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# **Influence Surfaces for Moments in Slabs Continuous over Flexible Cross Beams**

*Aires d'influence relatives aux moments fléchissants dans les dalles continues  
avec poutres transversales flexibles*

*Einflußflächen für Biegemomente von durchlaufenden Platten  
mit biegsamen Querträgern*

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## **I. Introduction**

In two recent articles [1, 2] influence surfaces for bending moments of continuous slabs have been presented. In reference [1] HOELAND developed solutions for a slab continuous over rigid and flexible cross beams in the form of infinite series<sup>1)</sup>. However, as the influence functions exhibit singular behavior at the influence point proper (i. e. point for which the influence functions are determined), such solutions are divergent at this point and slowly convergent in its neighborhood. For a discussion of the singularity and for exact numerical computations in the immediate vicinity, solutions in finite form are required.

Such solutions have been presented in reference [2] for the case of rigid cross beams. In this paper solutions in finite form for cases of flexible cross beams are developed. The singular behavior of the influence functions for the support moments over the cross beams is discussed. Finally, numerical solutions are presented in graphical form.

Whereas in references [1] and [2] solutions have been obtained by using the differential equation of a transversely loaded plate, the approach using the integral equation is presented in this paper. Apart from a different formulation of the problem this approach has the advantage of avoiding boundary conditions along the cross beam. This, in turn, shows that the boundary con-

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<sup>1)</sup> An error in the assumption of the boundary conditions expressed by the equation for  $q_y$  on page 127 of reference [1] will be pointed out shortly.

dition for the shearing forces along the cross beam in reference [1], equation for  $q_y$ , p. 127, are in error. Instead of using the expression for the boundary shear  $V_x$  containing the contribution of the twisting moment, e. g.

$$V_x = \left( Q_x + \frac{\partial M_{xy}}{\partial y} \right) = -D \left[ \frac{\partial^3 w}{\partial x^3} + (2-\nu) \frac{\partial^3 w}{\partial x \partial y^2} \right]$$

the expression for the shearing force  $Q_x$ , e. g.

$$Q_x = -D \left( \frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2} \right)$$

should have been used. As the twisting moments  $M_{xy}$  are continuous over the support beam, they should not appear in the boundary condition.

## II. Influence Surface for Deflection of Plate Strip Continuous over Flexible Cross Beam

An infinite plate strip with simply supported parallel edges is considered (fig. 1). At  $y=0$  the plate is continuous over an elastic cross beam with a constant bending stiffness  $EI$ . The coordinates of a point on the cross beam are taken as  $(z, 0)$  —  $z$  being the  $x$ -coordinate — in order to distinguish this

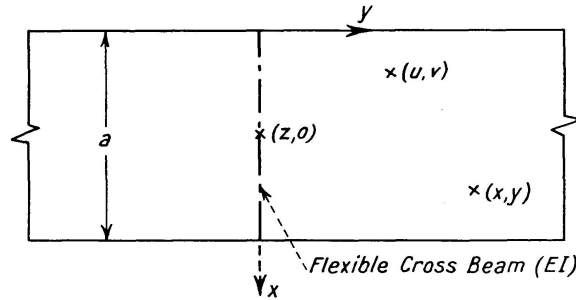


Fig. 1. Plate Strip Continuous Over Flexible Cross Beam.

$$\begin{aligned} \text{Non-dimensional Coordinates: } \frac{\pi u}{a} = \alpha; \quad \frac{\pi v}{a} = \beta; \\ \frac{\pi x}{a} = \xi; \quad \frac{\pi y}{a} = \eta; \quad \frac{\pi z}{a} = \zeta. \end{aligned}$$

point from a general point  $(x, y)$ . The deflection  $w$  of the plate at a given point  $(u, v)$ , referred to as the influence point, due to a concentrated load  $P$  at point  $(x, y)$  can be expressed by the following integral equation:

$$w(u, v; x, y) = P G(u, v; x, y) - \int_0^a EI \frac{\partial^4 w(z, 0; x, y)}{\partial z^4} G(u, v; z, 0) dz. \quad (1)$$

The function  $G(u, v; x, y)$  is the Green's function for the deflection of point  $(u, v)$  of an infinite plate strip with simply supported edges. It is given by the following series:

$$G(u, v; x, y) = \frac{a^2}{2\pi^3 D} \sum_{n=1}^{\infty} \frac{1}{n^3} \psi_n(u, v; x, y), \quad (2)$$

where

$$\psi_n(u, v; x, y) = \left[ 1 \mp \frac{n\pi}{a}(v-y) \right] e^{\pm \frac{n\pi}{a}(v-y)} \sin \frac{n\pi u}{a} \sin \frac{n\pi x}{a}$$

with the upper sign for  $v \leq y$ ,  
and the lower sign for  $v \geq y$ .

The first term under the integral sign in equation (1),  $E I \frac{\partial^4 w(z, 0; x, y)}{\partial z^4}$  expresses the distributed reaction of the cross beam acting on the plate. When multiplied by Green's function  $G(u, v; z, 0)$  and integrated over the length of the cross beam the integral constitutes the influence of this beam on the deflection at point  $(u, v)$ .

To simplify the following derivations, dimensionless coordinates are introduced [2], viz:

$$\begin{aligned} \frac{\pi u}{a} &= \alpha, & \frac{\pi v}{a} &= \beta, \\ \frac{\pi x}{a} &= \xi, & \frac{\pi y}{a} &= \eta, & \frac{\pi z}{a} &= \zeta. \end{aligned} \tag{3}$$

Assuming the deflection surface  $w$  in the form

$$w(\alpha, \beta; \xi, \eta) = \phi(\alpha, \beta; \xi, \eta) + P G(\alpha, \beta; \xi, \eta) \tag{4}$$

the function  $\phi$  is determined by substituting equation (4) into equation (1):

$$\phi(\alpha, \beta; \xi, \eta) = -\frac{\pi^3 E I}{a^3} \int_0^\pi \frac{\partial^4 w(\zeta, 0; \xi, \eta)}{\partial \zeta^4} G(\alpha, \beta; \zeta, 0) d\zeta. \tag{5}$$

Since  $\phi$  is a continuous function with respect to  $\alpha$  and  $\beta$ , it can be developed into eigen-functions associated with the Green's function  $G$  as follows:

$$\begin{aligned} \phi(\alpha, \beta; \xi, \eta) &= \sum_{n=1}^\infty a_n(\xi, \eta) \varphi_n(\alpha, \beta), \\ \varphi_n(\alpha, \beta) &= (1 + n\beta) e^{-n\beta} \sin n\alpha \end{aligned} \tag{6}$$

for positive values of  $\beta$ . Substituting into equation (5) and replacing  $G$  by equation (2) gives:

$$\sum_{n=1}^\infty a_n(\xi, \eta) \varphi_n(\alpha, \beta) = -\frac{\pi^3 E I}{a^3} \int_0^\pi \frac{\partial^4 w(\zeta, 0)}{\partial \zeta^4} \left\{ \frac{a^2}{2\pi^3 D} \sum_{m=1}^\infty \frac{1}{m^3} \psi_m(\alpha, \beta; \zeta, 0) \right\} d\zeta. \tag{7}$$

Multiplying both sides by  $\sin n\alpha$  and integrating with respect to  $\alpha$  from 0 to  $\pi$ , the orthogonality relations simplify equation (7) considerably. Taking into account that

$$\psi_m(\alpha, \beta; \zeta, 0) = \varphi_m(\alpha, \beta) \sin m\zeta$$

the following expression is obtained:

$$\frac{\pi}{2} a_n(\xi, \eta) (1 + n\beta) e^{-n\beta} = -\frac{\pi}{2} \cdot \frac{E I}{2n^3 a D} (1 + n\beta) e^{-n\beta} \int_0^\pi \frac{\partial^4 w(\zeta, 0)}{\partial \zeta^4} \sin n\zeta d\zeta. \tag{8}$$

With the substitution

$$\begin{aligned}\frac{\partial^4 w(\zeta, 0)}{\partial \zeta^4} &= \frac{\partial^4}{\partial \zeta^4} [\phi(\zeta, 0; \xi, \eta) + P G(\zeta, 0; \xi, \eta)] \\ &= \sum_{m=1}^{\infty} \left[ m^4 a_m(\xi, \eta) \sin m \zeta + \frac{P a^2}{2 \pi^3 D} m \psi_m(\zeta, 0; \xi, \eta) \right]\end{aligned}$$

the function  $a_n$  can be determined. Again the orthogonality relations are used.

Introducing the parameter

$$\rho = \frac{4 a D}{\pi E I}, \quad (9)$$

$$a_n = -\frac{P a^2}{2 n^3 \pi^3 D} \cdot \frac{n}{n + \rho} (1 + n \eta) e^{-n \eta} \sin n \xi. \quad (10)^2$$

The non-dimensional parameter  $\rho$  depends on the ratio of the bending stiffness of the plate,  $D = \frac{E h^3}{12(1-\nu^2)}$  to the bending stiffness of the cross beam  $E I$ . Substituting the pertinent values into equation (4) with  $P=1$  yields the influence function for deflection:

$$\begin{aligned}w(\alpha, \beta; \xi, \eta) &= \frac{a^2}{2 \pi^3 D} \sum_{n=1}^{\infty} \left[ \frac{1}{n^3} (1 \mp n(\beta - \eta)) e^{\pm n(\beta - \eta)} \right. \\ &\quad \left. - \frac{1}{n^2(n + \rho)} (1 + n \eta) (1 + n \beta) e^{-n(\beta + \eta)} \right] \sin n \alpha \sin n \xi \\ &\quad \text{upper sign } \eta \geq \beta, \\ &\quad \text{lower sign } \eta \leq \beta.\end{aligned} \quad (11)^2$$

The first term within the parenthesis represents the influence surface for the deflection of point  $(u, v)$  of a simply supported plate strip without cross beam. The second term expresses the influence of this beam. If the cross beam is infinitely rigid, viz.  $E I \rightarrow \infty$  and  $\rho \rightarrow 0$ , the coefficient of the second term reduces to:

$$\lim_{\rho \rightarrow 0} \frac{1}{n^2(n + \rho)} = \frac{1}{n^3}.$$

On the other hand, in the absence of a cross beam,  $E I \rightarrow 0$  and  $\rho \rightarrow \infty$  such that

$$\lim_{\rho \rightarrow \infty} \frac{1}{n^2(n + \rho)} = 0$$

and the second term will disappear.

