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Matrix-Analysis of Successive Moment-Distribution

Calcul matriciel de la répartition continue des moments

Matrix-Berechnung der fortlaufenden Momentenverteilung

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Introductory

In "A Study of Three-Dimensional Pile-Groups" (AIPC Mémoires 1947, p. 13) the author showed how one could by means of an infinite matrix series describe a sequence of successive approximations, useful in determining the forces in pile-groups. He added that it seemed possible in general to establish by simple matrix methods the coincidence in results between standard methods of successive approximations and the direct solution according to the classical theory of statically indeterminate structures.

The author particularly had in mind a proof for the statement that the method of successive moment distribution could be explained by an infinite matrix series whose convergent sum (for stable systems) would coincide with the state of stress of the structure, solved directly as an elastic system. That this must be the case is self-evident almost to triviality from a physical point of view. In the feeling that elementary matrix methods may in the future become more widely used for simply and clearly formulating the propositions of the theory of structures, a proof of the theorem stated was believed to be of some value. By-products of practical value may moreover emerge from the proof of quite abstract general statements.

It is a frequently observed paradox of applied matrix analysis that it is easier to treat a general case by general methods than special cases by special methods.

As far as is known by this author only one paper (S. Benscoter, "Matrix Analysis of Continuous Beams", Transactions ASCE, p. 1109) has undertaken to adapt the Cross method, to matrix treatment. This meritorious paper treats the practically important special case of continuous beams (with two bars connected to each joint) in such a manner that it would obviously not be easy to proceed to the general case along the same lines.

Chen ("Matrix Analysis of Pin-Connected Structures", Transactions ASCE 1949, p. 181) has studied trusses with pinned joints, by matrix methods. His solution is direct, without approximations or matrix series.

R. Oldenburger (Convergence of Hardy Cross's balancing process, Journ. of App. Mech. 1940, p. 166) has proved by a comparatively complicated matrix method the convergence of successive moment-distribution for beams. Ch. Massonnet (Sur la convergence de la méthode de Cross . . ., Révue universelle des mines, Liège 1946, no. 6) has treated analytically the convergence for frames.

The Method of Successive Moment Distribution

The method of successive moment-distribution (K. A. Čalisév, Technicki List, Zagreb 1922, and AIPC Mémoires 1936; Hardy Cross, Trans. ASCE 1932) is best suited to girders, frames, and trusses having joints that are rigid, rotatable, and having essentially no displacements in space. Complements to the method admit the analysis of joint translations and other special conditions.

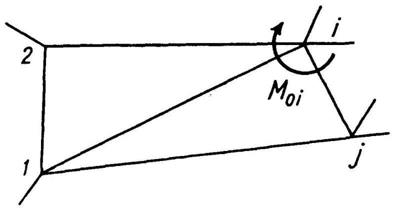

Fig. 1. Framework

Structures suited for treatment by this method are thus characterized by a number of almost stationary joints 1, 2, . . . i, j, . . . , at which straight or curved members with uniform or varying stiffness are rigidly connected. Hinges exist at such points where the stiffness EI of the members decreases to zero. The members connect various pairs of points i, j in this structure; no members run between many pairs of points, fig. 1.

The method of successive moment distribution is performed in the following manner: In the *unloaded* structure *all* joints are *locked* for rotation. Members (or joints directly) are *then* loaded by the external loads for which the computation of the real structure is required. In every case where external loads act upon a member ij, elementary statical methods are used to compute the fixed-end moments M_{0ij}, M_{0ji} at the (as yet fixed) ends ij and ji. All end-moments will be given positive signs, if they tend to turn *the joint* in a positive (clockwise) direction. The sum $\sum_j M_{0ij} = M_{0i}$ of all fixed-end moments M_{0ij} at a joint i and all moments M_{0ii} that act directly upon the joint i, is called the (primary) "unbalanced" moment M_{0i} acting upon the joint i.

