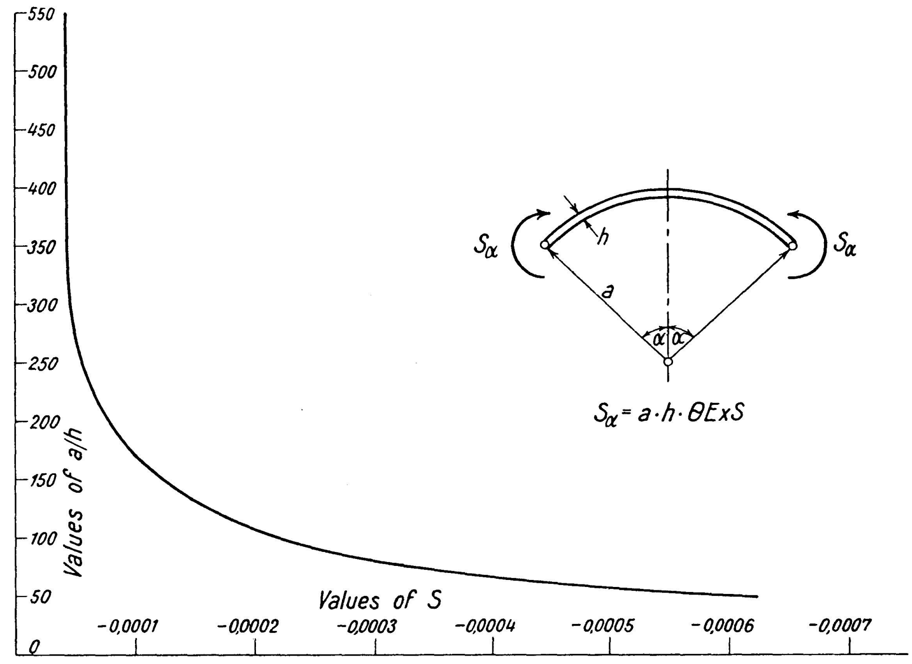


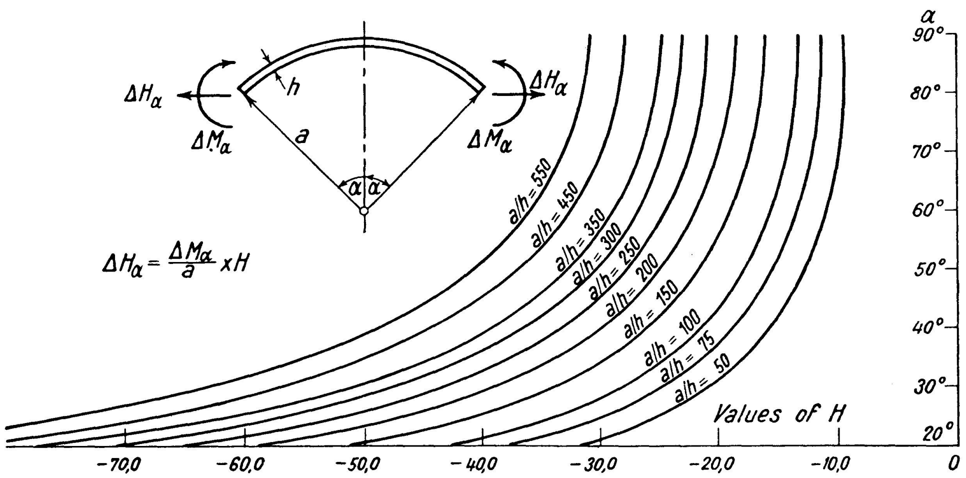
Fig. 3

Boundary forces and moment satisfying the continuity conditions are shown in Fig. 3.