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<sup>2)</sup> If  $\eta < 0$  the sign preceding  $\eta$  should be changed in equation (10) and in the second series of equation (11).

### III. Influence Surfaces for Bending Moments

The expressions for the influence functions of the bending moments  $M_x$  and  $M_y$  are obtained by differentiating equation (11) with respect to  $\alpha$  and  $\beta$  as follows (see for example [3], p. 260):

$M_x$ -influence surface:

$$\begin{aligned} m_x(\alpha, \beta) &= -\frac{\pi^2 D}{a^2} \left( \frac{\partial^2 w}{\partial \alpha^2} + \nu \frac{\partial^2 w}{\partial \beta^2} \right) \\ &= \frac{1}{2\pi} \sum_{n=1}^{\infty} \left[ \frac{1}{n} \{ (1+\nu) \mp (1-\nu) n(\beta-\eta) \} e^{\pm n(\beta-\eta)} \right. \\ &\quad \left. - \frac{1}{n+\rho} (1+n\eta) \{ (1+\nu) + (1-\nu) n\beta \} e^{-n(\beta+\eta)} \right] \sin n\alpha \sin n\xi. \end{aligned} \quad (12)^3$$

$M_y$ -influence surface:

$$\begin{aligned} m_y(\alpha, \beta) &= -\frac{\pi^2 D}{a^2} \left( \nu \frac{\partial^2 w}{\partial \alpha^2} + \frac{\partial^2 w}{\partial \beta^2} \right) \\ &= \frac{1}{2\pi} \sum_{n=1}^{\infty} \left[ \frac{1}{n} \{ (1+\nu) \mp (1-\nu) n(\beta-\eta) \} e^{\pm n(\beta-\eta)} \right. \\ &\quad \left. - \frac{1}{n+\rho} (1+n\eta) \{ (1+\nu) + (1-\nu) n\beta \} e^{-n(\beta+\eta)} \right] \sin n\alpha \sin n\xi. \end{aligned} \quad (13)^3$$

The influence function for the beam moment is proportional to the curvature of the cross beam, hence:

Influence surface for moment of cross beam:

$$\begin{aligned} m_b(\alpha, 0) &= -\frac{\pi^2 EI}{a^2} \frac{\partial^2 w(\alpha, 0)}{\partial \alpha^2} \\ &= \frac{2a}{\pi^2 \rho} \sum_{n=1}^{\infty} \left\{ \frac{1}{n} - \frac{1}{n+\rho} (1-\rho\eta) \right\} e^{-n\eta} \sin n\alpha \sin n\xi. \end{aligned} \quad (14)^3$$

For the particular case of the influence functions for the support moments over the cross beam, i. e.,  $\beta=0$ , the expressions for  $m_x$  and  $m_y$  reduce to:

$$m_x(\alpha, 0) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \left[ (1+\nu) \left\{ \frac{1}{n} - \frac{1}{n+\rho} (1-\rho\eta) \right\} - 2\nu\eta \right] e^{-n\eta} \sin n\alpha \sin n\xi, \quad (15)$$

$$m_y(\alpha, 0) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \left[ (1+\nu) \left\{ \frac{1}{n} - \frac{1}{n+\rho} (1-\rho\eta) \right\} - 2\eta \right] e^{-n\eta} \sin n\alpha \sin n\xi. \quad (16)$$

<sup>3</sup> If  $\eta < 0$  the sign preceding  $\eta$  should be changed in the second series of equation (12) and (13) and in equation (14).

The series with the terms  $2\nu\eta$  and  $2\eta$  in the above equations for  $m_x$  and  $m_y$  respectively, are the expressions for the influence functions in case of a rigid cross beam.

If  $\rho = 0$  the above equations reduce to:

$$m_x(\alpha, 0) = -\frac{\nu\eta}{\pi} \sum_{n=1}^{\infty} e^{-n\eta} \sin n\alpha \sin n\xi, \quad (17)$$

$$m_y(\alpha, 0) = -\frac{\eta}{\pi} \sum_{n=1}^{\infty} e^{-n\eta} \sin n\alpha \sin n\xi, \quad (18)$$

$$m_b(\alpha, 0) = 0. \quad (19)$$

Equation (18) checks with a previously obtained result, equation (12) of reference [2]. The value of  $m_x$  is  $\nu$  times the value of  $m_y$  as the curvature along the cross beam disappears. Finally, the expression for  $m_b$  reduces to zero as the cross beam will not deflect.

The other extreme case is obtained if the bending rigidity of the cross beam disappears,  $\rho = \infty$ , or

$$m_x(\alpha, 0) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} [(1+\nu) + (1-\nu)n\eta] e^{-n\eta} \sin n\alpha \sin n\xi, \quad (20)$$

$$m_y(\alpha, 0) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} [(1+\nu) - (1-\nu)n\eta] e^{-n\eta} \sin n\alpha \sin n\xi. \quad (21)$$

corresponding to already known results (see for example [2], equation (6)).

In general, it appears to be impossible to sum the series of equation (12) to (16) into finite expressions. However, at least for the two specific values of  $\rho = 1/2$  and  $\rho = 1$  such a summation is possible. Fortunately they correspond to practical values as will be shown presently. The details of the summation and also the finite expressions for the general case, i. e.,  $m_x(\alpha, \beta)$  and  $m_y(\alpha, \beta)$ , are given in the Appendix.

Considering the support moments for point  $(\alpha, 0)$  and the specific value  $\rho = 1$ , equation (14) to (16) takes the following form:

$$m_x(\alpha, 0) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \left[ (1+\nu) \left\{ \frac{1}{n} - \frac{1}{n+1} (1-\eta) \right\} - 2\nu\eta \right] e^{-n\eta} \sin n\alpha \sin n\xi, \quad (22)$$

$$m_y(\alpha, 0) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \left[ (1+\nu) \left\{ \frac{1}{n} - \frac{1}{n+1} (1-\eta) \right\} - 2\eta \right] e^{-n\eta} \sin n\alpha \sin n\xi, \quad (23)$$

$$m_b(\alpha, 0) = \frac{2a}{\pi^2\rho} \sum_{n=1}^{\infty} \left\{ \frac{1}{n} - \frac{1}{n+1} (1-n\eta) \right\} e^{-n\eta} \sin n\alpha \sin n\xi. \quad (24)$$

The finite expressions for point  $\left(\frac{\pi}{2}, 0\right)$  — i. e., influence point over the cross beam at half span — and the ratio  $\rho = 1$  are:

Case:  $\alpha = \frac{\pi}{2}$ ;  $\beta = 0$ ;  $\rho = 1$

$$\begin{aligned}
 m_x\left(\frac{\pi}{2}, 0\right) &= \frac{1}{8\pi} \left[ (1+\nu) \left\{ [1 - (\eta-1)e^\eta \sin \xi] \log(\cosh \eta + \sin \xi) \right. \right. \\
 &\quad - [1 + (\eta-1)e^\eta \sin \xi] \log(\cosh \eta - \sin \xi) - 2(\eta-1)e^\eta (\log 2 - \eta) \sin \xi \\
 &\quad \left. \left. + 2(\eta-1)e^\eta \cos \xi \left[ \tan^{-1}\left(\frac{\cos \xi}{e^\eta - \sin \xi}\right) - \tan^{-1}\left(\frac{\cos \xi}{e^\eta + \sin \xi}\right) \right] \right\} \right. \\
 &\quad \left. - 2\nu \eta \sinh \eta \left( \frac{1}{\cosh \eta - \sin \xi} - \frac{1}{\cosh \eta + \sin \xi} \right) \right], \quad (25)
 \end{aligned}$$

$$\begin{aligned}
 m_y\left(\frac{\pi}{2}, 0\right) &= \frac{1}{8\pi} \left[ (1+\nu) \left\{ [1 - (\eta-1)e^\eta \sin \xi] \log(\cosh \eta + \sin \xi) \right. \right. \\
 &\quad - [1 + (\eta-1)e^\eta \sin \xi] \log(\cosh \eta - \sin \xi) - 2(\eta-1)e^\eta (\log 2 - \eta) \sin \xi \\
 &\quad \left. \left. + 2(\eta-1)e^\eta \cos \xi \left[ \tan^{-1}\left(\frac{\cos \xi}{e^\eta - \sin \xi}\right) - \tan^{-1}\left(\frac{\cos \xi}{e^\eta + \sin \xi}\right) \right] \right\} \right. \\
 &\quad \left. - 2\eta \sinh \eta \left( \frac{1}{\cosh \eta - \sin \xi} - \frac{1}{\cosh \eta + \sin \xi} \right) \right], \quad (26)
 \end{aligned}$$

$$\begin{aligned}
 m_b\left(\frac{\pi}{2}, 0\right) &= \frac{a}{2\pi^2} \left\{ [1 - (\eta-1)e^\eta \sin \xi] \log(\cosh \eta + \sin \xi) \right. \\
 &\quad - [1 + (\eta-1)e^\eta \sin \xi] \log(\cosh \eta - \sin \xi) - 2(\eta-1)e^\eta (\log 2 - \eta) \sin \xi \\
 &\quad \left. \left. + 2(\eta-1)e^\eta \cos \xi \left[ \tan^{-1}\left(\frac{\cos \xi}{e^\eta - \sin \xi}\right) - \tan^{-1}\left(\frac{\cos \xi}{e^\eta + \sin \xi}\right) \right] \right\}. \quad (27)
 \end{aligned}$$

Similarly, expressions for the case of  $\rho = 1/2$  can be derived, leading finally to the following equations:

