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Spherical Domes Under Unsymmetrical Loading

Coupoles sphériques soumises à des charges asymétriques

Sphärische Kuppeln unter unsymmetrischer Belastung

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1. Scope

The determination of the internal forces in domes, having the form of a not very flat spherical calotte, is a problem that has already been solved, even under unsymmetrical loading. In such cases we can split up the actual loading in to others, which may be represented by trigonometrical functions of argument $n\theta$. Here, θ is the angle defined in § 2 and n is either zero or a positive integer.

In applications, approximate solutions of the equations are used, in many cases, as they are easier to employ. For axially symmetrical loadings we have BLUMENTHAL's solution [1], through a development in asymptotical series, and simpler processes, such as HETÉNYI's [2], in which the solution of the differential equation is obtained by disregarding the exponential function in respect of its second derivative[1]) and GECKELER's [4], in which the first derivative is also disregarded.

For unsymmetrical loadings, the solution[2]) corresponding to that of BLUMENTHAL is the HAVERS' method [5], which may be simplified to a close approximation (like GECKELER's) when the dome is not flat and has a small thickness in comparison with its radius, as shown in the present paper.

In studying this simplification, we shall consider separately the three states of internal forces which can be considered in superposition:

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[1]) GRAVINA [3] arrives at the same result by taking account of the first term only in BLUMENTHAL's asymptotical series.

[2]) Already studied by SCHWERIN for a particular case of loading [6].

1. — *the state of membrane*, in which, in addition to the active external forces, there exist only the reactive and internal forces acting in the planes tangent to the sphere;
2. — *the state of pure bending*, in which the sole forces acting on the edge and on the sections of the dome are bending and twisting moments; and
3. — *the state of disturbance of the edge*, when there are all types of forces acting on the edge and in adjacent sections, but which are propagated in a rapidly weakened, oscillating manner.

2. Notation

The points of the sphere (central surface of the dome) will be characterized by the angles φ and θ, measured from a vertical axis and from a horizontal reference axis respectively, as shown in Fig. 1. The constant value of φ, which corresponds to the edge of the dome, will be denoted by φ_c, and:

$$\omega = \varphi_c - \varphi. \tag{2.1}$$

The radius of the sphere is R and the uniform thickness of the shell is h. We denote by k the very small quantity:

$$k = \frac{h^2}{12 R^2} \tag{2.2}$$

and by α the expression:

$$\alpha = \sqrt[4]{\frac{1-\nu^2}{4} \frac{1+k}{k}} \cong \sqrt{\frac{R}{h} \sqrt{3(1-\nu^2)}} \cong 1{,}3 \sqrt{\frac{R}{h}}, \tag{2.3}$$

where ν is the Poisson's ratio of the material whose modulus of elasticity is denoted by E.

The displacements $\overline{u}$, $\overline{v}$ and $\overline{w}$ of a point on the dome are those which occur in the direction of the tangent to a parallel, in the direction of the tangent to a meridian and in the direction of the radius of the sphere, respectively. The latter is positive when it defines a displacement from the centre of the sphere and the two former are positive when directed in the same sense of positive angles θ and φ.

When these displacements correspond to a single term of the trigonometrical series which represents the external forces, with argument $n\theta$, they take the form:

$$\overline{u} = u \sin n\theta, \qquad \overline{v} = v \cos n\theta, \qquad \overline{w} = w \cos n\theta, \tag{2.4}$$

where, u, v and w are functions of φ alone.

The corresponding forces will be denoted by

$$\begin{aligned} &\overline{N}_\varphi = N_\varphi \cos n\theta, && \overline{N}_{\varphi\theta} = N_{\varphi\theta} \sin n\theta, && \overline{N}_\theta = N_\theta \cos n\theta, \\ &\overline{M}_\varphi = M_\varphi \cos n\theta, && \overline{M}_{\varphi\theta} = M_{\varphi\theta} \sin n\theta, && \overline{M}_\theta = M_\theta \cos n\theta, \\ &\overline{Q}_\varphi = Q_\varphi \cos n\theta, && && \overline{Q}_\theta = Q_\theta \sin n\theta, \end{aligned} \tag{2.5}$$

where: $\overline{N}_\varphi$ and $\overline{N}_\theta$ are the normal forces; $\overline{N}_{\varphi\theta}$ are the tangential forces; $\overline{Q}_\varphi$ and $\overline{Q}_\theta$ are the shearing forces; $\overline{M}_\varphi$ and $\overline{M}_\theta$ are the bending moments, and $\overline{M}_{\varphi\theta}$ the twisting moments, all of them per unit length of the section. The corresponding directions and positive senses are those indicated in fig. 2.