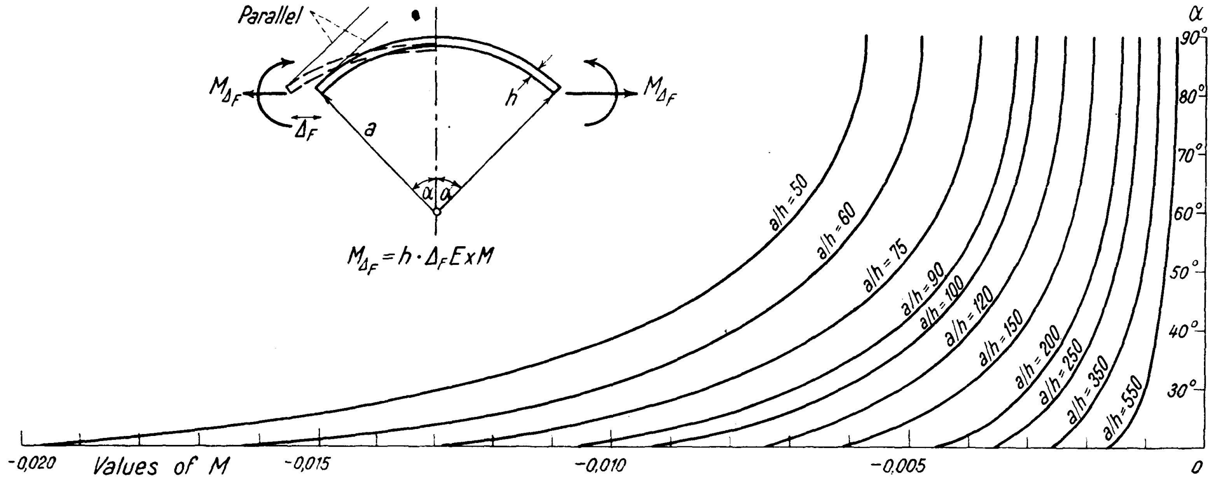
Angle $\alpha \doteq 20°$ was taken to illustrate the method. In practice these results are not reliable because of the reasons mentioned in Procedure of Analysis on p. 2.



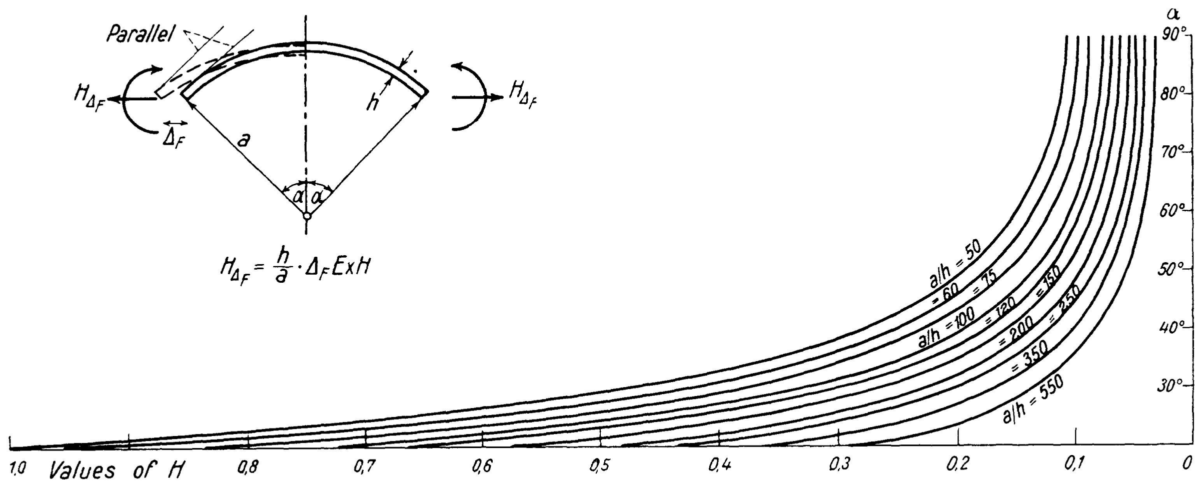
Graph No. 1. Stiffness of Spherical Shell