Case:  $\alpha = \frac{\pi}{2}$ ;  $\beta = 0$ ;  $\rho = 1/2$

$$\begin{aligned}
 m_x\left(\frac{\pi}{2}, 0\right) &= \frac{1}{8\pi} \left[ (1+\nu) \left\{ \log \frac{\cosh \eta + \sin \xi}{\cosh \eta - \sin \xi} + \left(\frac{\eta}{2} - 1\right) e^{\eta/2} \left[ \cos\left(\frac{\xi}{2} - \frac{\pi}{4}\right) \right. \right. \\
 &\quad \cdot \log \frac{\cosh \frac{\eta}{2} + \cos\left(\frac{\xi}{2} - \frac{\pi}{4}\right)}{\cosh \frac{\eta}{2} - \cos\left(\frac{\xi}{2} - \frac{\pi}{4}\right)} - \cos\left(\frac{\xi}{2} + \frac{\pi}{4}\right) \log \frac{\cosh \frac{\eta}{2} + \cos\left(\frac{\xi}{2} + \frac{\pi}{4}\right)}{\cosh \frac{\eta}{2} - \cos\left(\frac{\xi}{2} + \frac{\pi}{4}\right)} \\
 &\quad - 2 \sin\left(\frac{\xi}{2} + \frac{\pi}{4}\right) \tan^{-1}\left(\frac{\sin\left(\frac{\xi}{2} + \frac{\pi}{4}\right)}{\sinh \frac{\eta}{2}}\right) - 2 \sin\left(\frac{\xi}{2} - \frac{\pi}{4}\right) \\
 &\quad \left. \left. \cdot \tan^{-1}\left(\frac{\sin\left(\frac{\xi}{2} - \frac{\pi}{4}\right)}{\sinh \frac{\eta}{2}}\right) \right\} - 2\nu \eta \sinh \eta \left( \frac{1}{\cosh \eta - \sin \xi} - \frac{1}{\cosh \eta + \sin \xi} \right) \right], \quad (28)
 \end{aligned}$$



$$\begin{aligned}
m_y\left(\frac{\pi}{2}, 0\right) = & \frac{1}{8\pi} \left[ (1+\nu) \left\{ \log \frac{\cosh \eta + \sin \xi}{\cosh \eta - \sin \xi} + \left(\frac{\eta}{2} - 1\right) e^{\eta/2} \left[ \cos\left(\frac{\xi}{2} - \frac{\pi}{4}\right) \right. \right. \right. \\
& \cdot \log \frac{\cosh \frac{\eta}{2} + \cos\left(\frac{\xi}{2} - \frac{\pi}{4}\right)}{\cosh \frac{\eta}{2} - \cos\left(\frac{\xi}{2} - \frac{\pi}{4}\right)} - \cos\left(\frac{\xi}{2} + \frac{\pi}{4}\right) \log \frac{\cosh \frac{\eta}{2} + \cos\left(\frac{\xi}{2} + \frac{\pi}{4}\right)}{\cosh \frac{\eta}{2} - \cos\left(\frac{\xi}{2} + \frac{\pi}{4}\right)} \\
& - 2 \sin\left(\frac{\xi}{2} + \frac{\pi}{4}\right) \tan^{-1}\left(\frac{\sin\left(\frac{\xi}{2} + \frac{\pi}{4}\right)}{\sinh \frac{\eta}{2}}\right) - 2 \sin\left(\frac{\xi}{2} - \frac{\pi}{4}\right) \\
& \left. \left. \left. \cdot \tan^{-1}\left(\frac{\sin\left(\frac{\xi}{2} - \frac{\pi}{4}\right)}{\sinh \frac{\eta}{2}}\right) \right] \right\} - 2\eta \sinh \eta \left( \frac{1}{\cosh \eta - \sin \xi} - \frac{1}{\cosh \eta + \sin \xi} \right) \right] .
\end{aligned} \tag{29}$$

$$\begin{aligned}
m_b\left(\frac{\pi}{2}, 0\right) = & \frac{a}{\pi^2} \left\{ \log \frac{\cosh \eta + \sin \xi}{\cosh \eta - \sin \xi} + \left(\frac{\eta}{2} - 1\right) e^{\eta/2} \left[ \cos\left(\frac{\xi}{2} - \frac{\pi}{4}\right) \right. \right. \\
& \cdot \log \frac{\cosh \frac{\eta}{2} + \cos\left(\frac{\xi}{2} - \frac{\pi}{4}\right)}{\cosh \frac{\eta}{2} - \cos\left(\frac{\xi}{2} - \frac{\pi}{4}\right)} - \cos\left(\frac{\xi}{2} + \frac{\pi}{4}\right) \log \frac{\cosh \frac{\eta}{2} + \cos\left(\frac{\xi}{2} + \frac{\pi}{4}\right)}{\cosh \frac{\eta}{2} - \cos\left(\frac{\xi}{2} + \frac{\pi}{4}\right)} \\
& - 2 \sin\left(\frac{\xi}{2} - \frac{\pi}{4}\right) \tan^{-1}\left(\frac{\sin\left(\frac{\xi}{2} - \frac{\pi}{4}\right)}{\sinh \frac{\eta}{2}}\right) - 2 \sin\left(\frac{\xi}{2} + \frac{\pi}{4}\right) \\
& \left. \left. \left. \cdot \tan^{-1}\left(\frac{\sin\left(\frac{\xi}{2} + \frac{\pi}{4}\right)}{\sinh \frac{\eta}{2}}\right) \right] \right\} .
\end{aligned} \tag{30}$$

#### IV. Discussion of the Singularity of the Support Moments

Whereas it is impossible to express the influence functions in finite form except for specific values of  $\rho$ , a general discussion of the functional behavior in the immediate vicinity of the influence point  $\left(\frac{\pi}{2}, 0\right)$  can be given. In equation (15) and (16) the last term of the series for  $m_x$  and  $m_y$  can be summed [2]:

$$\sum_{n=1}^{\infty} e^{-n\eta} \sin n\alpha \sin n\xi = \frac{1}{4} \sinh \eta \left( \frac{1}{\cosh \eta - \cos(\alpha - \xi)} - \frac{1}{\cosh \eta - \cos(\alpha + \xi)} \right) .$$

Considering points in the immediate neighborhood of  $\left(\frac{\pi}{2}, 0\right)$  only, or

$$\begin{aligned}
\xi &= \frac{\pi}{2} + \epsilon, & \eta &= \delta, \\
\epsilon &\doteq 0, & \delta &\doteq 0
\end{aligned}$$

and neglecting higher order terms, the above equations reduce to:

$$m_x \sim -\frac{1}{2\pi} \left[ \frac{\nu \delta^2}{\epsilon^2 + \delta^2} - (1+\nu) J(\rho) \right], \tag{31}$$

$$m_y \sim -\frac{1}{2\pi} \left[ \frac{\delta^2}{\epsilon^2 + \delta^2} - (1 + \nu) J(\rho) \right], \tag{32}$$

$$m_b \sim \frac{2a}{\pi^2 \rho} J(\rho), \tag{33}$$

$$J(\rho) = \sum_{n=1,3,5\dots}^{\infty} \left( \frac{1}{n} - \frac{1}{n+\rho} \right).$$

Introducing polar coordinates  $\theta$  and  $r = \sqrt{\epsilon^2 + \delta^2}$  as shown in fig. 2 it follows that

$$\frac{\delta^2}{\epsilon^2 + \delta^2} = \sin^2 \theta,$$

$$\lim_{\substack{\epsilon \rightarrow 0 \\ \delta \rightarrow 0}} m_x \left( \frac{\pi}{2}, 0; \frac{\pi}{2} + \epsilon, \delta \right) = -\frac{1}{2\pi} [\nu \sin^2 \theta - (1 + \nu) J(\rho)], \tag{34}$$

$$\lim_{\substack{\epsilon \rightarrow 0 \\ \delta \rightarrow 0}} m_y \left( \frac{\pi}{2}, 0; \frac{\pi}{2} + \epsilon, \delta \right) = -\frac{1}{2\pi} [\sin^2 \theta - (1 + \nu) J(\rho)], \tag{35}$$

$$\lim_{\substack{\epsilon \rightarrow 0 \\ \delta \rightarrow 0}} m_b \left( \frac{\pi}{2}, 0; \frac{\pi}{2} + \epsilon, \delta \right) = \frac{2a}{\pi^2 \rho} J(\rho). \tag{36}$$

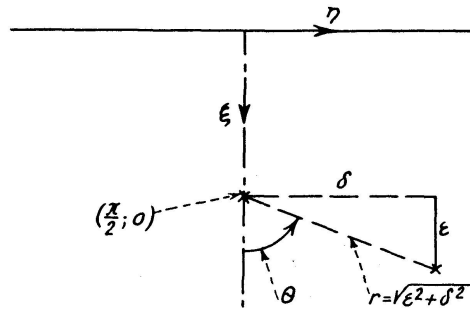


Fig. 2. Polar Coordinates Around Influence Point  $\left(\frac{\pi}{2}, 0\right)$ .