The deformations of the material in the central surface of the membrane are:

$$\bar{\epsilon}_u = \epsilon_u \cos n\,\theta, \qquad \bar{\epsilon}_v = \epsilon_v \cos n\,\theta, \qquad \bar{\gamma} = \gamma \sin n\,\theta, \tag{2.6}$$

where, $\bar{\epsilon}_u$ and $\bar{\epsilon}_v$ are the elongations in the u- and v-directions and $\bar{\gamma}$ is the shearing strain, that is:

$$\epsilon_u = \frac{1}{E\,h}(N_\theta - \nu N_\varphi), \qquad \epsilon_v = \frac{1}{E\,h}(N_\varphi - \nu N_\theta), \qquad \gamma = \frac{2\,(1+\nu)}{E\,h} N_{\varphi\theta}. \tag{2.7}$$

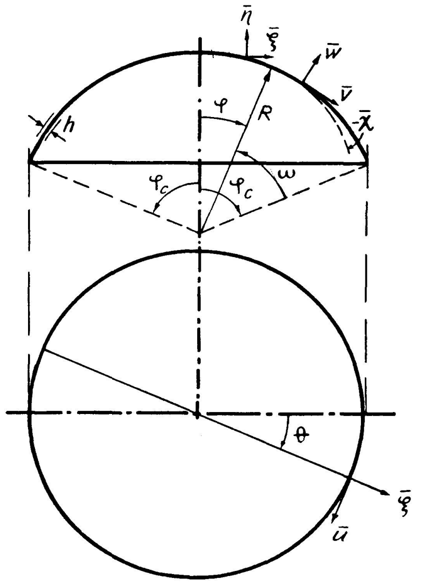

Fig. 1.

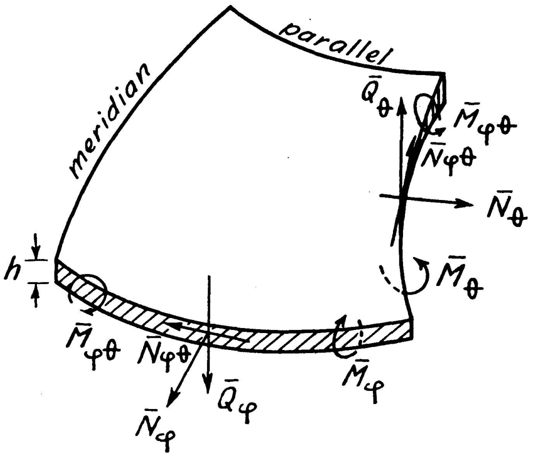

Fig. 2.

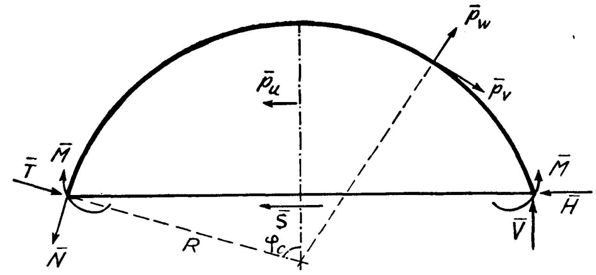

Fig. 3.

In expressions containing trigonometrical functions of argument $n\,\theta$, such as 2.4 to 2.6, it is possible to substitute simultaneously all the cosines by sines and all the sines by cosines without changing the results except, in certain cases, their sign. Should this occur in the following considerations, the double sign $\pm$ or $\mp$ will be used, the upper sign to be employed with the expressions 2.4 to 2.6 (to which correspond the expression 2.8 and the expressions 2.10 to 2.13) and the lower sign when the substitution has been made. The first case corresponds to symmetrical solutions with respect to the θ-reference axis, and the second case refers to anti-symmetrical solutions.