The method proceeds by *unlocking one* joint in the structure, while all the other joints remain locked for rotation. This joint i is first rotated through an angle of -1 radians (joint-rotation angles Θ in a clockwise direction are considered as positive) by the action of some external moment, fig. 2. Then the end ij ($=$ end i of member ij) will be bent by a moment S_{ij}, defining the *absolute end-stiffness* of the end ij.

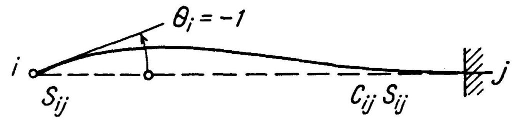

Fig. 2. End-stiffness S_{ij} and carry-over factor C_{ij}.

The opposite, fixed, end ji of the same member ij will simultaneously be bent by a moment $C_{ij}\,S_{ij}$. C_{ij} is termed the "carry-over factor" for the member ij (in the direction i to j).

End-stiffnesses S and carry-over factors C can be conveniently computed by simple statical methods. Tables are worked out of S and C for a great number of stiffness variations EI. It is possible, according to James (Principal Effects of Axial Load etc., NACA Techn. Note 534, 1935), to account for the influence upon S and C of an axial force in the members. For members of uniform stiffness this influence is expressible by Berry's functions. Diagrams by Hoff etc. (Buckling of Rigid-Jointed Plane Trusses, Transactions ASCE 1951, p. 958) reformed as in fig. 3, furnish values of S and C, taking into account

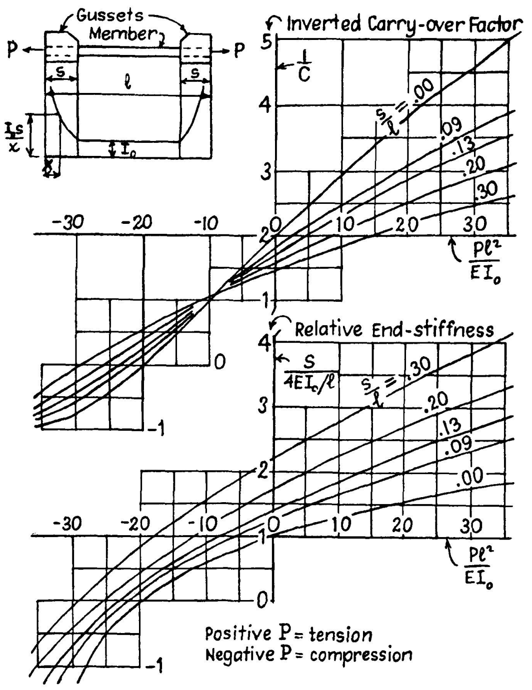

Fig. 3. Hoff's stiffness chart for gusseted member under axial load, rearranged.

extra joint rigidities caused by concurrent members or by gusset plates. Charts by J. E. Goldberg (Stiffness Charts for Gusseted Members under Axial Load, Transactions ASCE 1954, p. 43) can also be used for this purpose.

However, under the action of the "unbalanced" moment M_{0i} the joint i will rotate an angle Θ_{1i} instead of -1. This rotation Θ_{1i} induces in end ij a moment $-S_{ij}\Theta_{1i}$, fig. 2. All end-moments of members connected at i will together balance the acting moment M_{0i}: $M_{0i} - \sum_j S_{ij}\Theta_{1i} = 0$, or, denoting $\sum_j S_{ij}$ by S_i, $M_{0i} = S_i\Theta_{1i}$.

If S_i is written in a diagonal matrix S and M_{ni} and Θ_{ni} in column matrices M_n and Θ_n

$$M_n = \begin{bmatrix} M_{n1} \\ M_{n2} \\ \cdot \end{bmatrix}, \quad S = \begin{bmatrix} S_1 & 0 & \cdot \\ 0 & S_2 & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}, \quad \Theta_n = \begin{bmatrix} \Theta_{n1} \\ \Theta_{n2} \\ \cdot \end{bmatrix}$$

this may be written

$$M_0 = S\Theta_1, \text{ or } \Theta_1 = S^{-1}M_0 \tag{1}$$

The end-moments will thus be $-M_{0i}S_{ij}/S_i = -s_{ij}M_{0i}$, where the distribution number $s_{ij} = S_{ij}/S_i$.