The function  $J(\rho)$  can be computed as follows:

$$\begin{aligned} J(\rho) &= \sum_{n=1,3,5\dots}^{\infty} \left( \frac{1}{n} - \frac{1}{n+\rho} \right) = \frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+\rho} \right) [1 - (-1)^n] \\ &= \frac{1}{2} \left[ \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+\rho} \right) - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n+\rho} \right]. \end{aligned} \tag{37}$$

However, the theory of Gamma functions furnishes the following relationships (for example, [4], p. 458):

$$\begin{aligned} \frac{d}{d\rho} \log \Gamma(\rho) + \gamma &= \sum_{k=0}^{\infty} \left( \frac{1}{1+k} - \frac{1}{\rho+k} \right) = \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+\rho} \right) - \frac{1}{\rho}, \\ \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+\rho} \right) &= \frac{\Gamma'(\rho)}{\Gamma(\rho)} + \gamma + \frac{1}{\rho}. \end{aligned} \tag{38}$$

Making use of the relations (for example, [5], p. 28 (1)):

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -\log 2,$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n+\rho} = \int_0^1 \frac{x^{\rho-1}}{1+x} dx - \frac{1}{\rho} = \frac{1}{2} \left[ \frac{\Gamma'(\frac{\rho+1}{2})}{\Gamma(\frac{\rho+1}{2})} - \frac{\Gamma'(\frac{\rho}{2})}{\Gamma(\frac{\rho}{2})} \right] - \frac{1}{\rho}. \quad (39)$$

$J(\rho)$  becomes:

$$J(\rho) = \frac{1}{2} \left[ \frac{\Gamma'(\rho)}{\Gamma(\rho)} + \gamma + \log 2 + \frac{1}{2} \left\{ \frac{\Gamma'(\frac{\rho+1}{2})}{\Gamma(\frac{\rho+1}{2})} - \frac{\Gamma'(\frac{\rho}{2})}{\Gamma(\frac{\rho}{2})} \right\} \right],$$

$$= \frac{1}{2} \left[ \Psi(\rho) + \gamma + \log 2 + \frac{1}{2} \left\{ \Psi\left(\frac{\rho+1}{2}\right) - \Psi\left(\frac{\rho}{2}\right) \right\} \right], \quad (39)$$

where 
$$\Psi(\rho) = \frac{\Gamma'(\rho)}{\Gamma(\rho)} = \int_0^{\infty} \left( e^{-\alpha} - \frac{1}{(1+\alpha)^\rho} \right) \frac{d\alpha}{\alpha}.$$

Since  $\Psi(\rho)$  satisfies the following two relations:

$$\Psi\left(\frac{l}{n}\right) + \gamma = -\frac{\pi}{2} \cot \frac{l\pi}{n} + 2 \sum_{\nu=1}^{\lfloor \frac{n+1}{2} - 1 \rfloor} \left\{ \cos\left(\frac{2l\nu\pi}{n}\right) \log \sin\left(\frac{\nu\pi}{n}\right) \right\} - \log(2n),$$

$$(n = 2, 3, 4, \dots, l = 1, 2, \dots, (n-1)),$$

$$\Psi(\rho) + \gamma = \sum_{n=1}^{\rho-1} \frac{1}{n}, \quad (\rho = 1, 2, 3, \dots).$$

$J(\rho)$  can be readily computed. Values are shown graphically in fig. 3. It is a positive function which increases monotonically from zero to infinity with increasing  $\rho$ .

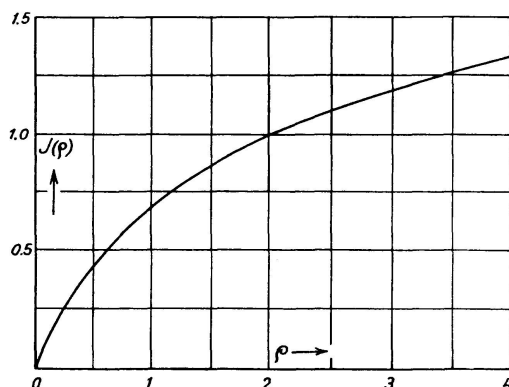


Fig. 3. Function  $J(\rho)$ .

<sup>4</sup>)  $\gamma = 0.5772156649 \dots$  (Euler's Constant).

The limit of  $m_x, m_y, m_b$ , can now be discussed. For simplicity the assumption  $\nu = 0$  is made, such that:

$$\lim_{\substack{\epsilon \rightarrow 0 \\ \delta \rightarrow 0}} m_x \left( \frac{\pi}{2}, 0; \frac{\pi}{2} + \epsilon, \delta \right) = \frac{1}{2\pi} J(\rho), \quad (40)$$

$$\lim_{\substack{\epsilon \rightarrow 0 \\ \delta \rightarrow 0}} m_y \left( \frac{\pi}{2}, 0; \frac{\pi}{2} + \epsilon, \delta \right) = -\frac{1}{2\pi} [\sin^2 \theta - J(\rho)], \quad (41)$$

$$\lim_{\substack{\epsilon \rightarrow 0 \\ \delta \rightarrow 0}} m_b \left( \frac{\pi}{2}, 0; \frac{\pi}{2} + \epsilon, \delta \right) = \frac{2a}{\pi^2 \rho} J(\rho). \quad (42)$$

As only the limit of  $m_y$  depends upon the angle of incidence  $\theta$ , the discussion will be restricted to this case. Inspection of fig. 3 and equation (41) shows that for  $J(\rho) > 1$ , no angle of incidence exists for which  $m_y$  will be zero. However for  $J(\rho) \leq 1$ ,  $m_y$  becomes zero for specific values  $\theta = \theta_0$ . Five cases are considered:

(I)  $\rho = 0$ :

$$J(\rho) = 0 \quad m_y \left( \frac{\pi}{2}, 0; \frac{\pi}{2}, 0 \right) = -\frac{1}{2\pi} \sin^2 \theta$$

$$\theta_0 = 0 \text{ Rad.}$$

(II)  $\rho = 1/2$ :

$$J\left(\frac{1}{2}\right) = \frac{1}{2} \left( \frac{\pi}{2} - \log 2 \right) = 0.4388$$

$$m_y \left( \frac{\pi}{2}, 0; \frac{\pi}{2}, 0 \right) = -\frac{1}{2\pi} (\sin^2 \theta - 0.4388)$$

and solving the equation:  $\sin^2 \theta - 0.4388 = 0$

$$\theta_0 = 0.7243 \text{ Rad.}$$

(III)  $\rho = 1$ :

$$J(1) = 0.69315 \quad m_y \left( \frac{\pi}{2}, 0; \frac{\pi}{2}, 0 \right) = -\frac{1}{2\pi} (\sin^2 \theta - 0.69315)$$

$$\theta_0 = 0.9818 \text{ Rad.}$$

(IV)  $\rho = 2$ :

$$J(2) = 1 \quad m_y \left( \frac{\pi}{2}, 0; \frac{\pi}{2}, 0 \right) = -\frac{1}{2\pi} (\sin^2 \theta - 1)$$

$$\theta_0 = \frac{\pi}{2} \text{ Rad.}$$

(V)  $\rho > 2$ :  $J(\rho) > 1$ , no  $\theta_0$  exists

and since  $\lim_{\rho \rightarrow \infty} J(\rho) = \infty$

$$\lim_{\rho \rightarrow \infty} m_y \left( \frac{\pi}{2}, 0; \frac{\pi}{2}, 0 \right) = +\infty,$$

This limiting case corresponds to an infinite strip with no cross beam.

The results are plotted graphically in fig. 4 (a) to (e). It can be seen that the angle of incidence  $\theta_0$  for the zero line increases successively from 0 to  $\frac{\pi}{2}$  corresponding to a change in  $\rho$  from zero to two. For values of  $\rho > 2$  no zero line passes through the influence point.

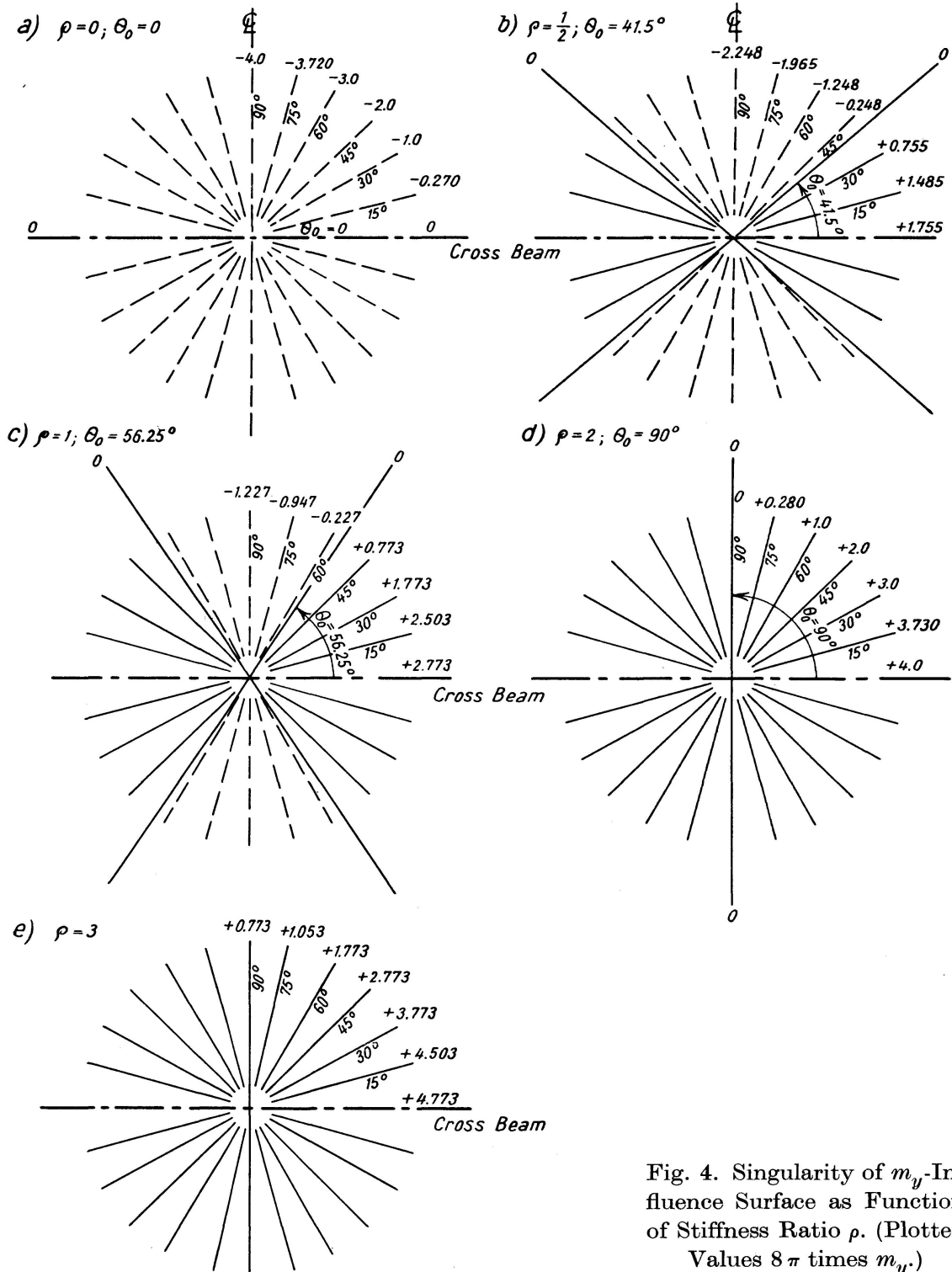


Fig. 4. Singularity of  $m_y$ -Influence Surface as Function of Stiffness Ratio  $\rho$ . (Plotted Values  $8\pi$  times  $m_y$ .)

