The Effect of Variable Repeated Loads in Building Structures Designed by the Plastic Theory

Influence des charges variables et répétées sur les ouvrages calculés d'après la théorie de la plasticité

Der Einfluß von veränderlichen und mehrmals aufgebrachten Lasten auf Bauwerke, welche nach der Plastizitäts-Theorie berechnet wurden

M. R. HORNE, M. A., Ph. D., A.M.I.C.E., Assistant Director of Research,
Department of Engineering, University of Cambridge

1. Introduction

The behaviour of mild steel structures beyond the elastic limit has been the subject of a long series of investigations [1] which have resulted in the evolution of the plastic theory. A number of techniques have been evolved for applying the plastic theory to the calculation of the static collapse loads of given structures, and to the design of structures to support given static loads [2—5]. The design procedure is so to proportion a structure that it will just support the "working loads" multiplied by a "load factor" greater than unity. It is then supposed that the probability of such loads occurring is so small as to be non-existent for all practical purposes.

While the ability of a structure to withstand these factored loads will normally ensure satisfactory behaviour at the working level, it may also be necessary to check the performance of the structure with respect to excessive deflections and repeated loading. The purpose of the present paper is to investigate the probability that a structure which has been designed for a certain static collapse load according to the plastic theory would actually prove unsuitable as a result of the action of repeated loads. The repeated loads envisaged are such that, if the structure were being designed by elastic methods, stresses would not be restricted by considerations of ordinary fatigue. Where fatigue conditions operate, stresses under working loads must be limited as in orthodox design, although the structure as a whole may still be designed according to the plastic theory. Excluding such conditions, it is nevertheless necessary to consider repeated loads which are not frequent enough to cause failure due to fatigue, since such loads may still cause trouble if they persistently produce small amounts of plastic deformation. These repeated loads may or may not be of greater intensity than the working loads, but are of less intensity than the collapse load.

The effect of variable repeated loads on structures in the plastic range has been fully described elsewhere [6—9]. A structure may be embarrassed by certain fibres being subjected alternately to plastic deformation in tension and compression. This is a condition known as "alternating yield". A structure may also be embarrassed due to the progressive growth of large deflections, leading to collapse, after various alternate loading combinations have been repeated a sufficient number of times. This condition is known as "incremental collapse". Both these dangerous conditions are avoided if the structure is able to "shake down", that is, reach such a state that all subsequent changes of stress can take place elastically. The maximum load factor at which a structure can "shake down" is termed the "shake-down load factor" and is denoted by λ_s. Trouble may therefore be expected if a structure is subjected successively to a number of loading combinations in which the loads are more than λ_s times the working loads. Methods of calculating λ_s are available in the literature and will not be dealt with here. A further discussion of the nature of alternating yield and incremental collapse is contained in sections 3 and 4 below.

Let it be supposed that a structure has been designed to collapse under the most critical combination of static loads at a load factor of λ_c. The load factor λ_c is so chosen that there is only a completely negligible probability that the corresponding loads will be reached during the expected life of the structure. The value of this probability is not in general calculable, but for the purposes of the present treatment, it is expedient to assume that it has some known value, which will be denoted by P_c. Consider now the probability that loads having load factors less than λ_c but greater than λ_s will occur a sufficient number of times to embarrass the structure as a result either of alternating yield or incremental collapse. Let P be the probability that the structure will cease to support the loads effectively due to t his cause. Then the structure will be at least as safe against the effect of variable repeated loads as against that of static loads provided $P < P_c$. Under such circumstances, no consideration need be given to the effect of repeated loads in choosing the design. Let λ_s' be the value of λ_s for a structure for which $P = P_c$, i. e. for a structure in which failure due to repeated loads and failure due to a simple static overload are equally probable. Then it is to be expected that any structure in which $\lambda_s > \lambda_s'$ (and hence $P < P_c$) would automatically be safe against repeated loads when designed to collapse under a static load factor of λ_c. Structures in which $\lambda_s < \lambda_s'$ would have to be designed for resistance to repeated loads. The aim of the present paper is to determine suitable values for λ_s' for practical loading conditions.

It will be realised that a discussion of the probable importance of repeated loads necessitates a knowledge of the relative probabilities of loads of various magnitudes occurring in actual structures. There is unfortunately very little data available on this subject. An investigation of live floor loads in buildings was made by the author [10] in the course of a study of the effect of area on mean loading intensity. The relative frequencies of various wind loads on buildings have been studied in another paper [11]. These papers have been used to provide data on floor and wind loads for the present investigation, and this information is described in section 2 below. The determination of the shakedown load factor λ_s' for alternating yield is considered in section 3, and for incremental collapse in section 4. The results are discussed in section 5.

2. Variations in the Loads on Building Structures

The loads to which a building structure is subjected are usually classified as either "dead" or "live" loads. The former, due to the weight of the building itself, including permanent partitions, are known within close limits when the building is designed, and do not vary from time to time. The live loads consist of superimposed floor loads, and loads on the outside walls and roof due to wind and snow. The nature and variability of the superimposed floor loads differs widely from one type of building to another; thus the floor of the records room of an office carries a load which varies only very slowly, while a place of assembly may carry a very heavy floor load for a few hours and then virtually nothing for a period of days or even weeks. There is also considerable variation in the certainty with which the maximum load ever likely to be experienced is known. In the case of wind loads, there is a large variation from time to time, but records of wind velocities enable reasonably close estimates to be made of the probable frequency of winds of various intensities over long periods.

It is apparent that no general treatment can cover all the widely varying types of loading to be expected in building structures. Attention is therefore confined in this paper to data which is available for wind loading, and for a certain type of floor loading. It is believed that these examples provide representative information on the relative frequencies of loads of different intensities on building structures in general.

Floor Loading Data

Data on the subject of live floor loads in office buildings has been given by White [12] and Dunham [13], and appears to be the type of live floor loading on which most information is available. This data has been subjected to statistical analysis [10] with the following results.

Suppose observations be made of the actual live floor loads on random occasions in a large number of offices. Consider the mean floor loads over floor areas of constant magnitude A, these areas being continuous and embracing if necessary several adjacent rooms. Then the probability δp of obtaining a mean floor load lying between w and $w+\delta w$ lb./ft.^{2} on any given area A is

$$\delta p = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \delta x \tag{1}$$

where

$$w = C\left\{1 + C' \frac{x}{\sqrt{A}}\right\} \tag{2}$$

and δx is the increment of x corresponding to an increment of δw in w. C and C' are constants. Equations (1) and (2) correspond to the Normal Probability Distribution

$$y = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

shown in Fig. 1, the probability δp being given by the shaded area. The curve is found to give satisfactory agreement with the observed data for values of w above the mean floor load. The mean floor load (for all the offices taken together) is represented by the quantity C.

The probability p that a load of w per square foot will be exceeded on any given floor area A on any one occasion is given by the area under the curve to the right of the ordinate w, i. e.

$$p = \frac{1}{\sqrt{2\pi}} \int_w^\infty e^{-\frac{x^2}{2}} dx$$

(see inset, Fig. 1). This probability p may be obtained from tables of the Normal Probability Function by entering with the corresponding value of the deviation x (equation (2)).

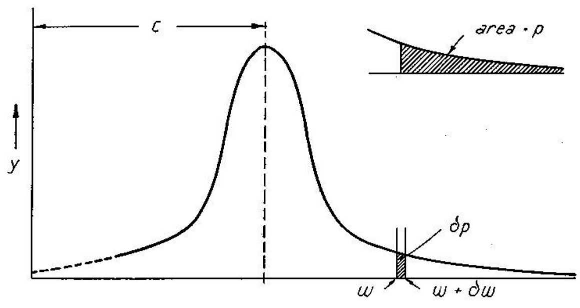


Fig. 1

The paper [10] from which equations (1) and (2) are taken was concerned with the effect of the area A on mean floor loads. The variation of w with A is irrelevant in the present investigation, and a constant value may be taken.