The far end of member ij will be bent by $-C_{ij}s_{ij}M_{0i} = -c_{ji}M_{0i}$, where $c_{ji} = C_{ij}s_{ij}$.

Consequently, an external moment $= -1$ acting on a single unlocked joint i will cause at end ij a moment s_{ij} and at end ji a moment c_{ji}, fig. 4. Unbalanced moments M_{0j} in the "neighbor joints" j of i will thus increase by $-c_{ji}M_{0i}$.

Now joint i is *locked* in its new equilibrium position Θ_{1i}. Some other joint, generally the one, j, having the largest total unbalanced moment (= the original M_{0j} plus all additional $-c_{ji}M_{0i}$ caused by the rotation of neighbor joints i) is unlocked, balanced, and relocked according to the same method.

This procedure is continued if it is convergent, until the largest remaining unbalanced moment is of no significance for the purpose of the analysis (check on strength, etc.).

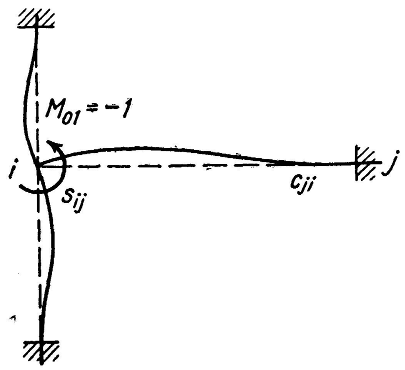

Fig. 4. Distribution of moment $M_{0i} = -1$ at joint i

It might be possible to describe this procedure by a series of matrix operations. However, from a matrix point of view the rule "largest remaining moment" for the choice of the "next joint" is extremely awkward. It will be advantageous to devise some other rule by which the moment-distribution proceeds to the same final result, but with simple matrix analysis.

Moment-Distribution in Stages

A short consideration will make clear that such a goal can hardly be attained by any means other *than by performing the moment-distribution in stages* in such a way that all joints are balanced in each stage *disregarding* in each stage *all* moments $-c_{ji} M_{0i}$ carried-over during *that* stage. Instead all these moments $-c_{ji} M_{0i}$ left behind will be applied as (only) unbalanced moments in the *next* stage.

Because of its more systematic character such a stage-sequence of operations is recommended in the practical application of the Cross method. Moment-distribution in stages, carried on until the largest remaining undistributed moment falls below a specified magnitude or percentage may imply a somewhat greater total number of balanced joints. Work can nevertheless be saved since the additions of carried-over moments before the balancing of each joint will be concentrated to the changes between the stages.

Very few papers on the moment-distribution method, or text-books on structural theory mention moment-distribution in stages. Instances are S. Hultin, Lecture Notes on the Calculation of Frames, Gothenburg 1949, and Wang C.-K., Statically Indeterminate Structures, New York 1953, p. 218. Distribution in stages is, however, currently used by a great number of European and American structural computers.

Matrix Analysis of Distribution in Stages

With this rule of balancing in stages it is easy to write down the unbalanced moments which were disregarded in the *first* stage in order to be balanced in the second. They are for the joint j

$$-c_{j1} M_{01} - c_{j2} M_{02} - c_{j3} M_{03} - \cdots$$

In this sum the j-th term, $-c_{jj} M_{0j}$, vanishes, c_{jj} being null. The sum just written down equals the element in the j-th row of the matrix product

$$M_1 = -c\,M_0 = -\begin{bmatrix} 0 & c_{12} & \cdot \\ c_{21} & 0 & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \cdot \begin{bmatrix} M_{01} \\ M_{02} \\ \cdot \end{bmatrix}$$

The *second* stage of balancing starts with the unbalanced moments $M_1 = -c\,M_0$. In the same manner as new unbalanced moments $-c\,M_0$ were formed during the balancing of the moments M_0 of the first stage, the unbalanced moments $M_2 = -c\,(-c\,M_0) = c^2\,M_0$ will appear after the moments $-c\,M_0$ have been balanced in the second stage, $-c^3\,M_0$ after the third stage, etc.