Knowing the behavior of  $m_y$  at the influence point  $(\frac{\pi}{2}, 0)$  and the tangents to the contour lines meeting at this point, the general shapes of the  $m_y$  influence surfaces can be easily visualized. Fig. 5 (a) to (e) illustrate the general appearance schematically. Cases (a) and (e) are the known cases for an infinitely rigid cross beam,  $\rho = 0$ , and no cross beam,  $\rho = \infty$ , respectively. Between the two cases a steady transition takes place.

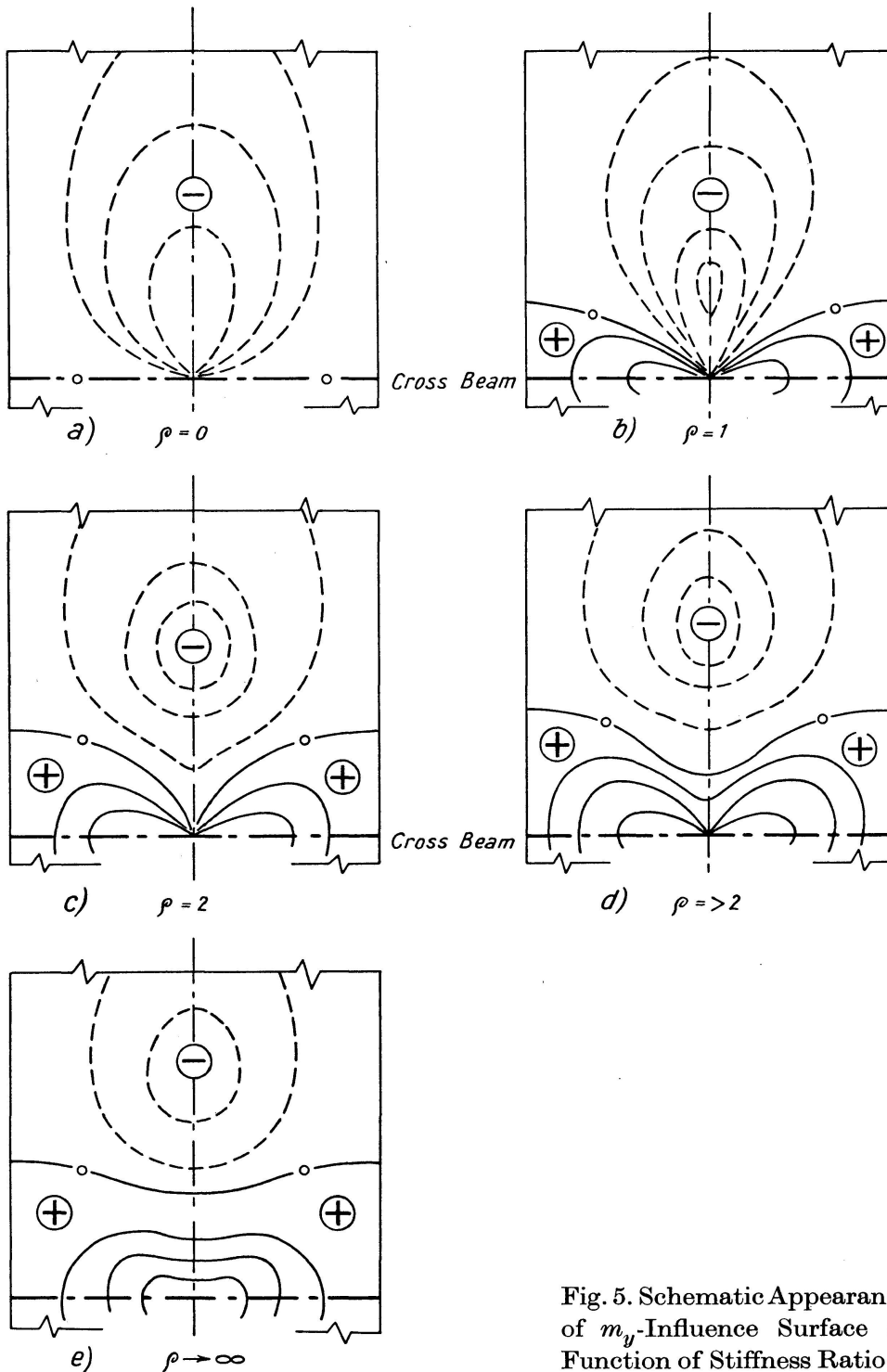


Fig. 5. Schematic Appearance of  $m_y$ -Influence Surface as Function of Stiffness Ratio  $\rho$ .

### V. Extension to the Case of Multiple Cross Beams

The method of the integral equation used in Chapter 2 for the case of a single cross beam can be extended to the case of a plate strip supported by a series of cross beams as indicated in fig. 6. If in addition to the bending resistance, consideration is given to the warping and the torsional resistance of the cross beams, the influence function for the deflection  $w$  of point  $(\alpha, \beta)$  takes the following form:

$$\begin{aligned}
 w(\alpha, \beta; \xi, \eta) = & G_1(\alpha, \beta; \xi, \eta) - \sum_{i=1}^k \frac{\pi^3}{a^3} \int_0^\pi E I_i \frac{\partial^4 w(\zeta_i, \gamma_i; \xi, \eta)}{\partial \zeta_i^4} G_1(\alpha, \beta; \zeta_i, \gamma_i) d\zeta_i \\
 & + \sum_{i=1}^k \frac{\pi^4}{a^4} \int_0^\pi \left\{ E I_{\omega i} \frac{\partial^5 w(\zeta_i, \gamma_i; \xi, \eta)}{\partial \zeta_i^4 \partial \gamma_i} - \frac{a^2}{\pi^2} G K_i \frac{\partial^3 w(\zeta_i, \gamma_i; \xi, \eta)}{\partial \zeta_i^2 \partial \gamma_i} \right\} \\
 & \cdot G_2(\alpha, \beta; \zeta_i, \gamma_i) d\zeta_i,
 \end{aligned} \quad (43)$$

where

$\zeta_i, \gamma_i$  = dimensionless coordinates of  $i$ -th cross beam.

$E I_i$  = bending stiffness of  $i$ -th cross beam.

$E I_{\omega i}$  = warping rigidity of  $i$ -th cross beam.

$G K_i$  = torsional rigidity (St. Venant's Torsion) of  $i$ -th cross beam.

$G_1(\alpha, \beta; \xi, \eta)$  = Green's function for deflection of point  $(\alpha, \beta)$  due to concentrated transverse load at point  $(\xi, \eta)$ .

$G_2(\alpha, \beta; \xi, \eta)$  = Green's function for deflection of point  $(\alpha, \beta)$  due to a concentrated moment at point  $(\xi, \eta)$  acting about an axis parallel to the  $\xi$ -axis.

The function  $G_1$  is identical with equation (2), or in non-dimensional form:

$$G_1(\alpha, \beta; \xi, \eta) = \frac{a^2}{2\pi^3 D} \sum_{n=1}^{\infty} \frac{1}{n^3} \psi_n(\alpha, \beta; \xi, \eta) \quad (44)$$

with  $\psi_n(\alpha, \beta; \xi, \eta) = [1 \mp n(\beta - \eta)] e^{\pm n(\beta - \eta)} \sin n\alpha \sin n\xi$ .

$G_2$  is the limiting case of the difference between  $P G_1(\alpha, \beta; \xi, \eta + \delta)$  and  $P G_1(\alpha, \beta; \xi, \eta)$  as  $P\delta$  approaches unity and  $\delta$  itself approaches zero, or:

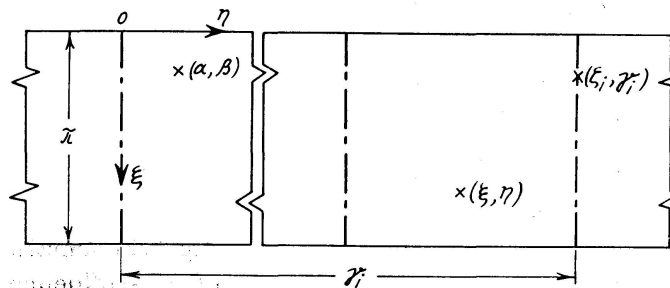


Fig. 6. Plate Strip With Multiple Cross Beams.

$$\begin{aligned}
 G_2(\alpha, \beta; \xi, \eta) &= \lim_{\substack{P \delta \rightarrow 1 \\ \delta \rightarrow 0}} [P G_1(\alpha, \beta; \xi, \eta + \delta) - P G_1(\alpha, \beta; \xi, \eta)] \\
 &= -\frac{(\beta - \eta)}{2\pi D} \sum_{n=1}^{\infty} \frac{1}{n} e^{\pm n(\beta - \eta)} \sin n\alpha \sin n\beta.
 \end{aligned} \tag{45}$$

The solution follows the pattern outlined in Chapter 2, the only difficulty being the extent of arithmetical operations.