For reactive forces, per unit length, we use, with $\varphi=\varphi_c$ (Fig. 3):

$$\overline{M} = \overline{M}_\varphi, \qquad \overline{N} = \overline{N}_\varphi, \qquad \overline{T} = \overline{Q}_\varphi + \frac{1}{R\sin\varphi_c}\frac{\partial \overline{M}_{\varphi\theta}}{\partial\theta} = T\cos n\theta,$$

$$\overline{S} = \overline{N}_{\varphi\theta} + \frac{1}{R}\overline{M}_{\varphi\theta} = S\sin n\theta, \qquad \overline{H} = \overline{T}\sin\varphi - \overline{N}\cos\varphi, \tag{2.8}$$

$$\overline{V} = -\overline{N}\sin\varphi - \overline{T}\cos\varphi,$$

hence (always with $\varphi=\varphi_c$):

$$M = M_\varphi, \qquad N = N_\varphi, \qquad T = Q_\varphi \pm \frac{n}{R\sin\varphi_c}M_{\varphi\theta}, \qquad S = N_{\varphi\theta} + \frac{1}{R}M_{\varphi\theta},$$

$$H = T\sin\varphi - N\cos\varphi, \qquad V = -N\sin\varphi - T\cos\varphi \tag{2.9}$$

and for active forces, per unit area (p_u, p_v and p_w are bounded functions):

$$\overline{p}_u = p_u\sin n\theta, \qquad \overline{p}_v = p_v\cos n\theta, \qquad \overline{p}_w = p_w\cos n\theta \tag{2.10}$$

in the u-, v- and w-directions respectively and with the same positive senses (Fig. 3).

The vertical displacements (positive upwards) and the radial-horizontal displacements (positive outwards) will be represented, respectively by (Fig. 1):

$$\eta\cos n\theta = \bar\eta = \overline{w}\cos\varphi - \overline{v}\sin\varphi, \tag{2.11}$$

$$\xi\cos n\theta = \bar\xi = \overline{w}\sin\varphi + \overline{v}\cos\varphi, \tag{2.12}$$

the angular displacement of the meridian (positive when in the opposite sense of positive φ) by:

$$\chi\cos n\theta = \bar\chi = \frac{1}{R}(\overline{w}' - \overline{v}) \tag{2.13}$$

with $\quad \eta = w\cos\varphi - v\sin\varphi, \qquad \xi = w\sin\varphi + v\cos\varphi, \qquad \chi = \frac{1}{R}(w' - v).$ (2.14)

Displacements of the edge are characterized by the index c.

3. The State of Membrane

In shells not clamped at the edges, we determine the internal forces — which are reduced to normal forces $\overline{N}_\theta$ and $\overline{N}_\varphi$ and to tangential forces $\overline{N}_{\varphi\theta}$ only — by the membrane theory, independently of the study of their deformation and considering the active forces defined in (2.10). The forces acting along the edge will be the reactive forces that we obtain with (2.9), where, with $\varphi=\varphi_c$:

$$M = 0, \qquad N = N_\varphi, \qquad T = 0, \qquad S = N_{\varphi\theta}. \tag{3.1}$$

The equilibrium equations give[3]):

$$N_\theta = p_w R - N_\varphi, \qquad N_\varphi = U(n) + U(-n), \qquad N_{\varphi\theta} = \pm[U(n) - U(-n)], \tag{3.2}$$

where:

$$U(m) = R\frac{\cot^m(\varphi/2)}{2\sin^2\varphi}\left\{\int\limits_\varphi^a \left[\left(p_v + \frac{|m|}{m}p_u\right)\sin\varphi - (m+\cos\varphi)\,p_w\right] \cdot \sin\varphi\tan^m(\varphi/2)\,d\varphi + C(m)\right\}, \tag{3.3}$$

being $a=0$, for $m \geqq -1$ and $a=\pi/2$ for $m<-1$.