In case this manipulation converges, the final balanced moments in every joint will be given by

$$M_f = \sum_{n=0}^{\infty} M_n = M_0 - c\,M_0 + c^2\,M_0 - c^3\,M_0 + \dots$$

$$M_f = (I + c)^{-1}\,M_0 \tag{2}$$

causing the final joint rotations $\Theta_f = \sum_{n=1}^{\infty} \Theta_n$, cf. (1),

$$\Theta_f = S^{-1}\,M_f = S^{-1}\,(I + c)^{-1}\,M_0 \tag{3}$$

Premultiplication by $(I + c)\,S$ yields

$$(I + c)\,S\,\Theta_f = M_0 \tag{4}$$

"Statically Indeterminate" Matrix Solution

The direct "statically indeterminate" solution without successive approximation is obtained by first locking the structure in its undeformed (fixed-end) condition. All members are now loaded, resulting in fixed-end moments M_{0ij}

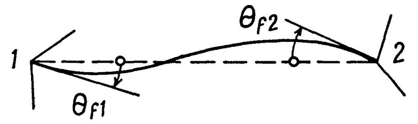

Fig. 5. Final rotations.

and unbalanced moments M_{0i} over the whole structure. All joints i of the structure are then unlocked and *finally* rotated through Θ_{fi} to equilibrium.

In member 12, fig. 5, the end	12	and end 21
was bent before the rotation by	M_{012}	M_{021}
Θ_{f1} adds the moment	$-S_{12}\,\Theta_{f1}$	$-C_{12}\,S_{12}\,\Theta_{f1}$
and Θ_{f2}	$-C_{21}\,S_{21}\,\Theta_{f2}$	$-S_{21}\,\Theta_{f2}$

Finally, end 12 will be bent by the moment

$$M_{f12} = M_{012} - S_{12}\,\Theta_{f1} - C_{21}\,S_{21}\,\Theta_{f2} \tag{5}$$

M_{f1} denoting the sum $\sum_{n=0}^{\infty} [(-c)^n\,M_0]_1$ of all the consecutive unbalanced moments at joint 1.

All moments at the ends $1j$ connected to joint 1 must balance the direct moment M_{011} at joint 1:

$$M_{011}+M_{012}+M_{013}+\cdots-S_{12}\Theta_{f1}-S_{13}\Theta_{f1}-\cdots$$
$$-C_{21}S_{21}\Theta_{f2}-C_{31}S_{31}\Theta_{f3}-\cdots=0$$

or, since $\sum_j S_{1j}=S_1$ and $C_{ij}S_{ij}=c_{ji}S_i$,

$$\sum_j M_{01j}=M_{01}=S_1\Theta_{f1}+c_{12}S_2\Theta_{f2}+c_{13}S_3\Theta_{f3}+\ldots$$

$$M_0=\begin{bmatrix} S_1 & c_{12}S_2 & c_{13}S_3 & \cdot \\ c_{21}S_1 & S_2 & c_{23}S_3 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}\begin{bmatrix}\Theta_{f1}\\ \Theta_{f2}\\ \cdot\end{bmatrix}=\begin{bmatrix} 1 & c_{12} & \cdot \\ c_{21} & 1 & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}\begin{bmatrix} S_1 & 0 & \cdot \\ 0 & S_2 & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}\Theta_f$$

$$M_0=(I+c)\,S\,\Theta_f \tag{6}$$

The agreement of (4) and (6) proves the statement that the Cross moment-distribution, if it converges, does so towards the classical direct solution of the statically indeterminate structure.