Since consideration is here being given to critical loads in a building as a whole, the value of A should certainly be taken greater than the average area of a single room — say greater than 200 sq. ft. — but it is difficult to decide the actual value which should be adopted. Fortunately the final results are not particularly sensitive to the precise value of A, and the value $A = 1000$ sq. ft. will be adopted. In the case of a large variety of office buildings in Britain it is found that $C' = 30$. Hence equation (2) becomes approximately

$$w = C(1+x) \tag{3}$$

Now let the "working floor load" (as given in the relevant code of practise) be w_0, and let $w_c = \lambda_c w_0$ and $w_s = \lambda_s w_0$. Hence w_c and w_s are respectively the "collapse" and "shake down" floor loads. Let p_0, p_c and p_s be the probabilities of observing load intensities of w_0, w_c and w_s on any *one* occasion, and let these same probabilities correspond to deviations in the Normal Law of x_0, x_c and x_s respectively. Then from equation (3)

$$\left.\begin{aligned} w_0 &= C(1+x_0) \\ w_c &= C(1+x_c) \\ w_s &= C(1+x_s) \end{aligned}\right\} \tag{4}$$

Hence

$$\left.\begin{aligned} \lambda_c &= \frac{1+x_c}{1+x_0} \\ \lambda_s &= \frac{1+x_s}{1+x_0} \end{aligned}\right\} \tag{5}$$

In general, if λ is any load factor, the deviation x in the Normal Law corresponding to this load factor is given by

$$\lambda = \frac{1+x}{1+x_0} \tag{6}$$

Hence the frequency distribution for load factors due to floor loads is a Normal Curve with mean value and standard deviation each equal to $\frac{1}{1+x_0}$.

Wind Loading Data

The frequencies of winds of varying velocities have been discussed in a previous paper [11]. During a single gale, gusts will cause repeated loads on a structure, but these loads will lie in an almost constant direction. Repeated loads only cause alternating yield or incremental collapse in a structure either when they act first in one sense and then in the opposite, or when they act alternately with different load combinations. Hence successive gusts within the same gale need only be treated as a single load application when considering shakedown effects. The frequencies of gales of various intensities are thus of greater relevance to the present problem.

Table 1. Highest mean hourly winds at Cardington, 1932—1944

Highest mean hourly wind: Miles per hour	Number of Months	Highest mean hourly wind: Miles per hour	Number of Months	Highest mean hourly wind: Miles per hour	Number of Months
15—19	1	40—44	19	53—54	2
20—24	12	45—46	8	55—56	0
25—29	31	47—48	6	57—58	0
30—34	35	49—50	1	59	1
35—39	35	51—52	1	60	0

The highest hourly mean winds at Cardington (England) monthly from 1932 to 1944 inclusive (complete monthly records only considered) are summarised in Table 1. These same results are shown by the histogram in Fig. 2, which also

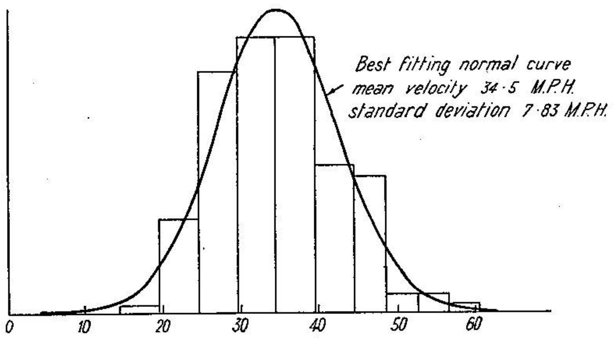


Fig. 2

shows the best fitting Normal Curve (mean velocity 34.5 miles per hour, standard deviation 7.83 m. p. h.). Assuming that the Normal Curve represents the "smoothed" results with sufficient accuracy, then the frequency of any maximum monthly velocity V m. p. h. is p where, if x is the corresponding deviation,

$$V = 34.5 + 7.83\,x \tag{7}$$

The maximum monthly velocities for sites other than Cardington would differ from the above, but it is reasonable to assume that the ratio of standard deviation to mean velocity would remain nearly the same. Hence in the general case,

$$V = D\,(1 + 0.23\,x) \tag{8}$$

where D is a constant. This formula may be used to give the variation of intensity of maximum wind loads at intervals of a month during the life of a structure. Thus let the "working" value of the wind load (as defined in the codes) correspond to a wind velocity V_0, the collapse load to a velocity V_c, and the shake down load to a velocity V_s. Let the corresponding probabilities that these velocities will be reached in any one month be p_0, p_c and p_s, the corresponding deviations in the Normal Law being x_0, x_c and x_s. Then

$$\left.\begin{aligned} V_0 &= D(1+0.23\,x_0) \\ V_c &= D(1+0.23\,x_c) \\ V_s &= D(1+0.23\,x_s) \end{aligned}\right\} \tag{9}$$

Since wind pressures are proportional to the square of the velocity, it follows from equation (9) that

$$\left.\begin{aligned} \lambda_c &= \left\{\frac{V_c}{V_0}\right\}^2 = \left\{\frac{1+0.23\,x_c}{1+0.23\,x_0}\right\}^2 \\ \lambda_s &= \left\{\frac{V_s}{V_0}\right\}^2 = \left\{\frac{1+0.23\,x_s}{1+0.23\,x_0}\right\}^2 \end{aligned}\right\} \tag{10}$$

In general, if λ is any load factor and x the corresponding deviation,

$$\lambda = \left\{\frac{1+0.23\,x}{1+0.23\,x_0}\right\}^2 \tag{11}$$

Hence $\sqrt{\lambda}$ has a Normal Frequency Distribution of mean value $\frac{1}{1+0.23\,x_0}$ and standard deviation $\frac{0.23}{1+0.23\,x_0}$.

The frequency distribution represented by equation (11) gives the distribution of load factors for the maximum monthly wind, whereas it is required to ascertain the frequency distribution for individual gales. In the present paper, it will be assumed that the same frequency distribution holds — an assumption which is equivalent to the approximation that during any one month, all the gales have an intensity equal to that of the maximum gale for that month. Any error that is involved in this approximation will be on the safe side, leading to an over-estimate of the damage caused to a structure by repeated loads.

3. Load Factors for Alternating Yield

A building structure will be subjected during its life to a large number of different load combinations. One load combination is in general found to be more critical than any other when considering collapse under static loading, and this combination of loads is used as the basis for design. Loads of less intensity than the loads causing collapse may produce some yielding and plastic deformation of the material, and different combinations of loads may cause yield to occur in the same fibres first in one sense and then in the other. Consider a structure in which alternating yield of this nature occurs for loads above the load factor λ_s, so that the structure will only support a limited number of load reversals above this level. The question arises as to the number of reversals which may be expected to cause fracture. Unfortunately the behaviour of structural assemblies under slowly alternating loads in the plastic range has scarcely been investigated at all experimentally, and no definite conclusions can be reached regarding the allowable number of stress reversals. It seems certain, however, that one could allow at least 10 over the range between yield in tension and yield in compression. This conservative figure is adopted in the calculations which follow.

As might be expected, it is found that alternating yield is only likely to be of importance in structures subjected to loads acting first in one direction, and then in the opposite. Since floor loads are necessarily unidirectional, consideration need only be given to the effects of wind. It will be assumed for the sake of simplicity that wind loads are the only loads acting on the structure — an assumption which will lead to a higher probability of alternating yield than can actually occur. A "safe" answer will thus be obtained.