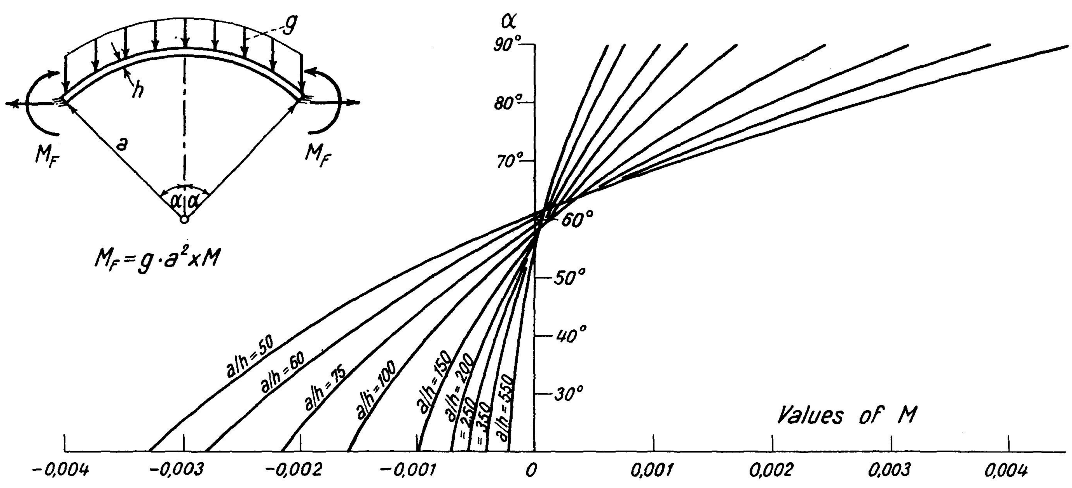
Graph No. 2. Change of Reaction due to Change of Moment Spherical Shell



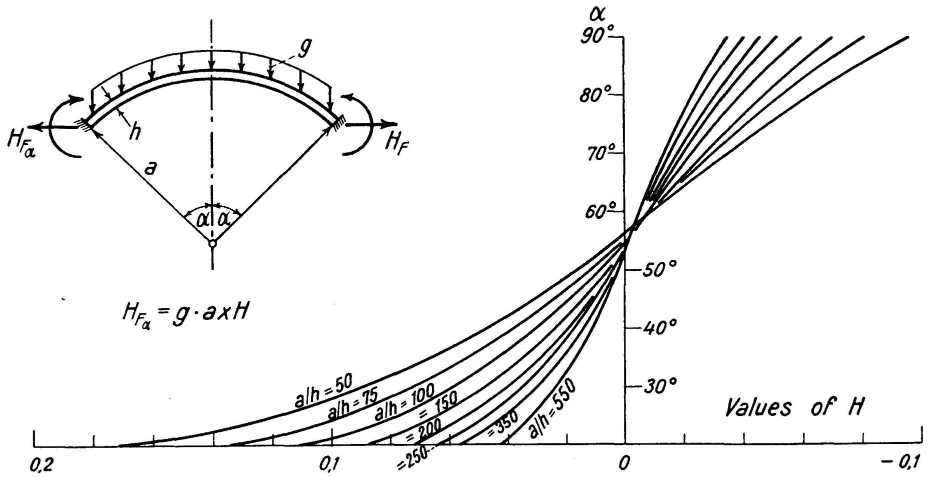
Graph No. 3. Fixed End Moment due to Δ_F Spherical Shell