Asymptotic Geometric Matrix Series

After the n-th stage there remains at joint i an unbalanced moment $M_{ni}=[(-c)^n M_0]_i$. It will be demonstrated that these terms are asymptotic to the terms of a geometric series. Denote by c_n an element with definite position in the matrix c^n and by λ the largest latent root of c (one such root is assumed; the following conclusions can easily be extended to apply also in the more improbable cases in which there are several equally large latent roots, cf. Frazer, Duncan, Collar, Elementary Matrices, Cambridge 1938, p. 135).

Sylvester's theorem easily proves that the ratio c_n/c_{n-1} between corresponding elements in the matrices c^n and c^{n-1} for increasing n *all* approach the largest latent root λ of c. As a consequence, if c^n and c^{n-1} are postmultiplied by the *same* column matrix M_0, the ratio of *all* corresponding elements in the two resulting column matrices obviously tend to λ with increasing n:

$$\lambda_{ni}=(c^n M_0)_i \,/\, (c^{n-1} M_0)_i \rightarrow \lambda \tag{6}$$

for all i.

This important theorem can be utilized in ordinary moment-distribution according to Cross, if it is performed in stages as clearly indicated by the matrix treatment itself. Each completed stage n concludes with numerical values of $M_{ni}=((-c^n)\,M_0)_i$. The calculator writes down a column with the before-mentioned ratios λ_{ni}. When he has found that this column of figures can be replaced without harmful inaccuracy by a column of *equal* numbers $\lambda_{n\,med}$ (<1), he will employ $\lambda_{n\,med}$ as an approximate value of the largest latent root λ, terminate the moment distribution, and calculate the sums of all remaining unbalanced moments in the geometric series $M_{ni}(-\lambda_{n\,med}+\lambda_{n\,med}^2-\ldots)$ by the formula

$$M_{rest\,i} = -M_{ni}\lambda_{n\,med}/(1+\lambda_{n\,med}).$$

It may sometimes be possible to do this at an early stage n of the moment distribution. In such a case a number of joint distributions can be saved which otherwise would have been necessary in a standard computation according to Cross. This saving of labor alone could in some cases motivate the procedure of moment-distribution in stages.

Arrangement of Numerical Computations

Matrix moment-distribution in stages can be carried through in the following form for the framework of fig. 6 to 12:

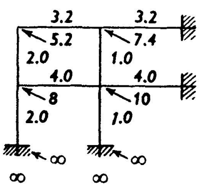

Fig. 6. Stiffnesses S_{ij} and $S_i = \sum_j S_{ij}$.

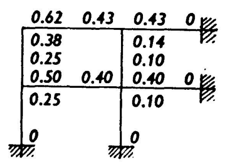

Fig. 7. Distribution numbers $s_{ij} = S_{ij}/S_i$.

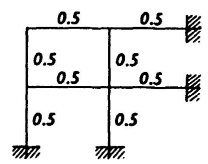

Fig. 8. Carry-over factors C_{ij}.

End-stiffnesses S_{ij} and joint stiffnesses S_i, distribution numbers s_{ij}, carry-over factors C_{ij}, matrix elements c_{ij}, and fixed-end moments M_{0ij} and primary unbalanced moments M_{0i} are noted upon framework sketches fig. 6 to 10 near the joints to which the magnitudes pertain. Stiffnesses, moments etc. that are the same at both ends of a member are noted near the middle of the member.

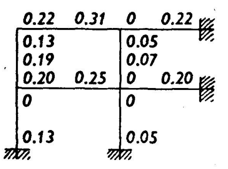

Fig. 9. Matrix elements $c_{ij} = C_{ji}s_{ji}$.

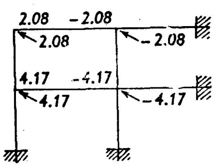

Fig. 10. Fixed-end moments M_{0ij} and primary unbalanced moments $M_{0i} = \sum_j M_{0ij}$.

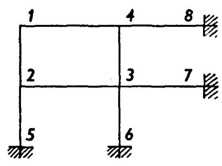

Fig. 11. Joint numbers i.

After numbering the joints, fig. 11, the elements c_{ij} are entered into a square matrix c, table 1. Blank elements are zero.