Let p_s denote the probability that a load factor of λ_s will be exceeded during any one gale, and let the number of gales to be expected during the life of the structure be n. Then the expected number of gales giving a load factor greater than λ_s will be $n\,p_s$. Since n will be numerically large and p_s small, the probability of obtaining t gales with a load factor greater than λ_s may be calculated from the Poisson Distribution. The value of the probability is

$$\frac{e^{-n p_s}(n\,p_s)^t}{t!}$$

Hence the probability of obtaining *at least* t gales of this intensity is P where

$$P = \sum_{y=t}^{y=n} \frac{e^{-n p_s}(n\,p_s)^y}{y!} \tag{12}$$

It is now possible to compute the load factor λ_s' at which a structure must be required to shake down in order to avoid trouble due to alternating yield. Suppose the structure has been designed so that there is a probability P_c that it will collapse under static loads at a load factor of λ_c. If p_c is the probability of the load factor λ_c being reached during any one of the n gales during the life of the structure, then since p_c is small,

$$P_c = n\,p_c \tag{13}$$

The value of the deviation x_c in the Normal Law corresponding to probability p_c may be determined

$$\left(p_c = \frac{1}{\sqrt{2\pi}} \int_{x_c}^{\infty} e^{-\frac{x^2}{2}}\,d\,x\right).$$

Hence, from equations (10), the value of x_0 is obtained. Taking any arbitrary value of x_s where $x_s < x_c$, the corresponding probability p_s is known, and also the value of λ_s (equations (10)). The probability P that the load factor λ_s will be exceeding during at least t out of the total of n gales is then given by equation (12). The value of P obtained will depend upon the value assumed for x_s and hence on the value of λ_s. That value of λ_s for which $P = P_c$ is the required shakedown load factor λ_s'.

Calculations have been performed for $t = 10$ and $n = 1000$ and 10,000. The latter figures represent lower and upper estimates of the number of gales to which a structure may be subjected during its life. To perform the calculations it is also necessary to define the probability P_c of collapse under static loading. The choice of a specific value for P_c is somewhat arbitrary, but fortunately it is found that the value chosen has, within wide limits, very little effect on the final results. A value for P_c of $1.0 \cdot 10^{-6}$ has been adopted in the calculations.

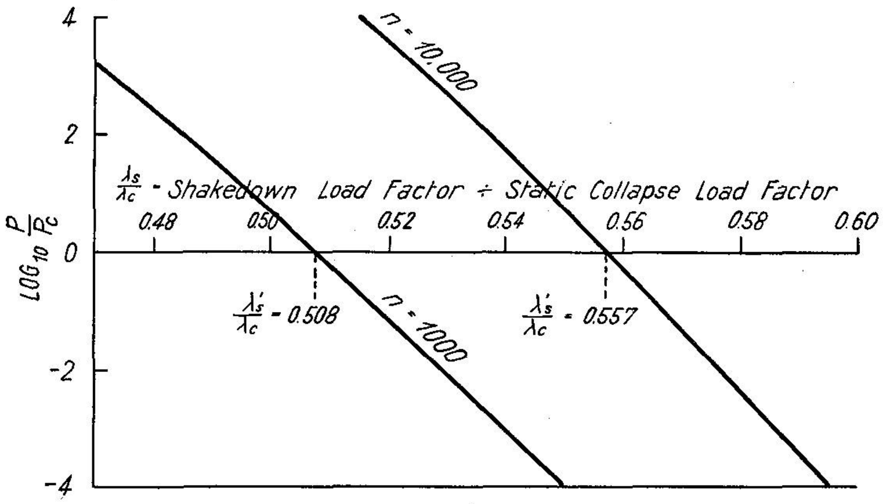


Fig. 3

The results are summarised in Fig. 3, in which $\log_{10}\left(\frac{P}{P_c}\right)$ is plotted vertically against $\frac{\lambda_s}{\lambda_c}$ horizontally. When $P = P_c$, $\log_{10}\left(\frac{P}{P_c}\right) = 0$, and hence $\frac{\lambda_s'}{\lambda_c} = 0.508$ when $n = 1000$ and 0.557 when $n = 10{,}000$. If $\lambda_c = 1.75$[1]), the values of λ_s' become 0.890 and 0.975 respectively.

The conclusion may therefore be drawn that provided the shakedown load factor λ_s of a structure is greater than 1.0, no complications due to the effects of alternating yield are to be expected. Experience has so far shown that shakedown load factors are considerably greater than unity for practical structures designed to collapse at a static load factor of 1.75. Hence the danger of alternating yield is of no importance in design, and may be entirely ignored.

---

[1]) This load factor is commonly used in Britain when designing structures by the plastic theory.

4. Load Factors for Incremental Collapse

The Phenomenon of Incremental Collapse

The phenomenon of incremental collapse is of greater importance, and more difficult to understand than that of alternating yield. It is best illustrated by reference to an example. The example chosen is rather artificial, and of little practical importance, but its simplicity is an advantage for the purpose of exposition.

A uniform beam AB, of length l, is fixed in position and direction at each end (Fig. 4 (I)). The beam carries *either* a load W_C at point C, distance $l/3$ from end A, *or* a load W_D at point D, distance $l/3$ from end B. The loads W_C and W_D are applied alternately, each having the maximum value W. The problem is to determine the behaviour of the beam when each individual application of load is sufficient to cause yield, but insufficient to cause static collapse.

The maximum elastic moment due to a load W applied at C occurs at A and has the value $\frac{4}{27} W l$. Let this moment be equal to the yield moment M_y, and let the corresponding load W be denoted by W_y. Then since $M_y = \frac{1}{\nu} M_p$ where ν is the shape factor for the beam and M_p the full plastic moment, $W_y = \frac{27}{4\nu} \frac{M_p}{l}$. The single load, applied at either C or D, which would be just sufficient to cause complete collapse (plastic hinges at A, B and under the load) has the value $9\frac{M_p}{l}$, and is denoted by W_p. If $\nu = 1.15$ (being a typical value for a rolled steel joist) it follows that $\frac{W_y}{W_p} = \frac{27}{36\nu} = 0.652$. Interest therefore centres on loads W which lie within the range $W_y < W < W_p$ or $0.652\, W_p < W < W_p$.

A complete analysis has been made of the cases $W = 0.85\, W_p$ and $W = 0.95\, W_p$, and the results are presented in Figs. 4 and 5 respectively. Considering first the load $W = 0.85\, W_p$, the horizontal axes in the graphs in Figs. 4 (II) to (V) may be regarded as identical axes of *time*, although not to any specific (or necessarily linear) scale. The loads W_C and W_D are shown in Fig. (II), the bending moments at A and C (M_A and M_C respectively) in Fig. (III), and the moments at B and D (M_B and M_D) in Fig. (IV). Under the first application of the load W_C, full plastic moment is reached at A (point j in Fig. (III)) before the full load ($W_C = 0.85\, W_p$) has been reached. Certain residual moments are left in the beam after the removal of the load (see points c in Figs. (III) and (IV)). The subsequent application of the load at D similarly causes full plasticity at B before W_D has reached its maximum value (see point k, Fig. (IV)). A certain amount of hinge rotation occurs at A when the load is applied at C, and at B when it is applied at D, and this continues to take place under subsequent applications of load. The central deflection of the beam is recorded in Fig. 4 (V), and it will be seen that after the load W_C has been removed for the first time (point c) there is a residual deflection of $\cdot 0037 \frac{M_p l^2}{EI}$ where I is the moment of