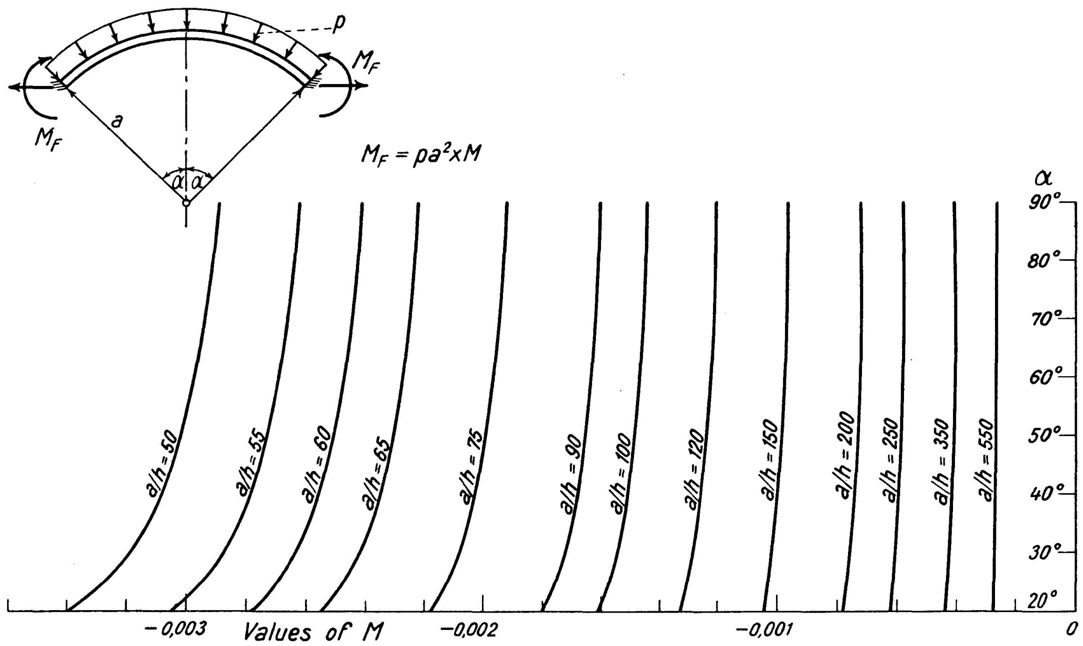
Graph No. 4. Fixed End Reaction due to Δ_F Spherical Shell



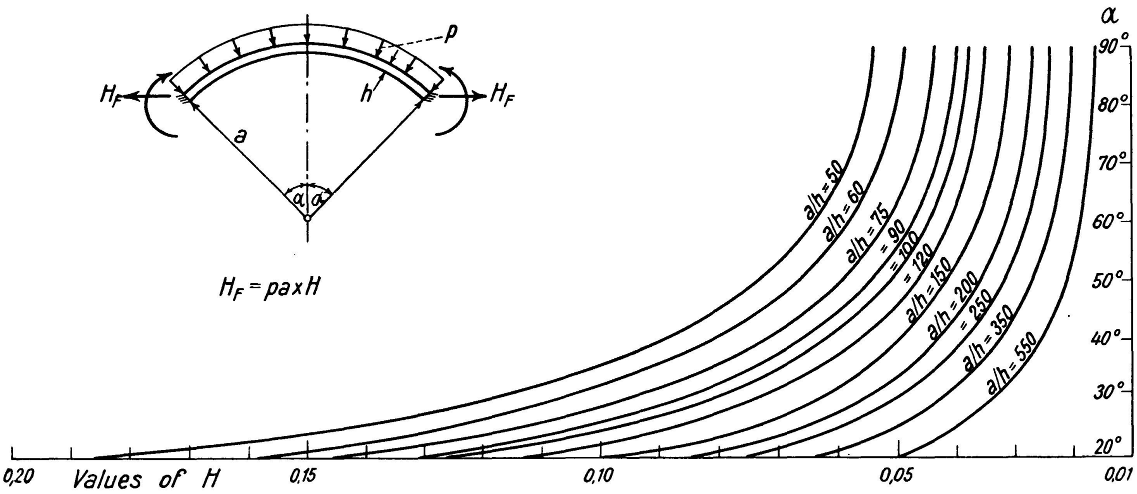
Graph No. 5. Fixed End Moment — Dead Load Spherical Shell