## VI. Two-Span Continuous Slab With Flexible Cross Beams

The slab as shown in fig. 7 is simply supported along its edges  $\xi = 0$  and  $\xi = \pi$ . The two end beams at  $\eta = \pm \gamma$  have the same constant bending rigidity  $E I$  as the cross beam at  $\eta = 0$ . As an example the influence surface for the

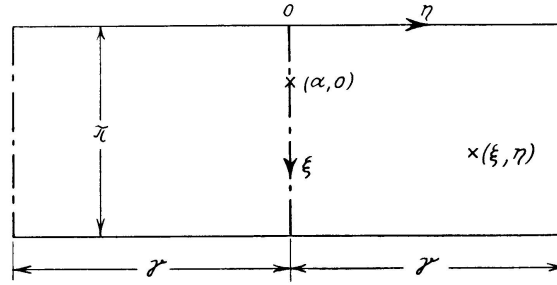


Fig. 7. Two-Span Continuous Slab With Flexible Cross Beams.

support moment  $M_y$  at point  $(\alpha, 0)$  will be computed. Referring to [2] the  $m_y$ -function is taken in two parts:

$$m_y = m_{y0} + m_{y1}, \tag{46}$$

where  $m_{y0}$  is the solution of the infinite plate strip with an elastic cross beam at  $\eta = 0$  as given by equation (16) or (27) and  $m_{y1}$  is a solution of the homogeneous plate equation:

$$m_{y1} = \sum_{n=1}^{\infty} (a_n \sinh n\eta + b_n \cosh n\eta + c_n n\eta \sinh n\eta + d_n n\eta \cosh n\eta) \sin n\alpha \sin n\xi. \tag{47}$$

The sum  $(m_{y0} + m_{y1})$  must fulfill all boundary conditions. Considering symmetry with respect to  $\eta = 0$  these conditions are: At  $\eta = 0$ :

$$\text{a) } \frac{\partial m_y}{\partial \eta} = 0. \tag{48}$$

However as  $\frac{\partial m_{y0}}{\partial \eta} = 0$  the condition becomes:

$$\frac{\partial m_{y1}}{\partial \eta} = 0, \tag{49}$$

$$\text{b) } -\frac{\pi^3 D}{a^3} \left( \frac{\partial^3 m_y}{\partial \eta^3} + \frac{\partial^3 m_y}{\partial \xi^2 \partial \eta} \right) = \frac{\pi^4 E I}{2 a^4} \frac{\partial^4 m_y}{\partial \xi^4}. \tag{50}$$



Equation (50) expresses the condition that the "shearing force"<sup>5)</sup> at  $+0$  is equal to half the reaction of the cross beam. However as  $m_{y0}$  fulfills this condition separately the equation reduces to:

$$-\frac{\pi^3 D}{a^3} \left( \frac{\partial^3 m_{y1}}{\partial \eta^3} + \frac{\partial^3 m_{y1}}{\partial \xi^2 \partial \eta} \right) = \frac{\pi^4 EI}{2a^4} \frac{\partial^4 m_{y1}}{\partial \xi^4}. \quad (51)$$

At  $\eta = \frac{\pi c}{a} = \gamma$ :

$$c) \quad \frac{\partial^2 m_y}{\partial \eta^2} = 0. \quad (52)$$

$$d) \quad -\frac{\pi^3 D}{a^3} \left( \frac{\partial^3 m_y}{\partial \eta^3} + (2-\nu) \frac{\partial^3 m_y}{\partial \xi^2 \partial \eta} \right) = -\frac{\pi^4 EI}{a^4} \frac{\partial^4 m_y}{\partial \xi^4}. \quad (53)$$

The first condition assumes the edge free of bending moments  $M_y$  whereas the second condition postulates the identity between the plate boundary shear and the loading of the cross beam. Introducing the expressions (16) and (47) for  $m_{y0}$  and  $m_{y1}$  into the boundary conditions (49), (51), (52) and (53) furnishes a system of equations for the determination of the constants  $a_n$  to  $d_n$ . Explicit expressions for these constants in general form are too complex. However, their actual computation for a given example does not offer any particular difficulty.

## VII. Numerical Examples

The foregoing derivations were used for computation of two examples:

### a) Plate Strip Continuous over Flexible Cross Beam

Choosing the influence point  $\left(\frac{\pi}{2}, 0\right)$  and a stiffness ratio  $\rho = \frac{4aD}{\pi EI} = 1$ , equation (14), (15) and (16) were used to compute the influence surfaces for the beam moment  $m_b$  and the bending moments  $m_x$  and  $m_y$ . Furthermore, the assumption of Poisson's ratio  $\nu = 0$  makes the expressions for  $m_b$  and  $m_x$  identical except for a constant multiplier. The results are plotted in two graphs (9 and 10). It should be noted that the plotted values correspond to  $8\pi$  times the influence values for  $m_x$  and  $m_y$  and  $2\pi^2/a$  times for  $m_b$ . For cases other than  $\nu = 0$ , the graphs 9 and 10 are still applicable. Due to the fact that  $\nu$  does not enter the boundary conditions, the influence surface can be taken in the following form:

$$m_y = (m_y)_{\nu=0} + \nu (m_x)_{\nu=0}$$

and similarly for  $m_x$ . However for other cases where  $\nu$  influences the boundary conditions (e. g., free edges, elastically supported edges, etc.) such a procedure is not rigorously applicable and its accuracy must be investigated from case to case.

<sup>5)</sup> As  $m_y$  must fulfill the same boundary conditions as the deflection  $w$  such terminology seems appropriate.

A remark is indicated concerning the choice of  $\rho = 1$ . A cross section through the cross beam shown in fig. 8 is considered. The dimensions are given in terms of the span length  $a$ . The interaction between the beam and the slab is usually taken into account by determining an effective width of the slab acting as a flange of the beam (see e. g., [3], p. 119). However, for the present qualitative consideration, it is sufficiently accurate to assume that the action of the slab forces the neutral axis of the beam to coincide with the middle plane of the slab  $n-n$ . Computing the moment of inertia  $I$  of the rectangular beam  $a/10 \times a/25$  with respect to  $n-n$  the value of  $\rho$  is determined,  $\rho = 0.98$ .

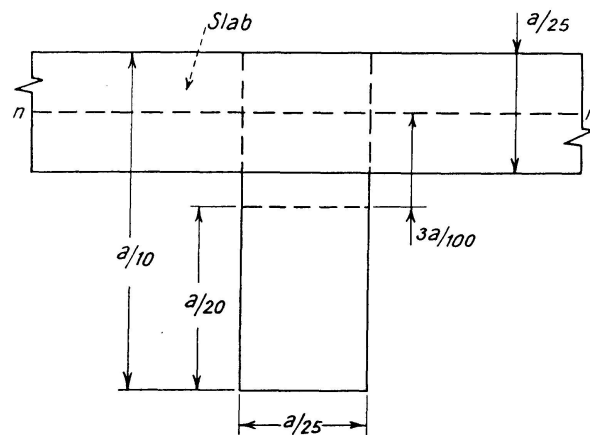


Fig. 8. Section Through Cross Beam.

Span Length =  $a$ . Stiffness Ratio  $\rho = \frac{4 a D}{\pi E I} = 0.98$ .

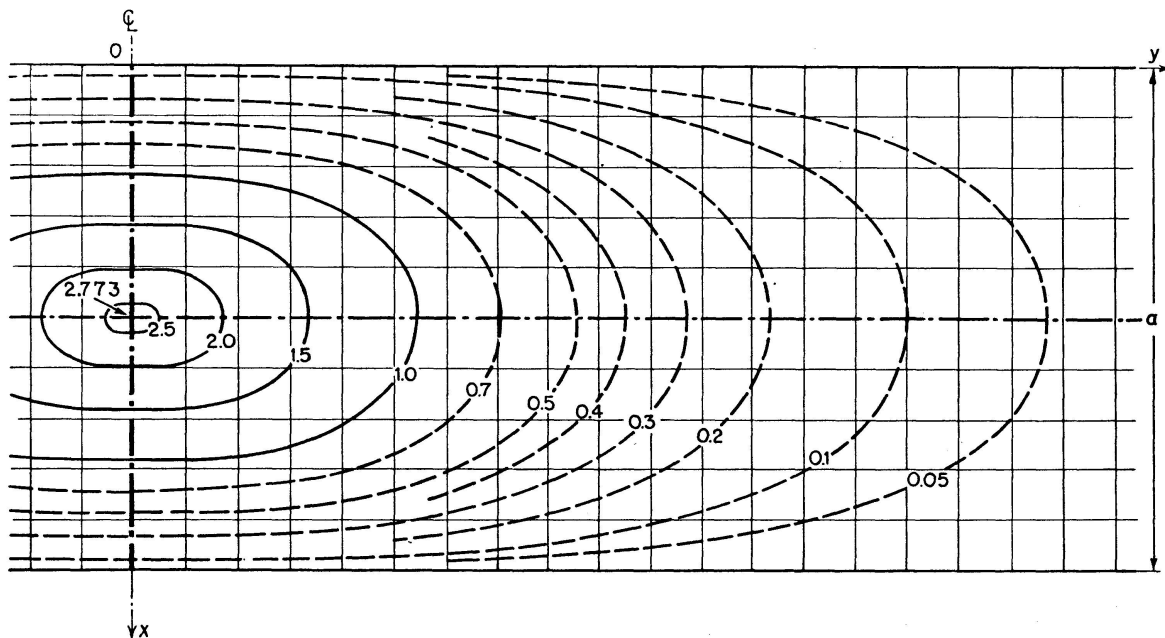


Fig. 9. Plate Strip with Flexible Cross Beam ( $\rho = \frac{4 a D}{\pi E I} = 1; \nu = 0$ ).

$m_x$  = Influence Surface for Plate Moment  $m_x$  over Cross Beam ( $8 \pi$  times).

$m_y$  = Influence Surface for Bending Moment of Cross Beam ( $2 \pi^2/a$  times).

A more rigorous investigation would lead to a neutral axis slightly below  $n-n$  and hence to a smaller  $I$  and a somewhat higher value of  $\rho$ . Nevertheless the example shows that the case  $\rho=1$  will correspond in practice to a rather flexible cross beam, the usual cases being limited between  $0 < \rho < 1$ .