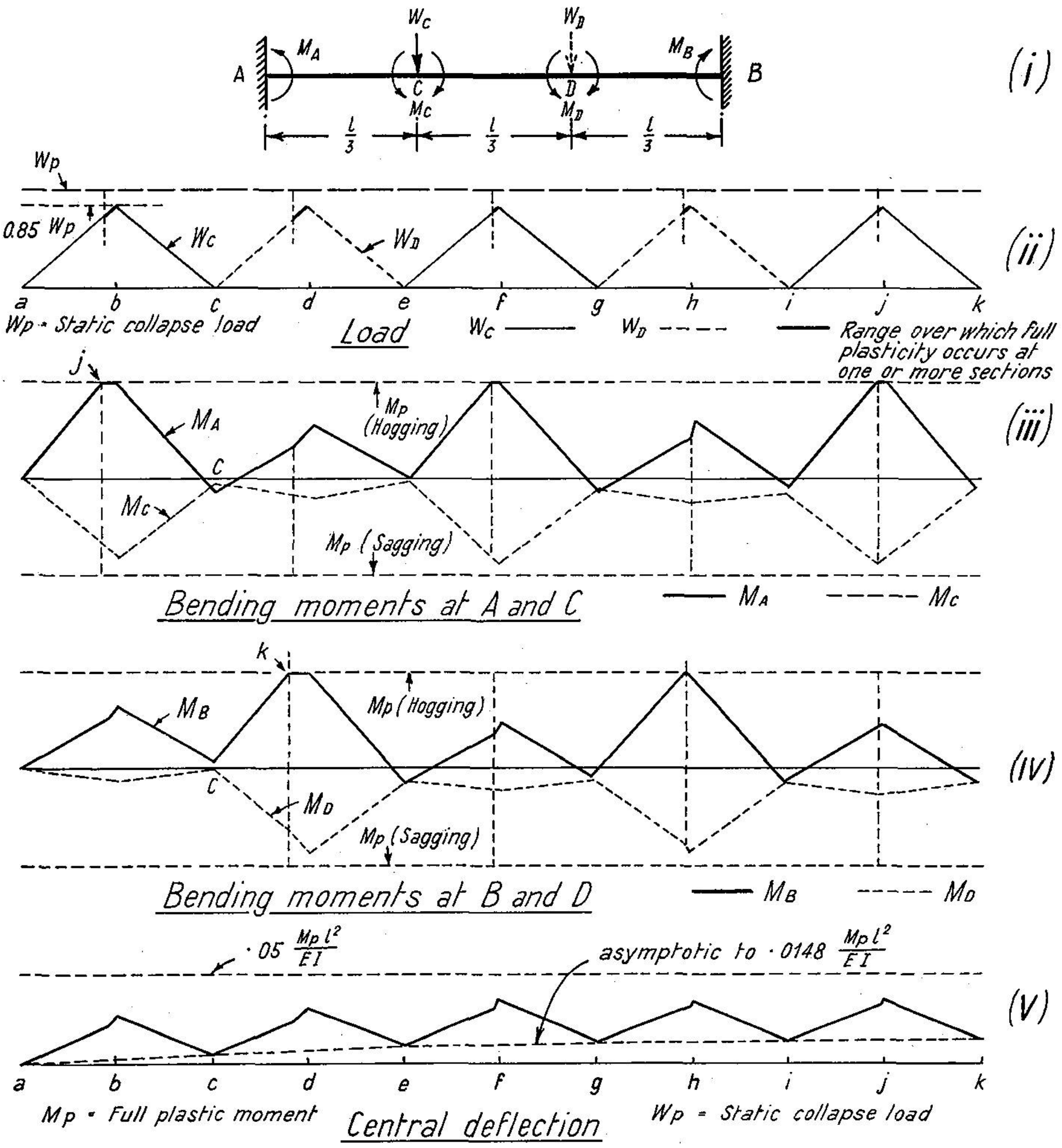


Fig. 4

inertia of the beam about the axis of bending. After the first application of the load W_D, the residual central deflection rises to $\cdot 0093 \frac{M_p l^2}{EI}$.

As the loads are repeated the range of load over which full plasticity occurs at any section decreases (see heavy lines in Fig. (II)), and the residual deflections increase more and more slowly (Fig. (V)). Ultimately all the changes of stress in the beam occur elastically and the residual central deflection reaches a constant value of $\cdot 0148 \frac{M_p l^2}{EI}$.

Consider now the case when the loads W_C and W_D alternately reach the maximum value of 0.95 W_p (see Fig. 5). Except on the first application of load

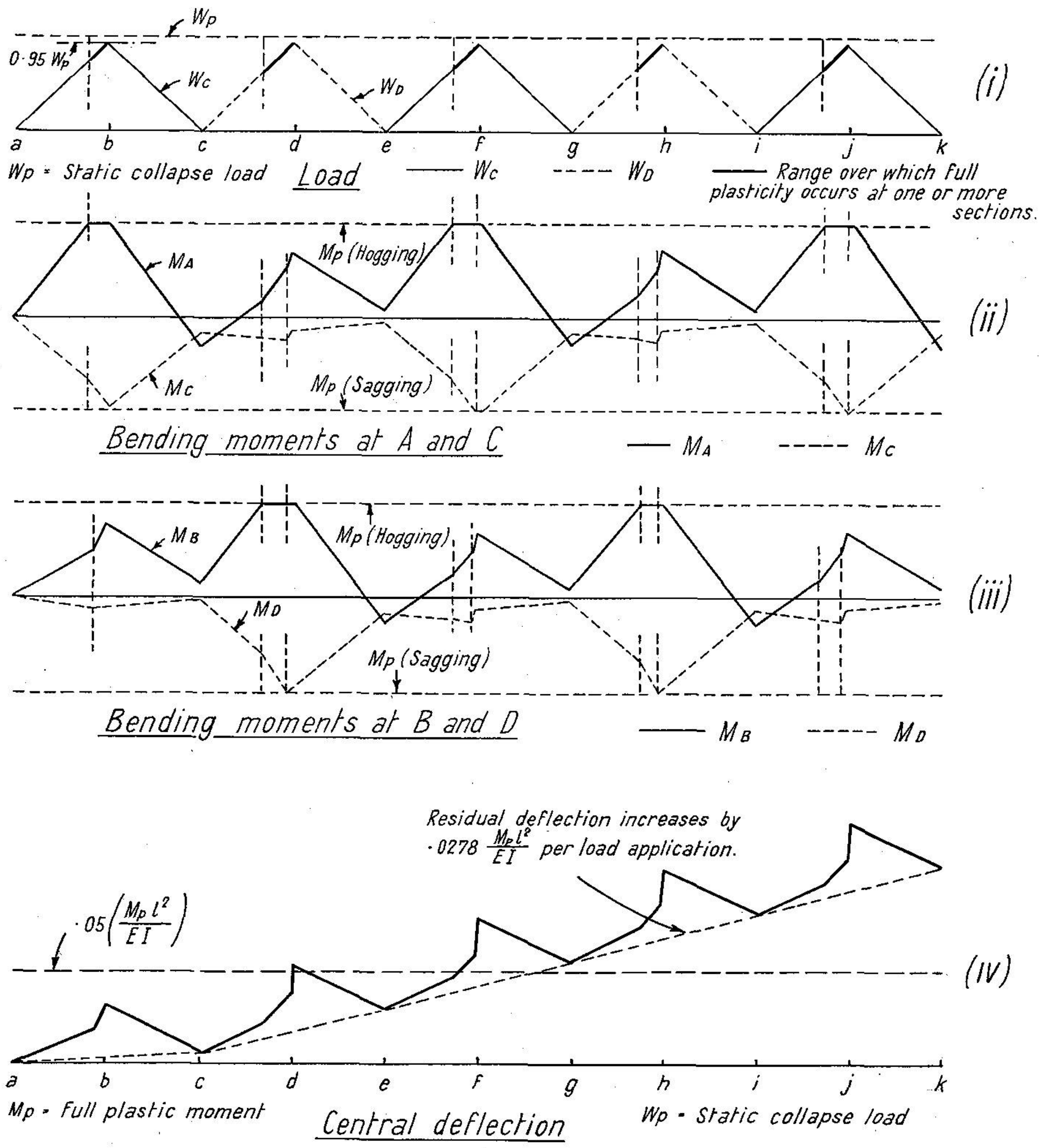


Fig. 5

W_C, full plasticity is reached under this load at both A and C (Fig. (II)). Under each application of the load W_D, full plasticity is reached at both B and D (Fig. (III)). There is no tendency for the range of load over which full plasticity occurs to decrease, and the residual deflections increase progressively (Fig. (IV)). After a few applications of load, the rate of increase of residual deflection assumes a uniform value of $\cdot 0278 \dfrac{M_p l^2}{EI}$ per load application, and the structure reaches a deformed state as shown in Fig. 6.