Table 1. Matrix moment-distribution

Row i (Joint)	Columns j (opposite end) 1	2	3	4	M_0	$c\,M_0$	$c^2 M_0$	$c^3 M_0$	M_f
1, $c =$		.13		.22	2.08	.08	.04	.04	2.00
2	.19		.20		4.17	–.44	.19	–.01	4.81
3		.25		.07	–4.17	.88	–.08	.05	–5.18
4	.31		.05		–2.08	.44	.07	.01	–2.46

The primary unbalanced moments M_{0i} are entered into a column matrix M_0 at the right hand side of c. Cumulative multiplication and addition on an ordinary desk calculating machine expedites the consecutive evaluation of the matrix products $c\,M_0$, $c^2 M_0, \ldots$. The final sum of all the distributed moments in each joint is $M_f = M_0 - c\,M_0 + c^2 M_0 + \ldots$. The final end-moments M_{fij} of the members $i\,j$ can be computed according to

$$M_{fij} = M_{0ij} - s_{ij}\,M_{fi} - c_{ij}\,M_{fj} \tag{7}$$

cf. (5), (1), or by balancing only the moments M_{fi} in the fixed-end condition of fig. 10, which is carried through in fig. 12.

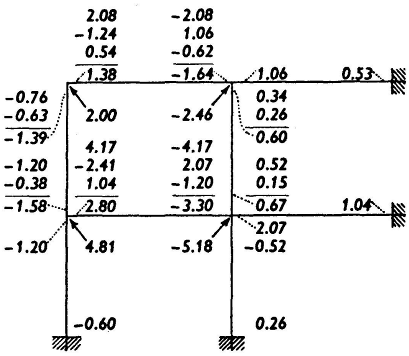

Fig. 12. Distribution of final end-moments.

Only insignificant unbalanced moments should remain after one balancing stage. This furnishes a check upon the numerical computations. In that stage the correct final end-moments M_{fij} of all members $i\,j$ have also been evaluated.

Framework Buckling Criterion

Hoff's buckling criterion (Stable and Unstable Equilibrium of Plane Frameworks, J. Aeron. Sci. 1941, p. 115) is based upon the convergence or divergence of the moment-distribution, using James's (l. c.) stiffnesses and carry-over factors. In investigating stability, it also appears preferable to perform the moment-distribution in stages. Instead of a more or less vague idea of convergence or divergence developed during an irregular wandering about among the joints of the structure, a distribution in stages furnishes after each stage, through the column λ_{ni}, a better-defined indication of whether the critical root tends to a value below or above 1. This determines the convergence or divergence of the remaining series $M_{ni} = \Sigma((-c)^n M_0)_i$ which is asymptotic to a geometric series with ratio $-|\lambda|$.

Successive stages of the Cross moment-distribution correspond to successive powers of c. A large number of moment-distribution stages can be saved by repeatedly squaring the matrix c, to c^2, c^4, c^8, c^{16} etc., by means of simple matrix multiplication: $c^8 c^8 = c^{16}$. Approximate values of the largest latent root then are, for instance, $|\lambda|_{appr.} = (c^{16}{}_{ij} / c^8{}_{ij})^{1/8}$.

High powers of the matrix c will contain only proportional rows. This implies that any originally applied virtual moments will after a number of distribution stages approach proportionality to the related elements in any column of c.

Numerical Example on Framework Buckling

As another example of matrix treatment of successive moment distribution the buckling load of the framework shown in fig. 13 will be computed. This framework 1234 has members of equal lengths 1 and equal stiffnesses EI.

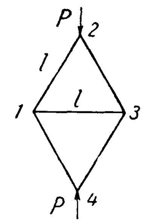

Fig. 13. Buckling of framework.

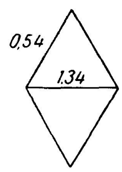

Fig. 14. $S_{ij} . 4EI_0/l_0$.

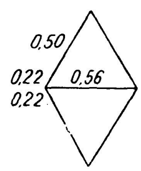

Fig. 15. s_{ij}.