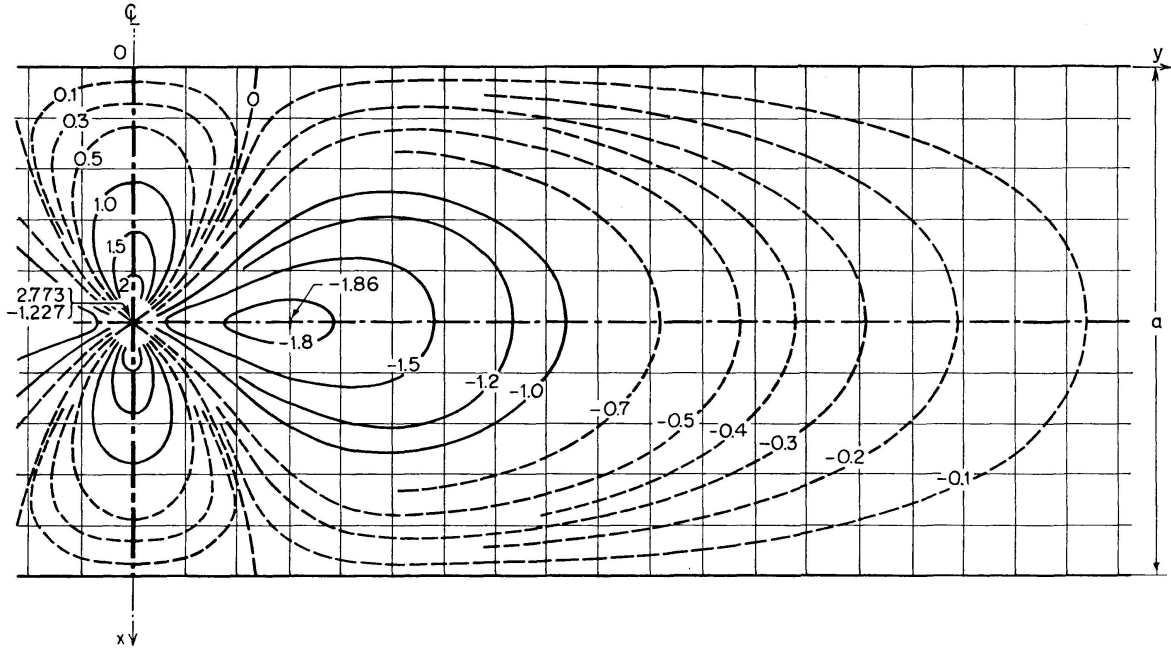


Fig. 10. Plate Strip with Flexible Cross Beam ( $\rho = \frac{4aD}{\pi EI} = 1; \nu = 0$ ).  
 $m_y$  = Influence Surface for Plate Moment over Cross Beam ( $8\pi$  times).

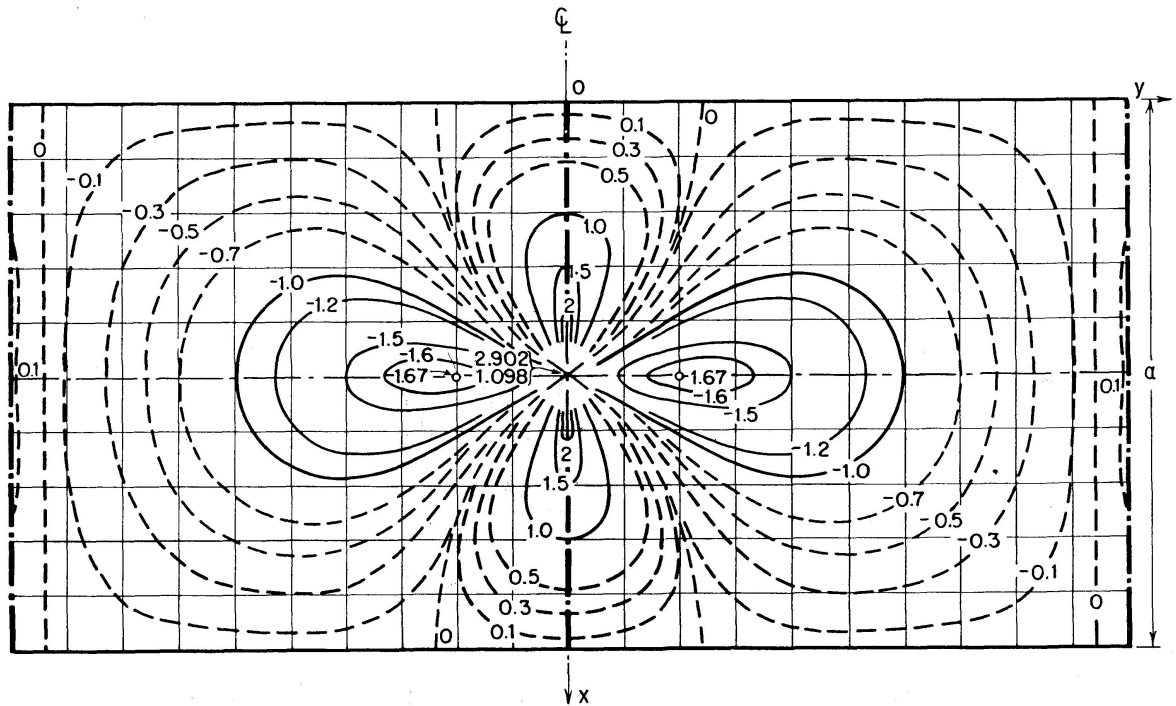


Fig. 11. Two Span Continuous Slab with Flexible Cross Beams ( $\rho = \frac{4aD}{\pi EI} = 1; \nu = 0$ ).  
 $m_y$  = Influence Surface for Plate Moment over Middle Cross Beam ( $8\pi$  times).

*b) Two-Span Continuous Slab with Flexible Cross Beams*

The plates were chosen as two square plates,  $\gamma = \pi$ , simply supported along the edges  $\xi = 0$  and  $\xi = \pi$  (fig. 7). Taking the ratio  $\rho = 1$ , equal bending stiffness  $EI$  for all three cross beams was assumed. Only the influence surface for  $m_y\left(\frac{\pi}{2}, 0\right)$  was computed. It should be pointed out that the calculations were done for  $\nu = 0$ . Using the method outlined in Chapter 6 the computed constants  $a_n$  to  $d_n$  of equation (47) for the first and third term of the series are given in the following table:

$a_1$	$5.151 \times 10^{-3}$	$a_3$	$7.674 \times 10^{-10}$
$b_1$	$5.151 \times 10^{-3}$	$b_3$	$2.558 \times 10^{-10}$
$c_1$	$2.830 \times 10^{-3}$	$c_3$	$7.538 \times 10^{-10}$
$d_1$	$-5.151 \times 10^{-3}$	$d_3$	$-7.674 \times 10^{-10}$

In calculating the  $m_{y1}$  values, only the first term of the series was considered. The influence surface for  $m_y$  is shown in fig. 11.

### VIII. Acknowledgements

The results presented in this paper were obtained in the course of research on moment influence surfaces sponsored by the National Science Foundation (Grant NSF-G 2949). The numerical computations and the drawings of the influence surfaces were prepared by Mr. ROBERT G. SARUBBI, Instructor in the Department of Civil Engineering.

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### Appendix

*a) Summation Formulæ*

For summation of equations (12) to (14) use of the following formulæ was made:

$$\sum_{n=1}^{\infty} \frac{r^n}{n} \cos nx = -\frac{1}{2} \log(1 - 2r \cos x + r^2), \quad (\text{A})$$

$$\sum_{n=1}^{\infty} \frac{r^n}{n} \sin nx = \tan^{-1} \left( \frac{r \sin x}{1 - r \cos x} \right), \quad (\text{B})$$

$$\sum_{n=1,3,5,\dots}^{\infty} \frac{r^n}{n} \cos nx = \frac{1}{4} \log \frac{1 + 2r \cos x + r^2}{1 - 2r \cos x + r^2}, \quad (\text{C})$$

$$\sum_{n=1,3,5,\dots}^{\infty} \frac{r^n}{n} \sin nx = \tan^{-1} \left( \frac{2r \sin x}{1 - r^2} \right), \quad (\text{D})$$

$$\sum_{n=1}^{\infty} r^n \cos nx = \left( \frac{1 - r \cos x}{1 - 2r \cos x + r^2} - 1 \right) = \frac{1}{2} \left( \frac{1 - r^2}{1 - 2r \cos x + r^2} - 1 \right), \quad (\text{E})$$

$$\sum_{n=1}^{\infty} r^n \sin nx = \frac{r \sin x}{1 - 2r \cos x + r^2}, \quad (\text{F})$$

$$\sum_{n=1}^{\infty} n r^n \cos nx = \frac{r \{ (1 + r^2) \cos x - 2r \}}{(1 - 2r \cos x + r^2)^2}, \quad (\text{G})$$

for values of  $|r| < 1$ . Equations (A), (B), (D) may be found in reference [6], p. 190. The derivation of equation (E) is given in reference [2], p. 496, Appendix. Equations (C), (F), (G) were developed similarly.