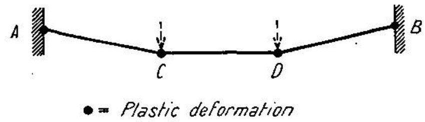


Fig. 6

There is thus a fundamental difference in the behaviour of the beam under alternating loads at the two load levels of $W = 0.85\ W_p$ and $W = 0.95\ W_p$. Under the former load the beam "shakes down", or reaches such a state that all *changes* of bending moments and deflection occur as in an elastic structure. Under alternating loads at the level $W = 0.95\ W_p$, incremental collapse takes place, the deflections building up to indefinitely large values. The "shake down" load for incremental collapse is the largest load at which behaviour of the type shown in Fig. 4 can occur, and for the beam under consideration is found to have the value $0.90\ W_p$. Methods of calculating shake-down loads are given in the literature [6—9].

In an actual structure, the successive maximum values of the various loads will differ. It is therefore necessary, in order to assess the danger of increment collapse, to find the increase of permanent deflection for a load application of any given intensity for the general case. Only an approximate solution is possible, since accurate calculations involve tracing the behaviour of particular structures through particular loading histories. A safe approximate rule will therefore be given, and its use justified by reference to the beam problem considered above.

Let the deflection of a certain point in a structure under static loads which are just sufficient to cause yield be denoted by S_y. Let static collapse be *defined* as the attainment of a corresponding *permanent* deflection of S_c, and let the load factor at collapse be λ_c. Let the shake down load factor be λ_s. Then a single load application at a load factor level of λ may be assumed to produce an increase in permanent deflection of $\left\{\frac{\lambda-\lambda_s}{\lambda_c-\lambda_s}\right\} S_c$. If q denotes the proportion of the permanent deflection at collapse produced by a single load application, then it follows that

$$q = \frac{\lambda - \lambda_s}{\lambda_c - \lambda_s} \tag{14}$$

When λ is equal to λ_c, equation (14) shows that $q = 1$, thus agreeing with the definition of static collapse which has been adopted.

The safety of equation (14) with reference to the beam already considered may be demonstrated. The complete history of the beam has been computed for alternate applications of loads at a number of constant load factors, and the load applications (N) required to produce critical permanent deflections S_c of $2\,S_y$, $4\,S_y$ and $8\,S_y$ are shown in Fig. 7 (the deflection at yield S_y has the value $0.0214\,\frac{M_p l^2}{E I}$). None of these values of S_c can be produced by loads of less than the shake down value $0.900\ W_p$. The reciprocal of N is shown plotted in Fig. 8, and gives q, the mean proportion of the critical deflection S_c produced per application of load. Equation (14) corresponds to the straight line OA, which is seen to give high (that is, safe) values of q except for low values of the load ratio $\frac{W}{W_p}$ when $S_c = 2\,S_y$. Even in the latter case, the deflection is underestimated

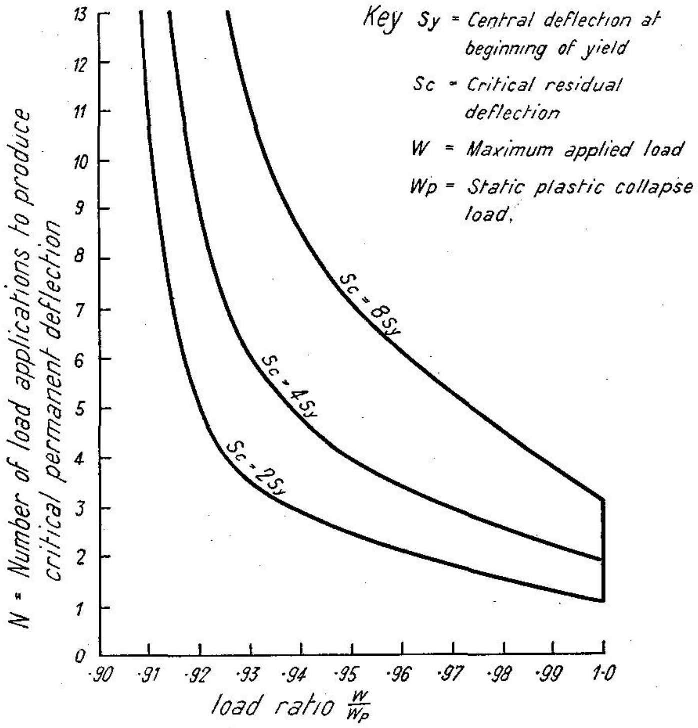


Fig. 7

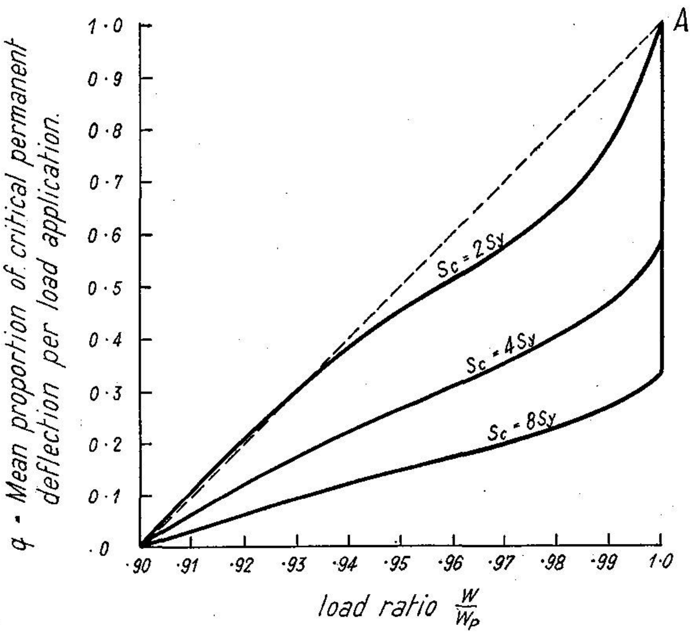


Fig. 8

when using the approximate formula by less than 5%. Judging therefore from this particular example, equation (14) appears to give safe results except when the permanent deflection at collapse is assumed to be less than about twice the elastic deflection at yield. Provided deflections in the elastic range are not so large that the design has to be based on limiting deflections at working loads, deflections at "collapse" (as usually conceived) will be at least three or four times the deflection at yield. Equation (14) has also been tested by applying it to results (both experimental and theoretical) obtained for the shake down behaviour of a fixed base portal frame subjected to the loading cycle shown in Fig. 9. The approximate expression is again found to give safe results, and will

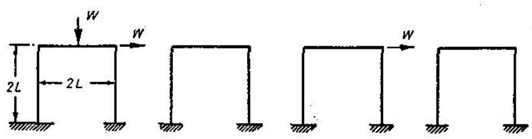


Fig. 9

therefore be assumed to be satisfactory for all structures except for those in which *limiting deflections at working loads are a controlling factor in choosing* the design.