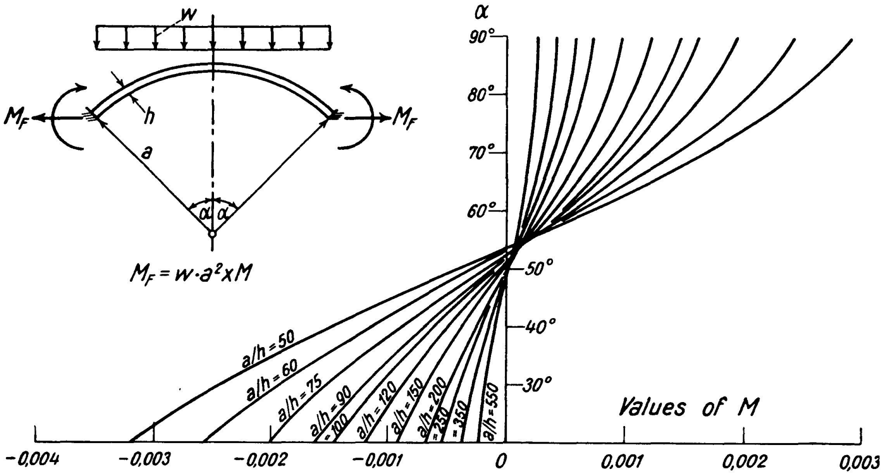
Graph No. 6. Fixed End Reaction — Dead Load Spherical Shell



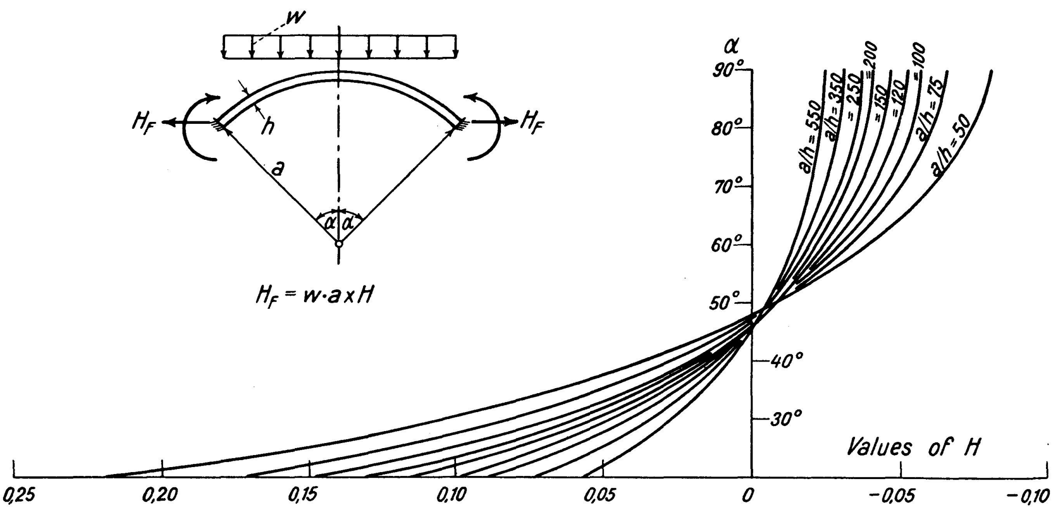
Graph No. 7. Fixed End Moment — Normal Load Spherical Shell