The terms $C(n)$, corresponding to $U(n)$, and $C(-n)$, related to $U(-n)$, are the integration constants, which, in such cases, where the domes have the shape of a spherical calotte (therefore with a single edge) should never provide infinite values for the forces on the crown. Hence, we have $C(n)=0$ for any value of n, and $C(-n)=0$ for $n=0$ and $n=1$.

For $n \geqq 2$, the constant $C(-n)$ can have any finite value, corresponding to the application of normal and tangential forces along the edge. These forces balance one another without disturbing the other external forces, and give rise to new internal forces, characterized by (Fig. 4):

$$N_{\varphi 1} = -N_{\theta 1} = \mp N_{\varphi\theta 1} = A_1\frac{\tan^n(\varphi/2)}{\sin^2\varphi} \tag{3.4}$$

with $A_1 = 0{,}5\,R\,C(-n)$.

Denoting by the index 0 the values of the forces (3.2) when we make $C(n)=C(-n)=0$ in (3.3), we have:

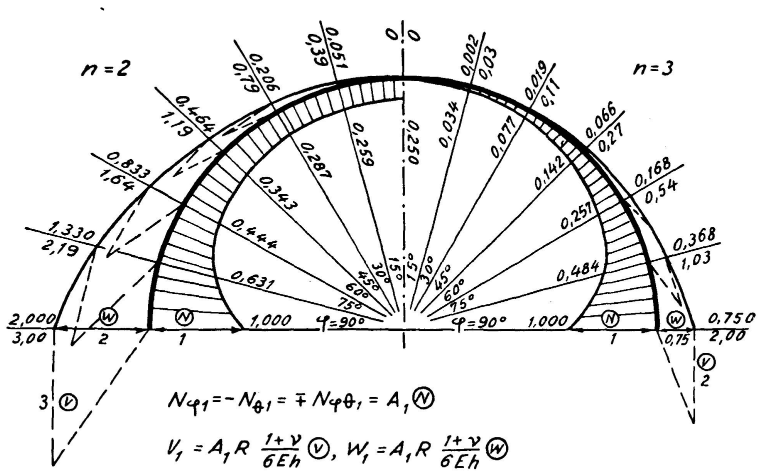

Fig. 4.

[3]) All the formulæ in this § 3, except expression (3.13) and the subsequent expressions may be obtained from those given in FLÜGGE's paper [6], where the corresponding proof is to be found.

$$N_\varphi = N_{\varphi 0}+N_{\varphi 1}, \qquad N_\theta = N_{\theta 0}+N_{\theta 1}, \qquad N_{\varphi\theta} = N_{\varphi\theta 0}+N_{\varphi\theta 1}, \tag{3.5}$$

where, the first term of the second member corresponds to the effect of the loading on the dome and the second term contains the constant A_1 to be determined in accordance with the boundary conditions. It should be noted, however, that $A_1=0$ for $n=0$ and $n=1$ [4]).

The displacements of the membrane are defined by the following equations (the accents indicate the derivatives in relation to φ):

$$w+v' = R\,\epsilon_v, \tag{3.6}$$

$$w\sin\varphi+v\cos\varphi\pm n\,u = R\,\epsilon_u\sin\varphi, \tag{3.7}$$

$$u'\sin\varphi-u\cos\varphi\mp n\,v = R\,\gamma\sin\varphi, \tag{3.8}$$

where ϵ_u, ϵ_v and γ can be obtained from the forces N, already calculated, by means of (2.7).

By eliminating u and w from Eqs. (3.6) to (3.8), we arrive at the single equation with variable v:

$$v''-v'\cot\varphi+v\,(1-n^2)\operatorname{cosec}^2\varphi = R\,\phi, \tag{3.9}$$

in which

$$\phi = \epsilon_v'-\epsilon_u'\pm n\,\gamma\operatorname{cosec}\varphi = \frac{1+\nu}{E\,h}\left(N_\varphi'-N_\theta'\pm\frac{2n}{\sin\varphi}N_{\varphi\theta}\right). \tag{3.10}$$

With v, we find w and u from (3.6) and (3.7) (for $n=0$ we use (3.8) instead of (3.12)):

$$w = R\,\epsilon_v-v', \tag{3.11}$$

$$\pm n\,u = R\,(\epsilon_u-\epsilon_v)\sin\varphi+v'\sin\varphi-v\cos\varphi. \tag{3.12}$$