An expression having been obtained for the mean proportion of critical deflection produced per application of load for loads applied at any constant load factor, the further approximation will be made that the permanent deflections are simply additive when loads of varying load factors are applied successively. According to this assumption, the resultant value of q after a given number of load applications may be derived by simple addition of the values for the individual loads. This has been tested by performing detailed calculations for the sequence of loading shown at the foot of Fig. 10 for the beam shown in Fig. 4 (I). The resultant values of q for the three cases $S_c = 2\,S_y$, $S_c = 4\,S_y$ and $S_c = 8\,S_y$ are shown graphically in Fig. 10. The correctly calculated values (continuous lines) are compared with those obtained from the curves in Fig. 8. It will be seen that the assumption that permanent deflections are simply additive leads to a close estimate of the residual deflections. Hence, if equation (14) is used in place of the curves in Fig. 8, again with the assumption that the deflections are additive, safe results will be obtained.

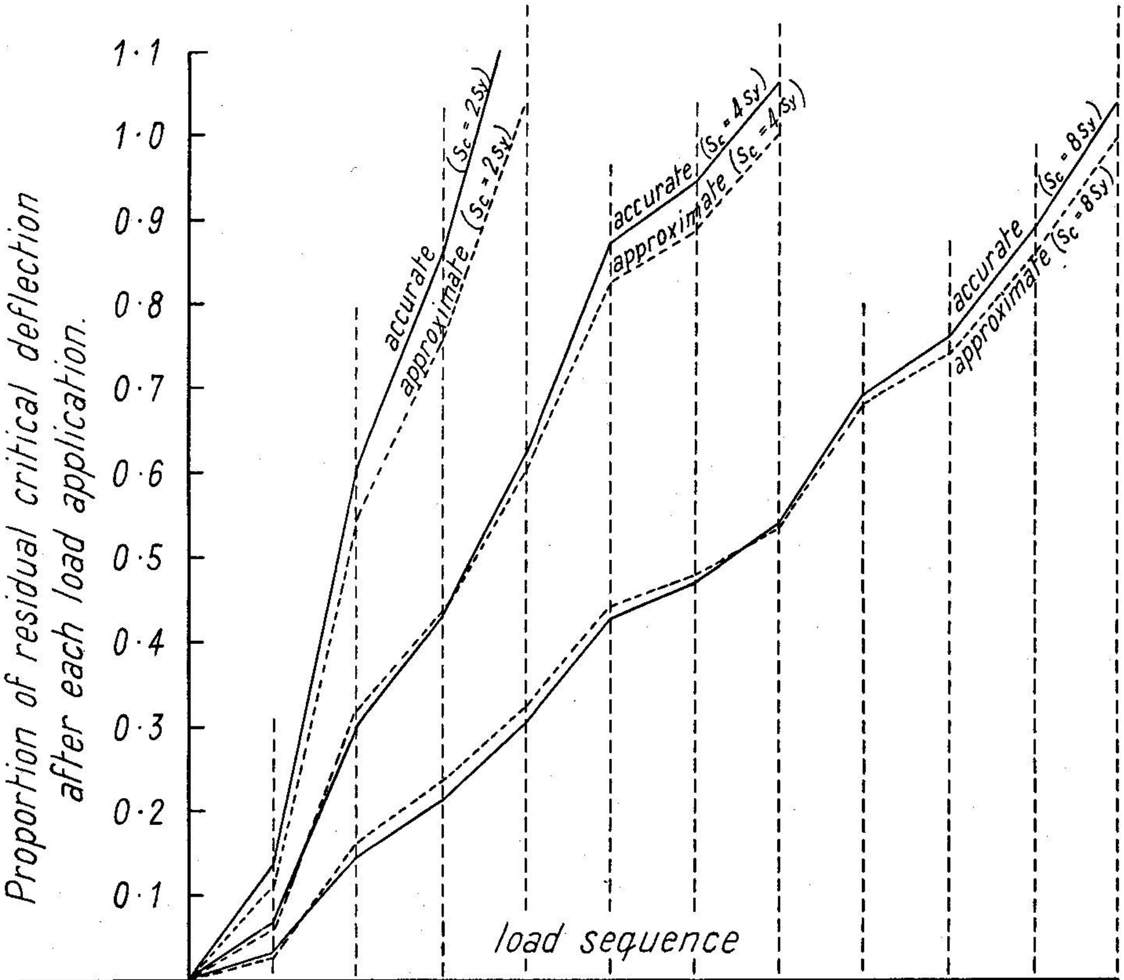
A complete study of the effect of variable repeated loads in producing incremental collapse would involve a consideration of the combined effect of floor and wind loads. The simultaneous use of two different load frequency curves leads however to unmanageable calculations, and it will therefore be assumed in turn:

1. That wind loads have the same frequency distribution as floor loads.
2. That floor loads have the same frequency distribution as wind loads.

Whichever procedure gives the more critical result will then be accepted as "safe".

Load Factors Calculated from Floor Loading Data

Let a structure be designed to collapse under static loads at a load factor of λ_c, and let its shake down load factor (for incremental collapse) be λ_s. The frequency curve of load factors derived from floor loading data (see equation (6))



load at C	0·91	0	0·92	0	0·94	0	0·92	0	0·92	0	0·95
load at D	0	0·95	0	0·93	0	0·91	0	0·95	0	0·94	0

Fig. 10

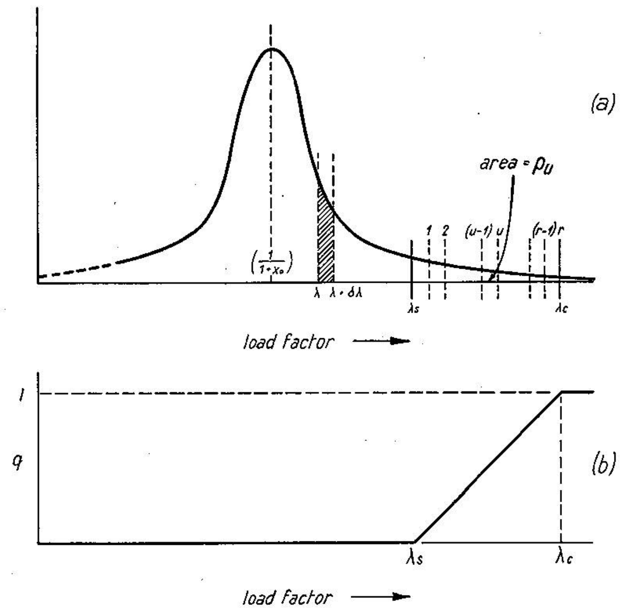


Fig. 11

is given in Fig. 11 (a). The probability that a single load application will have a load factor lying between λ and $\lambda+\delta\lambda$ is given by the shaded area, the area under the whole curve being unity. Let the load factor range from λ_s to λ_c be divided into r equal intervals, and let the area under the curve contained between the ordinates $\lambda=\frac{r-(u-1)}{r}\lambda_s+\frac{(u-1)}{r}\lambda_c$ and $\frac{r-u}{r}\lambda_s+\frac{u}{r}\lambda_c$ be denoted by p_u. The symbol u denotes any integar from 1 to r inclusive.

Consider now the increase in permanent deflection produced by a single load application. The horizontal scale in Fig. 11 (b) is identical with that in Fig. 11 (a), while the vertical scale gives the proportion (q) of the critical deflection produced per load application. It is zero for load factors less than λ_s, and varies uniformly with load factor up to a value of unity when $\lambda=\lambda_c$ (equation (14)). Above a load factor of λ_c a single load application is sufficient to produce the critical permanent deflection (since there is no interest in deflections greater than the critical, q is given the value unity when $\lambda>\lambda_c$). A load with a load factor between $\frac{r-(u-1)}{r}\lambda_s+\frac{(u-1)}{r}\lambda_c$ and $\frac{r-u}{r}\lambda_s+\frac{u}{r}\lambda_c$ will have a mean value of q of $\frac{u-\frac{1}{2}}{r}$.