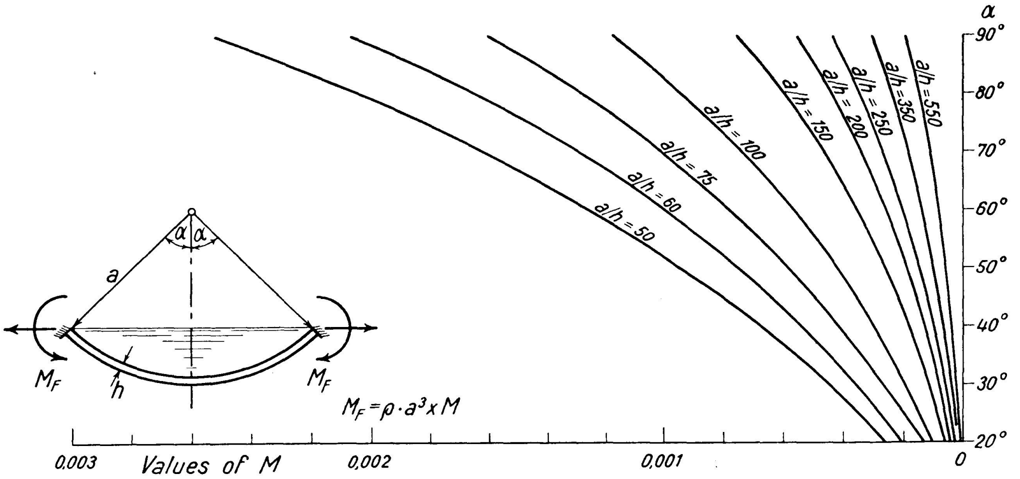
Graph No. 8. Fixed End Reaction — Normal Load Spherical Shell



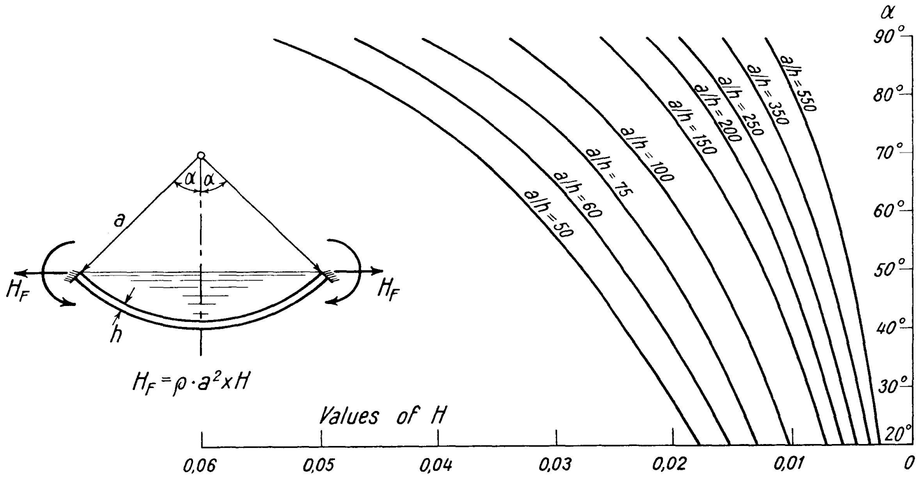
Graph No. 9. Fixed End Moment — Live Load Spherical Shell



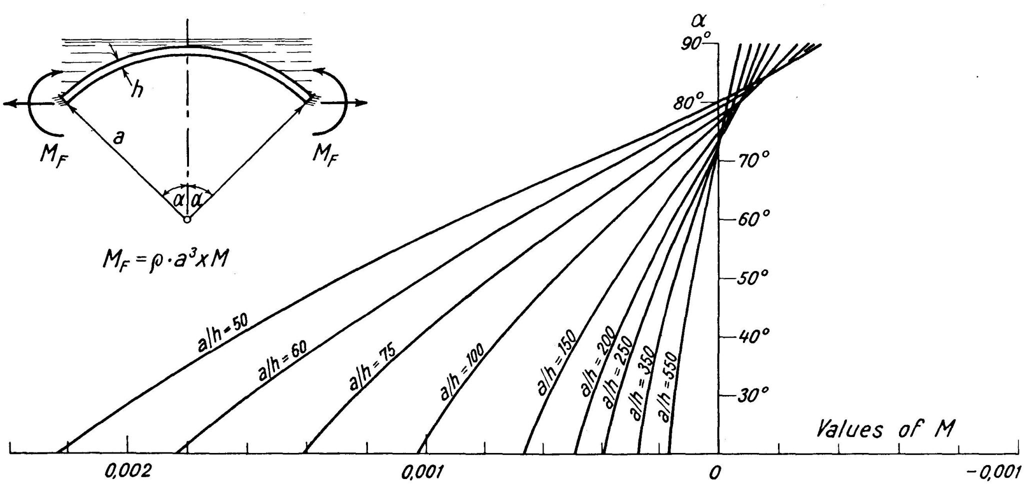
Graph No. 10. Fixed End Reaction — Live Load Spherical Shell