The rigidity caused by the triangular framework makes the displacements of the joints 1 and 3 negligible. A trial axial load of $P = 20\, EI/l^2$ induces in the members 12 and 23 forces $P_{12} = -11.5\, EI/l^2$ and $P_{13} = +11.5\, EI/l^2$. End-stiffnesses and carry-over factors from fig. 3 for $s/l = 0$ (no gussets) are noted in fig. 14 and 16; s_{ij} and c_{ij} are computed in fig. 15 and 17. Symmetric magnitudes are not noted.

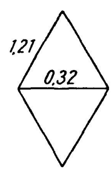

Fig. 16. C_{ij}.

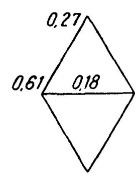

Fig. 17. c_{ij}.

For antisymmetric virtual applied moments $M_{01}^{\cdot} = M_{03}^{\cdot}$, $M_{02}^{\cdot} = M_{03}^{\cdot}$.

$$c^{\cdot} M_0^{\cdot} = \begin{bmatrix} & .61 & .18 & .61 \\ .27 & & .27 & \\ .18 & .61 & & .61 \\ .27 & & .27 & \end{bmatrix} \begin{bmatrix} M_{01}^{\cdot} \\ M_{02}^{\cdot} \\ M_{03}^{\cdot} \\ M_{04}^{\cdot} \end{bmatrix}$$

is equivalent to

$$c\, M_0 = \begin{bmatrix} .18 & 1.22 \\ .54 & \end{bmatrix} \begin{bmatrix} M_{01} \\ M_{02} \end{bmatrix}$$

Hence the stability of the system characterized by c is the same as of that characterized by $c^{\cdot}$. Simple matrix multiplication yields

$$c^2 = \begin{bmatrix} .69 & .22 \\ .10 & .66 \end{bmatrix}, \quad c^4 = \begin{bmatrix} .50 & .30 \\ .14 & .46 \end{bmatrix}, \quad c^8 = \begin{bmatrix} .29 & .29 \\ .13 & .25 \end{bmatrix}, \quad c^{16} = \begin{bmatrix} .11 & .16 \\ .07 & .10 \end{bmatrix}$$

$$|\lambda|_{appr.} = \begin{bmatrix} .88 & .93 \\ .93 & .90 \end{bmatrix}$$

the elements of the matrix of $|\lambda|_{appr.}$ being computed as $(c^{16}{}_{ij} / c^{8}{}_{ij})^{1/8}$.

For a matrix c of order two, as in this instance, the largest latent root would have been more easily computed by a direct solution of the characteristic equation of c:

$$\begin{bmatrix} \lambda - .18 & -1.22 \\ -.54 & \lambda \end{bmatrix} = 0, \quad \lambda^2 - .18\lambda - .66 = 0$$

with the roots $\lambda = .91$ and $.73$.

After a large number of distribution stages, all "later" unbalanced virtual moments and all "later" deflections of the structure are multiplied by $-\lambda$ for each new distribution-stage. When $|\lambda| < 1$ the structure will reach stable equilibrium under the action of any initial virtual disturbances M_{0i}. If the actual forces upon the structure are increased by a factor K, it seems plausible from a physical point-of-view that the multiplier λ of the "later" virtual moments should increase at least approximately by the same factor K. To make λ equal to the buckling value 1 the loads upon the frame should thus be increased approximately by the factor $1/\lambda$.

Choosing an axial load of $P = 22.5\, E\, I/l^2$ for a possibly diverging moment-distribution, fig. 14 to 17 will be modified as shown in fig. 18 to 21.

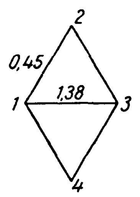

Fig. 18. S_{ij} $4 E I_0 / l_0$.

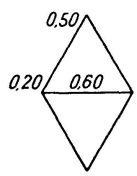

Fig. 19. s_{ij}.

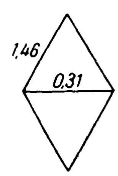

Fig. 20. C_{ij}.