After some number n of load applications there will be a certain probability of getting a resultant value of q equal to or greater than unity, i. e. a certain probability, to be denoted by P, that the structure will collapse. This probability may be calculated by using a direct extension of the Binomial Distribution as dealt with in the theory of probability.

Let

$$\left[(1-p_s)+\sum_{u=1}^{u=r} p_u z^{\frac{(u-\frac{1}{2})}{r}}\right]^n = \sum_{h=0}^{h=(2r-1)n} C_h z^{\frac{h}{2r}} \tag{15}$$

where p_s is the probability of a single load application producing a load factor greater than λ_s, and h is any integer. The coefficients C_h may be determined by expanding the expression on the left.

Then

$$P = \sum_{h=2r}^{h=(2r-1)n} C_h \tag{16}$$

The probability of static collapse P_c is given by $P_c=n\,p_c$ where p_c is the probability of the collapse load factor λ_c being exceeded on a single application of load. The values of P and P_c may be calculated for any given values of λ_c, λ_s and n. For given values of λ_c and n, that value of λ_s for which $P=P_c$ is the required shake down load factor λ_s'. Structures in which $\lambda_s>\lambda_s'$ will have a lower probability of collapse under repeated loads than under a single load application.

The value of $\frac{P}{P_c}$ has been calculated for a continuous range of values of $\frac{\lambda_s}{\lambda_c}$ both for $n=1000$ and $n=100{,}000$, the value of P_c being kept constant at $1.0\cdot10^{-6}$. This value of P_c has been chosen somewhat arbitrarily, but as in the case of failure due to alternating yield, a large variation in P_c has little effect on the final result. The symbol n denotes the number of occasions on which a structure suffers a critical combination of loads, and is difficult to determine. The lower limit of 1000 might represent a critical load combination once a month for 80 years, while the upper limit of 100,000 might correspond to an average of two critical combinations per day for 150 years. The results are shown in Fig. 12,

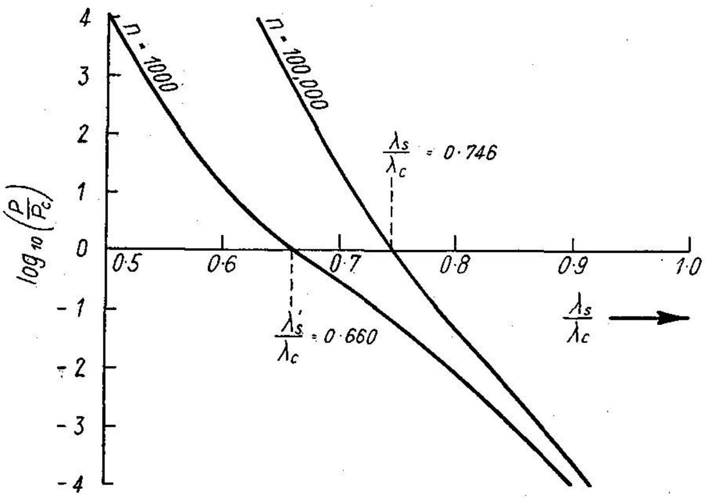


Fig. 12

in which $\log_{10}\left(\frac{P}{P_c}\right)$ is plotted against $\frac{\lambda_s}{\lambda_c}$. The condition $P = P_c$ (i. e. $\log_{10}\left(\frac{P}{P_c}\right) = 0$) gives $\frac{\lambda_s'}{\lambda_c} = 0.660$ when $n = 1000$ and $\frac{\lambda_s'}{\lambda_c} = 0.746$, when $n = 100{,}000$. If $\lambda_c = 1.75$, the values of λ_s' become 1.15 and 1.31 respectively, and incremental collapse will not be of importance unless a structure has a shake down load factor lower than these figures.

Load Factors Calculated from Wind Loading Data

The frequency distribution for load factors obtained from wind loading data is a normal curve with $\sqrt{\lambda}$ as the variate (see equation (11)), and may be represented as shown in Fig. 13 (a). The proportion (q) of critical permanent deflection due to each load application is shown to the same horizontal scale in Fig. 13 (b), the curve $A\,B\,C$ being obtained on the assumption of a linear relationship between q and λ in the range λ_s to λ_c (see equation (14)). The assumption that within the latter range there is a linear relationship between q and $\sqrt{\lambda}$ is represented by the straight line $A\,B'\,C$. The error involved in making this approximation is small, and is in any case on the safe side. When $\frac{\lambda_s}{\lambda_c} = 0.5$ the maximum error is 9% and when $\frac{\lambda_s'}{\lambda_c} = 0.8$ the maximum error is only 3%.

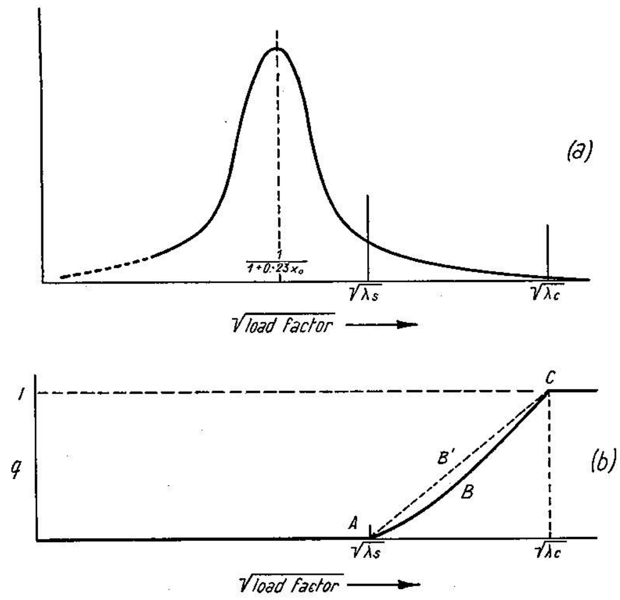


Fig. 13

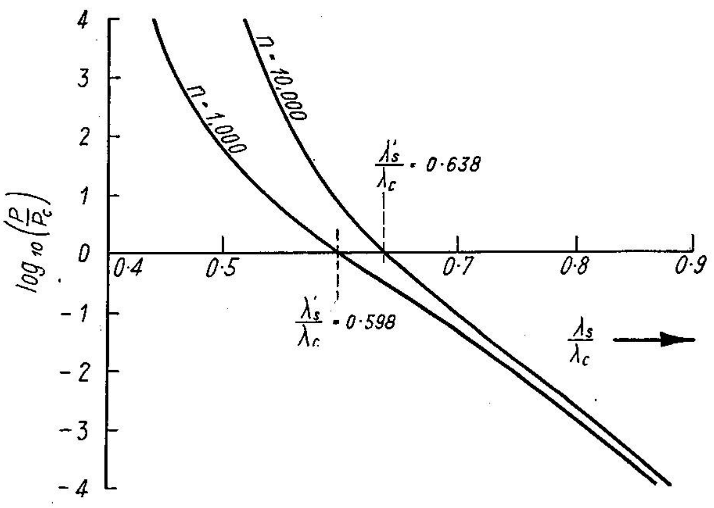


Fig. 14

The above simplification enables the analysis already given for the case of floor loading data to be used (equations (15 and (16)), the only modification required being the replacement of λ by $\sqrt{\lambda}$. The relations between λ_c, λ_s and x_0, x_c and x_s are obtained from equations (10). Calculations have been performed on the assumption that $P_c = 10^{-6}$, and that n, the number of gales during the life of the structure, is either 1000 or 10,000. The results are presented in Fig. 14. The probability of incremental collapse is equal to that of static collapse when $\frac{\lambda_s'}{\lambda_c} = 0.598$ $(n = 1000)$ or $\frac{\lambda_s'}{\lambda_c} = 0.638$ $(n = 10{,}000)$. Hence if $\lambda_c = 1.75$, the shake down load factor has the value 1.05 when $n = 1000$ and 1.12 when $n = 10{,}000$.