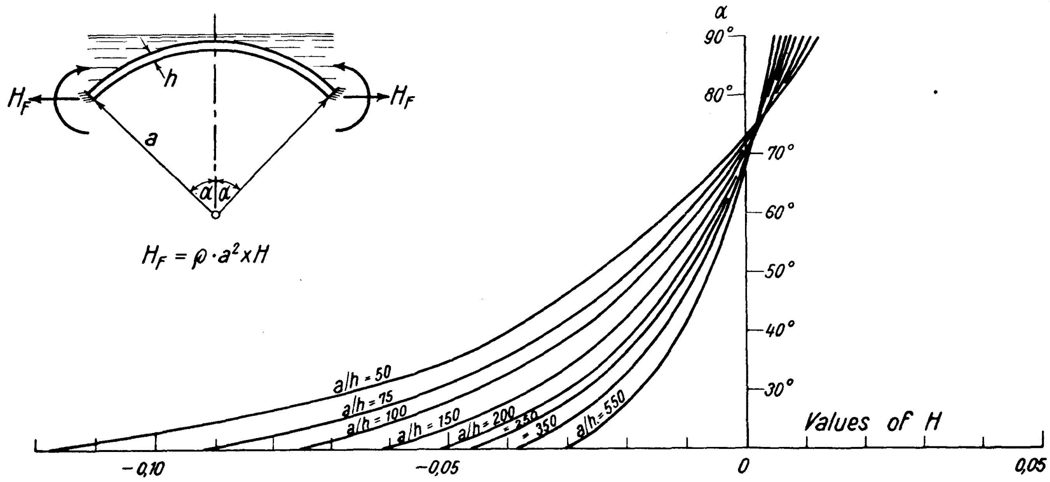
Graph No. 11. Fixed End Moment — Hydrostatic Load Inverted Spherical Shell



Graph No. 12. Fixed End Reaction — Hydrostatic Load Inverted Spherical Shell



Graph No. 13. Fixed End Moment — Hydrostatic Load Erect Spherical Shell



Graph No. 14. Fixed End Reaction — Hydrostatic Load Erect Spherical Shell

Summary

A short paper has been presented here to introduce a method of successive approximations in the solution of spherical shells continuous at their boundary. This is a practical method meant for the use of design offices doing work on concrete tanks etc. A set of charts has been included to aid in computation work.

Résumé

L'auteur expose brièvement un procédé qui permet de résoudre par approximations successives le problème du calcul des parois sphériques minces encastrées sur leur périphérie. Il s'agit d'un procédé pratique, destiné au bureau de calcul et à tous les cas de couverture de réservoirs en béton armé. Un certain nombre de diagrammes facilitent les calculs.

Zusammenfassung

In einem kurzen Abriß wird ein Verfahren eingeführt, welches mittelst stufenweiser Annäherungen das Problem der an ihrem Umfang eingespannten Kugelschale löst. Es ist dies ein handliches Verfahren, bestimmt zum Gebrauch im Rechnungsbüro und wo auch immer man sich mit der Frage von Abdeckungen von Eisenbetonbehältern usw. zu befassen hat. Eine Anzahl von Diagrammen erleichtern die Rechenarbeit.