The general solution of (3.9) is:

$$v = \frac{R\sin\varphi}{2n}\left[\tan^n\frac{\varphi}{2}\left(\int_{\pi/2}^{\varphi}\phi\cot^n\frac{\varphi}{2}\,d\varphi+C_1\right)-\cot^n\frac{\varphi}{2}\left(\int_{0}^{\varphi}\phi\tan^n\frac{\varphi}{2}\,d\varphi+C_2\right)\right], \tag{3.13}$$

in which the constants C_1 and C_2 must be chosen so as not to give rise to moments along the edge, for we assume the permanence of the state of membrane (details are given in § 4, where it is also shown, that for $n=0$ and $n=1$, such moments never occur, the constants corresponding to displacements of the dome as a rigid body).

The expression (3.13), for $n=0$, assumes an indeterminate form, which can be shown to be equivalent to:

$$v = \frac{1+\nu}{E\,h}R\sin\varphi\left(\int\frac{N_\varphi-N_\theta}{\sin\varphi}\,d\varphi+C_0\right). \tag{3.14}$$

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[4]) Physically, this is explained by the fact that it is impossible for a system of balanced forces, distributed along the edge proportionally to $\sin n\,\theta$ or $\cos n\,\theta$ if $n=0$ or $n=1$, to exist.

Denoting by v_0 the value of v corresponding to the forces $N_{\varphi 0}$, $N_{\theta 0}$ and $N_{\varphi\theta 0}$, and by v_1 that related to the forces (3.4), we have:

$$v = v_0 + v_1, \tag{3.15}$$

where (introducing (3.4) into (3.10) and the latter into (3.13)), with $n > 1$ (Fig. 4) [5]):

$$v_1 = A_1 R \frac{1+\nu}{E h} \tan^n \frac{\varphi}{2} \, \frac{n^2 + n \cos\varphi - \sin^2\varphi}{n\,(n^2-1)\sin\varphi}. \tag{3.16}$$

It should be noted that for $n=0$ and $n=1$, v_1 has zero values for there are no forces (3.4), as has already been said ($A_1 = 0$).

In applications it is useful to know that to v_1 there correspond (Fig. 4):

$$\chi_1 = 0, \qquad w_1 = A_1 R \frac{1+\nu}{E h} \tan^n \frac{\varphi}{2} \, \frac{n + \cos\varphi}{n\,(n^2-1)}, \qquad u_1 = \mp w_1 \frac{n}{\sin\varphi}. \tag{3.17}$$

4. The State of Pure Bending and the Displacement of the Dome as a Rigid Body

When in (3.13) integration constants are used that differ of $A_2 \cdot 2\,n/R$ and $A_3 \cdot 2\,n/R$ from those employed, the displacements which are added to the former ones are:

$$v = v_2 + v_3, \tag{4.1}$$

where
$$v_2 = A_2 \sin\varphi \tan^n \frac{\varphi}{2}, \qquad v_3 = A_3 \sin\varphi \cot^n \frac{\varphi}{2}. \tag{4.2}$$

For $n > 1$, we have necessarily $A_3 = 0$, so that be not $v = \infty$ with $\varphi = 0$. To the remaining term v_2 (Fig. 5) there corresponds, according to HAVERS [5], the state of pure bending in which (in addition to $N_\varphi = N_\theta = N_{\varphi\theta} = 0$ and $\epsilon_u = \epsilon_v = \gamma = 0$):

$$M_{\theta 2} = -M_{\varphi 2} = \pm M_{\varphi\theta 2} = A_2 \frac{E h}{1+\nu} k\, n\,(n^2-1) \frac{\tan^n(\varphi/2)}{\sin^2\varphi} \tag{4.3}$$

and, from (3.11), (3.12) and the last of the expressions (2.14) (Fig. 5):

$$w_2 = -v_2' = -A_2 (n + \cos\varphi) \tan^n \frac{\varphi}{2}, \tag{4.4}$$

$$\pm u_2 = v_2, \qquad \chi_2 = \frac{n\, w_2}{R \sin\varphi}. \tag{4.5}$$

The moments $\overline{M}_\varphi$ and $\overline{M}_{\varphi\theta}$, when $\varphi = \varphi_c$, are those acting along the edge; they constitute a balanced system (always assuming $n > 1$) [6]).