*b) Summation of Equations (12) to (14)*

With the aid of the formulæ (A) to (G) the series of equations (12) to (14) for the specific value  $\rho = 1$  are summed. The results are given without repeating the arithmetical operations. For  $\rho = 1$ :

$$\begin{aligned} 8\pi m_x(\alpha, \beta; \xi, \eta) = & (1 + \nu) \log \frac{\cosh(\beta - \eta) - \cos(\alpha + \xi)}{\cosh(\beta - \eta) - \cos(\alpha - \xi)} + (1 - \nu) (\beta - \eta) \sinh(\beta - \eta) \\ & \cdot \left( \frac{1}{\cosh(\beta - \eta) - \cos(\alpha - \xi)} - \frac{1}{\cosh(\beta - \eta) - \cos(\alpha + \xi)} \right) + (1 + \nu) \left\{ \mp \eta \sinh(\beta \pm \eta) \right. \\ & \cdot \left( \frac{1}{\cosh(\beta \pm \eta) - \cos(\alpha - \xi)} - \frac{1}{\cosh(\beta \pm \eta) - \cos(\alpha + \xi)} \right) + (\pm \eta - 1) e^{\beta \pm \eta} \\ & \cdot \left[ \cos(\alpha + \xi) \log[\cosh(\beta \pm \eta) - \cos(\alpha + \xi)] - \cos(\alpha - \xi) \log[\cosh(\beta \pm \eta) - \cos(\alpha - \xi)] \right. \\ & \left. - 2(\log 2 \mp \eta) \sin \alpha \sin \xi - 2 \sin(\alpha + \xi) \tan^{-1} \left( \frac{\sin(\alpha + \xi)}{e^{\beta \pm \eta} - \cos(\alpha + \xi)} \right) + 2 \sin(\alpha - \xi) \right. \\ & \left. \cdot \tan^{-1} \left( \frac{\sin(\alpha - \xi)}{e^{\beta \pm \eta} - \cos(\alpha - \xi)} \right) \right] \left. \right\} - (1 - \nu) \beta \left\{ (1 \mp \eta) e^{\beta \pm \eta} \left[ \cos(\alpha + \xi) \log[\cosh(\beta \pm \eta) \right. \right. \right. \\ & \left. \left. - \cos(\alpha + \xi)] - \cos(\alpha - \xi) \log[\cosh(\beta \pm \eta) - \cos(\alpha - \xi)] - 2(\log 2 \mp \eta) \sin \alpha \sin \xi \right. \right. \\ & \left. \left. - 2 \sin(\alpha + \xi) \tan^{-1} \left( \frac{\sin(\alpha + \xi)}{e^{\beta \pm \eta} - \cos(\alpha + \xi)} \right) + 2 \sin(\alpha - \xi) \tan^{-1} \left( \frac{\sin(\alpha - \xi)}{e^{\beta \pm \eta} - \cos(\alpha - \xi)} \right) \right] \right\} \end{aligned}$$

$$\pm \eta \left( \frac{\cosh(\beta \pm \eta) \cos(\alpha - \xi) - 1}{[\cosh(\beta \pm \eta) - \cos(\alpha - \xi)]^2} - \frac{\cosh(\beta \pm \eta) \cos(\alpha + \xi) - 1}{[\cosh(\beta \pm \eta) - \cos(\alpha + \xi)]^2} \right) + (1 \mp \eta) \sinh(\beta \pm \eta) \cdot \left( \frac{1}{\cosh(\beta \pm \eta) - \cos(\alpha - \xi)} - \frac{1}{\cosh(\beta \pm \eta) - \cos(\alpha + \xi)} \right) \Bigg\}, \tag{H}$$

$$\begin{aligned} 8 \pi m_y(\alpha, \beta; \xi, \eta) = & (1 + \nu) \log \frac{\cosh(\beta - \eta) - \cos(\alpha + \xi)}{\cosh(\beta - \eta) - \cos(\alpha - \xi)} - (1 - \nu) (\beta - \eta) \sinh(\beta - \eta) \\ & \cdot \left( \frac{1}{\cosh(\beta - \eta) - \cos(\alpha - \xi)} - \frac{1}{\cosh(\beta - \eta) - \cos(\alpha + \xi)} \right) + (1 + \nu) \left\{ \mp \eta \sinh(\beta \pm \eta) \right. \\ & \cdot \left( \frac{1}{\cosh(\beta \pm \eta) - \cos(\alpha - \xi)} - \frac{1}{\cosh(\beta \pm \eta) - \cos(\alpha + \xi)} \right) + (\pm \eta - 1) e^{\beta \pm \eta} \\ & \cdot \left[ \cos(\alpha + \xi) \log[\cosh(\beta \pm \eta) - \cos(\alpha - \xi)] - \cos(\alpha - \xi) \log[\cosh(\beta \pm \eta) - \cos(\alpha + \xi)] \right] \\ & - 2(\log 2 \mp \eta) \sin \alpha \sin \xi - 2 \sin(\alpha + \xi) \tan^{-1} \left( \frac{\sin(\alpha + \xi)}{e^{\beta \pm \eta} - \cos(\alpha + \xi)} \right) + 2 \sin(\alpha - \xi) \\ & \cdot \tan^{-1} \left( \frac{\sin(\alpha - \xi)}{e^{\beta \pm \eta} - \cos(\alpha - \xi)} \right) \Bigg\} + (1 - \nu) \beta \left\{ (1 \mp \eta) e^{\beta \pm \eta} \left[ \cos(\alpha + \xi) \log[\cosh(\beta \pm \eta) \right. \right. \\ & - \cos(\alpha + \xi)] - \cos(\alpha - \xi) \log[\cosh(\beta \pm \eta) - \cos(\alpha - \xi)] - 2(\log 2 \mp \eta) \sin \alpha \sin \xi \\ & \left. \left. - 2 \sin(\alpha + \xi) \tan^{-1} \left( \frac{\sin(\alpha + \xi)}{e^{\beta \pm \eta} - \cos(\alpha + \xi)} \right) + 2 \sin(\alpha - \xi) \tan^{-1} \left( \frac{\sin(\alpha - \xi)}{e^{\beta \pm \eta} - \cos(\alpha - \xi)} \right) \right] \right\} \\ & \pm \eta \left( \frac{\cosh(\beta \pm \eta) \cos(\alpha - \xi) - 1}{[\cosh(\beta \pm \eta) - \cos(\alpha - \xi)]^2} - \frac{\cosh(\beta \pm \eta) \cos(\alpha + \xi) - 1}{[\cosh(\beta \pm \eta) - \cos(\alpha + \xi)]^2} \right) + (1 \mp \eta) \sinh(\beta \pm \eta) \\ & \cdot \left( \frac{1}{\cosh(\beta \pm \eta) - \cos(\alpha - \xi)} - \frac{1}{\cosh(\beta \pm \eta) - \cos(\alpha + \xi)} \right) \Bigg\}, \tag{I} \end{aligned}$$

$$\begin{aligned} \frac{\pi^2}{2a} m_b(\alpha, 0; \xi, \eta) = & \log \frac{\cosh \eta - \cos(\alpha + \xi)}{\cosh \eta - \cos(\alpha - \xi)} + (\pm \eta - 1) e^{\pm \eta} \left\{ \cos(\alpha + \xi) \log[\cosh \eta - \cos(\alpha + \xi)] \right. \\ & - \cos(\alpha - \xi) \log[\cosh \eta - \cos(\alpha - \xi)] - 2(\log 2 \mp \eta) \sin \alpha \sin \xi - 2 \sin(\alpha + \xi) \\ & \cdot \tan^{-1} \left( \frac{\sin(\alpha + \xi)}{e^{\pm \eta} - \cos(\alpha + \xi)} \right) + 2 \sin(\alpha - \xi) \tan^{-1} \left( \frac{\sin(\alpha - \xi)}{e^{\pm \eta} - \cos(\alpha - \xi)} \right) \Bigg\}. \tag{K} \end{aligned}$$

It may be pointed out that the above expressions hold anywhere on the plate strip, fig. 1. The equations for the support moments of point  $(\frac{\pi}{2}, 0)$  follow either by introducing  $\alpha = \frac{\pi}{2}, \beta = 0$  into the above equation (H) to (K) or by summing equations (22) to (24), expressed in series form, directly with the use of the formulæ (A) to (G). Both procedures will lead to equations (25) to (27).

For the case of  $\rho = 1/2$  summation is also possible. The results for the

support moments are given by equations (28) to (30). However, as indicated before, no summation for a general value of  $\rho$  is possible.

### Summary

The influence functions for bending moments of a simply supported plate strip continuous over a flexible cross beam, are derived in closed form. A complete discussion of the singularities of the support moments (influence point above cross beam) is given, and numerical values for different ratios of slab to beam stiffness are presented. The  $m_x$ - and  $m_y$ -surfaces for the support moments and the  $m_b$ -surface for the bending moment of the cross beam are plotted in graphs, readily usable for design.

As an extension, the cases of a two-span continuous slab with flexible cross beams and of a plate strip with multiple cross beams, are treated. A graph shows the  $m_y$ -surface for the support moment of the two-span slab.

### Résumé

Les fonctions d'influence relatives aux moments fléchissants dans une dalle de grande longueur portant librement et comportant une poutre transversale flexible ont été déterminées sous une forme finie. Les singularités des moments aux appuis (point d'influence au-dessus de la poutre transversale) sont étudiées d'une manière détaillée et concrétisées par des valeurs numériques se rapportant à différentes conditions relatives de rigidité de la dalle et de la poutre transversale. Pour l'utilisation directe, dans la pratique, les aires d'influence  $m_x$  et  $m_y$  des moments aux appuis et l'aire  $m_b$  relative au moment fléchissant de la poutre transversale sont représentées.

A titre d'extension, les auteurs étudient le cas de la dalle à deux travées avec poutres transversales flexibles et celui de la dalle de grande longueur avec plusieurs poutres transversales. Enfin, ils donnent la représentation de l'aire  $m_y$  du moment aux appuis d'une dalle à deux travées.

### Zusammenfassung

Die Einflußfunktionen für die Biegemomente eines einfach gelagerten Plattenstreifens mit einem biegsamen Querträger werden in geschlossener Form abgeleitet. Die Singularitäten der Stützenmomente (Einflußpunkt über dem Querträger) werden ausführlich besprochen und durch numerische Werte für verschiedene Verhältnisse der Platten- zur Querträgersteifigkeit veranschaulicht. Zur direkten Benutzung in der Praxis sind die  $m_x$ - und die  $m_y$ -Einflußflächen der Stützenmomente und die  $m_b$ -Fläche für das Biegemoment des Querträgers dargestellt.

In einer Erweiterung werden die Fälle der Zweifeldplatte mit biegsamen Querträgern und des Plattenstreifens mit mehreren Querträgern besprochen. Schließlich ist die  $m_y$ -Fläche des Stützenmomentes einer Zweifeldplatte dargestellt.