5. Conclusions

Calculations have been made of the minimum shake down load factor which may be tolerated in a structure which has been designed on the basis of the plastic theory to collapse at a static load factor of 1.75. The results are summarised in Table 2, lower and upper estimates being given in each case. It has been

Table 2. Minimum allowable shake-down load factors assuming a static load factor at collapse of 1.75

	Alternating yield	Incremental Collapse	
		Floor Load Data	Wind Load Data
Lower limit	0.89	1.15	1.05
Upper limit	0.98	1.31	1.12

assumed in the analysis for incremental collapse that deflections at working loads are not the ruling factor in the choice of a design. When the static load factor at collapse is some value λ_c other than 1.75, the corresponding shake down load factors may be obtained by multiplying the values in the table by $\frac{\lambda_c}{1 \cdot 75}$.

Throughout the analysis leading to these results, numerous approximations are made. Care has been taken, however, that all approximations are on the safe side, and for this reason it is probably satisfactory to accept the lower shake down load factors quoted in Table 2. If this course is adopted, collapse due to alternating yield is seen to be a most unlikely occurrence ($\lambda_s' = 0.89$), while incremental collapse is only liable to occur if the shake down load factor is less than 1.15. Even when the higher values in Table 2 are taken the critical load factor for alternating yield is as low as 0.98, and for incremental collapse, 1.31. The lowest calculated shake down load factor (corresponding to $\lambda_c = 1.75$) for alternating yield reported in the literature is 1.22 (see example of portal frame with wind load in either direction, Fig. 24 and equations (15) and (16), reference (14)). The lowest reported load factor for incremental collapse is 1.56 (see example of double bay portal frame, reference (8)). While lower values than these could undoubtedly be obtained, particularly for structures containing members of highly unequal stiffness, there appears to be sufficient grounds for stating that variable repeated loads are most unlikely to be of importance in the design of structures by the plastic theory.

The results of the analysis contained in this paper may be summarised as follows. It is necessary to distinguish between two types of repeated loading. The first type refers to loads which would be dealt with, when designing structural components by orthodox elastic methods, by specifying safe ranges of stress determined by endurance under fatigue. Crane rails subjected to heavy duty afford an example. When such components occur in structures designed by the plastic theory, they should still be checked for fatigue conditions by limiting the elastic range of stress under working conditions. Excluding such structures or structural components, the analysis shows that the effect of repeated loads on behaviour in the plastic range is most unlikely to be of importance in design. For all practical purposes therefore it is unnecessary to perform a shakedown analysis (6), (7) — a most valuable conclusion in view of the complexity of such calculations.

The work described in this report was carried out at the Engineering Laboratory, Cambridge University, under the direction of Professor J. F. Baker, Head of the Department of Engineering. It forms part of a general investigation into the behaviour of steel structures in the plastic range being carried out at Cambridge with the assistance of the British Welding Research Association.

References

1. Baker, J. F.: "A Review of Recent Investigations into the Behaviour of Steel Frames in the Plastic Range." J. Inst. Civ. Engrs., Vol. 31 (1948/49), p. 188.
2. Baker, J. F.: "The Design of Steel Frames." Structural Engineer, Vol. 27 (1949), p. 397.
3. Horne, M. R.: "A Moment Distribution Method for the Analysis and Design of Structures by the Plastic Theory." Proc. Inst. Civ. Engrs., Part III, Vol. 3. (1954), p. 51.
4. Neal, B. G., and Symonds, P. S.: "The Rapid Calculation of the Plastic Collapse Load for a Framed Structure." Proc. Inst. Civ. Engrs., Part III, Vol. 1 (1952), p. 58.
5. Neal, B. G., and Symonds, P. S.: "The Calculation of Plastic Collapse Loads for Plane Frames." Prelim. Pub., 4th Congress, Int. Assn. Bridge and Struct. Engrg. (1952), p. 75.
6. Horne, M. R.: "The Effect of Variable Repeated Loads in the Plastic Theory of Structures." Research, Engineering Structures Supplement (1949), p. 141.
7. Symonds, P. S., and Neal, B. G.: "The Calculation of Failure Loads on Plane Frames under Arbitrary Loading Programmes." J. Inst. Civ. Engrs., Vol. 35 (1950/51), p. 41.
8. Neal, B. G., and Symonds P. S.: "A Method for Calculating the Failure Load for a Framed Structure subjected to Fluctuating Loads." J. Inst. Civ. Engrs., Vol. 35 (1950/51), p. 186.
9. Neal, B. G.: "The Behaviour of Framed Structures under Repeated Loading." Quart. J. Mech. and Appl. Maths., Vol. 4 (1951), p. 78.
10. Horne, M. R.: "The Variation of Mean Floor Loads with Area." Engineering, Vol. 171 (1951), p. 179.
11. Horne, M. R.: "Wind Loads on Structures." J. Inst. Civ. Engrs., Vol. 33 (1949/50), p. 155.
12. White, C. M.: "Survey of Live Loads in Offices." 1st Interim Report, Steel Structures Research Committee (1931), p. 45. H. M. Stationery Office.
13. Dunham, J. W.: "Design Live Loads in Buildings." Trans. A.S.C.E., Vol. 112 (1947), p. 725.
14. Baker, J. F.: "Comportement et Conception des Ossatures Métalliques." Revue de la Soudure, Numéro 3 (1951), p. 128.

Summary

The plastic theory of structures has been evolved with reference to loads which are only applied once. It has nevertheless been realised that repeated loads may cause collapse at a lower intensity of loading than a single load application, even when fatigue effects are entirely absent. The importance of such an effect can only be assessed by a study of the relative frequencies of loads of various intensities, and the theory of probability thus has to be introduced. The paper applies the theory of probability to the derivation of design load factors for repeated loads in structures which are to be designed by the plastic theory.

Résumé

La théorie de la plasticité des ouvrages a été mise au point en tablant sur des charges qui ne font l'objet que d'une seule application. L'on s'est néanmoins rendu compte qu'une charge répétée peut provoquer la déformation ou la rupture sous des efforts plus faibles que dans le cas d'une application unique de la charge, même lorsque les effets de fatigue sont entièrement absents. L'importance d'un tel effet ne peut être déterminée que par une étude des fréquences relatives de charges de différentes intensités et par introduction de la théorie des probabilités. L'auteur applique cette dernière théorie au calcul des coefficients de charge théorique, pour des charges répétées appliquées à des ouvrages qui doivent être calculés d'après la théorie de la plasticité.

Zusammenfassung

Die Theorie über das plastische Verhalten der Bauwerke wurde für Belastungen entwickelt, die nur ein einziges Mal zur Wirkung gelangten. Man gab sich aber Rechenschaft, daß bei einer mehrmals aufgebrachten Last Formänderung oder Bruch für kleinere Belastungswerte auftreten, als wenn die gleiche Last nur einmal wirkt, selbst wenn keine Ermüdung des Materials eintritt. Die Bedeutung einer solchen Wirkung kann nur durch eine Untersuchung über das Verhalten bei der in verschiedenen Abständen wiederholten Belastung, deren Größe ebenfalls variiert wird, erfaßt werden; ferner unter Anwendung der Wahrscheinlichkeitsrechnung.

Der Verfasser wendet diese Theorie für die Bestimmung der Koeffizienten der theoretischen Last auf die Fälle von wiederholter Belastung bei Bauwerken an, welche nach der Plastizitätstheorie zu berechnen sind.