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[5]) The integration constants used here, in the application of (3.13), are those that give rise to zero moments, in agreement with HAVERS [5].

[6]) For the reasons mentioned in § 6.

For $n=1$, A_2 as well as A_3 may be different from zero. The corresponding displacements are those of the dome as a rigid body (therefore without additional internal forces) when it is displaced horizontally or rotates around a horizontal axis:

$$v = A_2 \sin\varphi \tan\frac{\varphi}{2} + A_3 \sin\varphi \cot\frac{\varphi}{2} = A_2(1-\cos\varphi)+A_3(1+\cos\varphi), \qquad (4.6)$$

$$w = -v' = -A_2\sin\varphi + A_3 \sin\varphi, \qquad (4.7)$$

$$\pm u = A_2(1-\cos\varphi) - A_3(1+\cos\varphi), \qquad (4.8)$$

$$R\chi = -(A_2+A_3). \qquad (4.9)$$

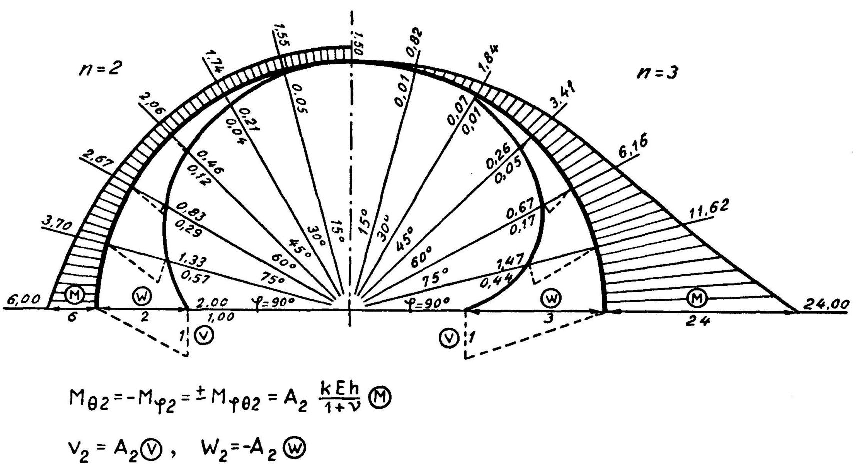

Fig. 5.

The horizontal displacement (which occurs in the direction of the θ-reference axis, when we use the expression (2.4), or in the direction perpendicular to this axis if the trigonometrical functions are changed) has the value:

$$\zeta_a = 2A_3 \qquad (4.10)$$

and the angular displacement (which occurs around the horizontal axis through the crown of the dome and perpendicular to the direction of horizontal displacement):

$$\psi_a = \frac{A_2+A_3}{R}. \qquad (4.11)$$

For $n=0$, we have in (4.2) the same function of φ for v_2 and v_3. Thus, (4.1) will simply be:

$$v = v_2 = A_2 \sin\varphi, \qquad (4.12)$$

which is equivalent to making, in (3.14), $C_0 = A_2 E h / R(1+\nu)$. To this solution correspond the following equalities, if the values of (2.4) are used:

$$\bar{v} = v_2, \qquad \bar{w} = w_2 = -v_2' = -A_2\cos\varphi, \qquad \bar{u} = 0. \qquad (4.13)$$