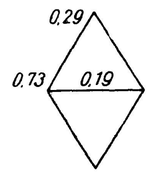

Fig. 21. c_{ij}.

$$c^{\cdot}=\begin{bmatrix} & .73 & .19 & .73 \\ .29 & & .29 & \\ .19 & .73 & & .73 \\ .29 & & .29 & \end{bmatrix},\quad c=\begin{bmatrix} .19 & 1.46 \\ .58 & \end{bmatrix},\quad c^2=\begin{bmatrix} .88 & .28 \\ .11 & .85 \end{bmatrix}$$

$$c^4=\begin{bmatrix} .80 & .48 \\ .19 & .75 \end{bmatrix},\quad c^8=\begin{bmatrix} .73 & .74 \\ .29 & .65 \end{bmatrix},\quad c^{16}=\begin{bmatrix} .75 & 1.02 \\ .40 & .64 \end{bmatrix}$$

$$|\lambda|_{appr.}=\begin{bmatrix} 1.00 & 1.04 \\ 1.04 & 1.00 \end{bmatrix}$$

Loads of $P=20$ and $22.5\,EI/l^2$ have thus yielded largest latent roots of about .91 and 1.02. By interpolation, a latent root of 1 can be expected for $P=22.1\,EI/l^2$. This load can be accepted as the buckling load, or as initial force in a second iteration.

Conclusions

The trivial theorem that the Hardy Cross successive moment-distribution approaches the direct classical solution of the elastic structure has been proved by matrix analysis. The simple proof was in fact made possible by studying a *general* case and by considering a moment-distribution *in stages*. The advantages of the matrix treatment of a moment-distribution in stages suggests that the same procedure could also be advantageous in practical numerical applications.

The matrix treatment of moment-distribution in stages also discloses that the stage-series of unbalanced moments is asymptotic to a geometric series, the ratio and sum of which can be approximated; this property may sometimes be useful in practical numerical work and in the numerical application of Hoff's buckling criterion for frameworks.

Whenever the matrix technique can be employed, the rationalization of the computation procedure thereby achieved leads to advantages of a time-saving systematizing of numerical work, a more definite appraisal of convergence, or the possibility of an approximate estimation of the sums of all the remaining corrections.

Summary

The procedure of successive moment-distribution is interpreted by an infinite matrix series whose definite sum is also given. It is shown that this sum coincides with the corresponding finite expression deduced without approximations by the classical theory of elastic structures. It is finally demonstrated that the matrix series for moment-distribution in stages (as defined) is asymptotic to geometric series whose common ratio and various sums can be approximated. The matrix method given is exemplified on an ordinary frame problem and on a frame-buckling problem.

Résumé

L'auteur traite la méthode de la répartition continue des moments à l'aide d'une série matricielle infinie, dont il indique également la somme. Il montre que cette somme concorde avec l'expression finie correspondante, qui peut être déduite sans approximations de la théorie classique des corps élastiques. Il montre enfin que la série matricielle correspondant à la répartition progressive des moments (suivant définition) présente une allure asymptotique par rapport à une série géométrique, dont la solution générale et les différentes sommes peuvent être déterminées approximativement. La méthode matricielle exposée fait l'objet d'exemples portant sur un problème de cadre ordinaire et sur un problème de flambage de cadre.

Zusammenfassung

Die Methode der fortlaufenden Momenten-Verteilung wird mit einer unendlichen Matrix-Reihe samt deren Summe dargestellt. Der Verfasser zeigt, daß diese Summe mit dem entsprechenden endlichen Ausdruck übereinstimmt, der ohne Annäherung von der klassischen Theorie elastischer Körper abgeleitet wird. Dann weist er nach, wie die Matrix-Reihe für die stufenweise Momentenverteilung asymptotisch zu einer geometrischen Reihe verläuft, deren allgemeine Lösung und verschiedene Summen näherungsweise bestimmt werden können. Die praktische Anwendung ist an einer gewöhnlichen Rahmenberechnung und an einem Rahmen-Knickproblem ersichtlich.

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