This represents the displacement of the dome, by translation, in the direction of its axis (positively upwards):

$$\eta_a = w_2 \cos\varphi - v_2 \sin\varphi = -A_2. \tag{4.14}$$

If the trigonometrical functions are changed in (2.4) (u_2 of (3.8) with $\gamma = 0$):

$$\bar{v} = \bar{w} = 0, \qquad \bar{u} = u_2 = C_3 \sin\varphi. \tag{4.15}$$

This means a rotation of the dome around its own axis of

$$\psi_b = \frac{C_3}{R}. \tag{4.16}$$

5. The State of Disturbance of the Edge

The displacements and forces that occur along the edge of the dome and are propagated to the adjacent regions in a rapidly convergent oscillating manner, may be represented, according to HAVERS [5], by combinations of functions of the type:

$$\phi_c = \varphi_0 e^{-\alpha\varphi_1} \cos(\alpha\varphi_2 + \varphi_3), \qquad \phi_s = \varphi_0 e^{-\alpha\varphi_1} \sin(\alpha\varphi_2 + \varphi_3), \tag{5.1}$$

where φ_0, φ_1, φ_2 and φ_3 are functions of φ, and α is given by (2.3). In the case of domes which are not flat and have small values for h/R, we have, approximately, if n is not large:

$$\varphi_0(\varphi) \approx \varphi_0(\varphi_c), \qquad \varphi_1 \approx \varphi_2 \approx \varphi_c - \varphi = \omega, \qquad \varphi_3 \approx 0. \tag{5.2}$$

Then we may write, by eliminating the constant factor:

$$\phi_c = e^{-\alpha\omega} \cos\alpha\omega, \qquad \phi_s = e^{-\alpha\omega} \sin\alpha\omega, \tag{5.3}$$

whose derivatives in relation to φ are:

$$\phi_c' = \alpha(\phi_s + \phi_c), \qquad \phi_s' = \alpha(\phi_s - \phi_c). \tag{5.4}$$

In such cases, all displacements and internal forces may be represented by the function

$$K_c \phi_c + K_s \phi_s \tag{5.5}$$

in which K_c and K_s have the values given in the following table, when $1/\alpha^2$, in them, is neglected as compared with unity (K_1 and K_2 are integration constants).

Along the edge, the displacements and forces are determined by considering that, there, $\phi_c = 1$ and $\phi_s = 0$; it can be shown that the characteristic quantities u_c, v_c, w_c, χ_c, M and N are equivalent to the constant K_c of the corresponding expressions and that for ξ_c, η_c, T, H, V and S we may use the expressions (2.8) to (2.12), which take the form of the terms that contain K_1 and K_2 in (6.5) to (6.8) and (6.13) to (6.14).

The expression (5.5) may be represented in the following form:

$$\begin{aligned} K_s\phi_s+K_c\phi_c &= K_s e^{-\alpha\omega}\sin\alpha\omega+K_c e^{-\alpha\omega}\cos\gamma\omega \\ &= \pm\sqrt{K_s^2+K_c^2}\,e^{-\alpha\omega}\cos(\alpha\omega+\psi) \end{aligned} \tag{5.6}$$

with

$$0\leqq\psi=\arctan\frac{-K_s}{K_c}<\pi. \tag{5.7}$$

Their absolute values are always smaller than

$$\phi_0=\sqrt{K_s^2+K_c^2}\,e^{-\alpha\omega}. \tag{5.8}$$

This expression may be useful for the design of the structure. It should be noted that the highest absolute value of (5.6) will never exceed the value K_c of the edge, if $\psi<0{,}4195\,\pi$ or $\psi\geqq 0{,}75\,\pi$, and, otherwise, will reach the maximum

$$\sqrt{\frac{K_s^2+K_c^2}{2}}\,e^{\psi-0{,}75\pi}=0{,}0670\,e^{\psi}\sqrt{K_s^2+K_c^2} \tag{5.9}$$

for $\alpha\omega+\psi=0{,}75\,\pi$ (see Fig. 6).

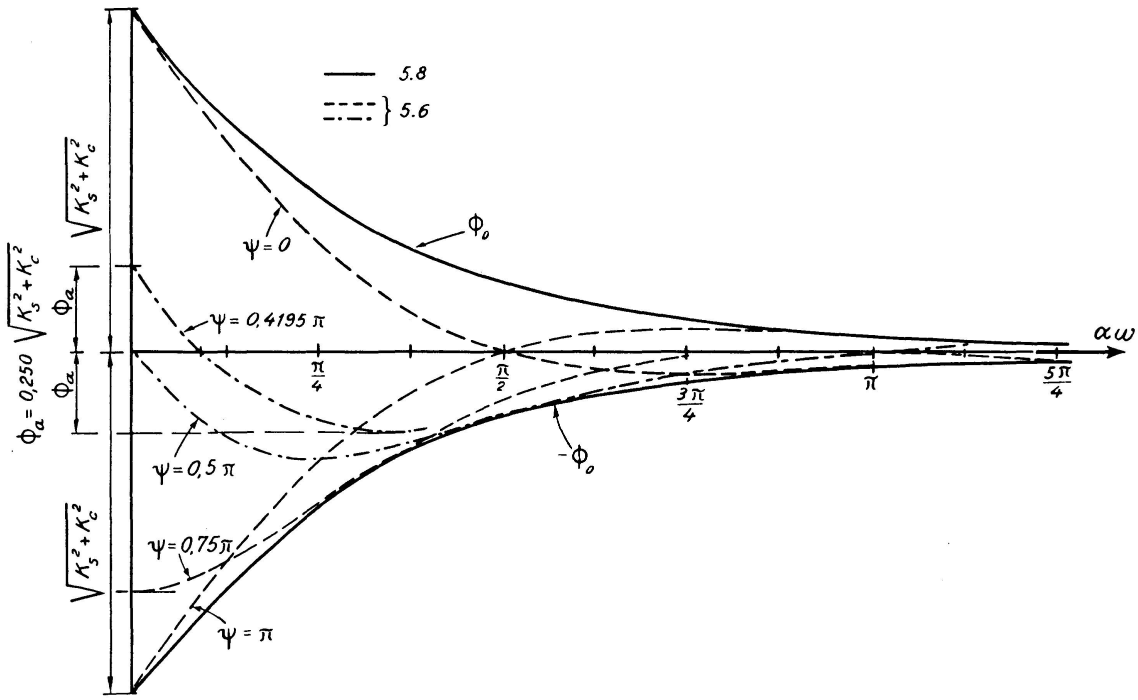

Fig. 6.

6. Superposition of States

By superimposing the three states which were studied in §§ 3 to 5, we obtain the solution of the problem of domes having the form of a spherical calotte. That solution contains four integration constants, when $n>1$:

$$A_1,\ A_2,\ K_1,\ K_2,$$

for	K_c	K_s	$\tan\psi$ (5.7)
w	$2\alpha^2 R K_1$	$2\alpha^2 R K_2$	$-\frac{K_1}{K_2}$
v	$\alpha R(1+\nu)(K_2-K_1)$	$-\alpha R(1+\nu)(K_2+K_1)$	$\frac{K_2-K_1}{K_2+K_1}$
u	$\mp R(1+\nu)\,n\,K_2 \operatorname{cosec}\varphi_c$	$\pm R(1+\nu)\,n\,K_1 \operatorname{cosec}\varphi_c$	$\frac{K_2}{K_1}$
χ	$2\alpha^3(K_2+K_1)$	$2\alpha^3(K_2-K_1)$	$\frac{K_1+K_2}{K_1-K_2}$
$\frac{N_\varphi}{Eh}$	$\alpha(K_1-K_2)\cot\varphi_c+\frac{n^2}{\sin^2\varphi_c}K_2$	$\alpha(K_1+K_2)\cot\varphi_c-\frac{n^2}{\sin^2\varphi_c}K_1$	$-\frac{K_s}{K_c}$
$\frac{N_{\varphi\theta}}{Eh}$	$B_4=\pm\frac{n}{\sin\varphi_c}\cdot[\alpha(K_1-K_2)+K_2\cot\varphi_c]$	$B_5=\pm\frac{n}{\sin\varphi_c}\cdot[\alpha(K_1+K_2)-K_1\cot\varphi_c]$	$-\frac{B_4}{B_5}$
$\frac{M_\varphi}{Eh}$	$R(B_1K_2-B_2K_1)$	$-R(B_1K_1+B_2K_2)$	$\frac{B_1K_2-B_2K_1}{B_1K_1+B_2K_2}$
$\frac{M_{\varphi\theta}}{Eh}$	$-B_5 R\frac{1-\nu}{2\alpha^2}$	$B_4 R\frac{1-\nu}{2\alpha^2}$	$\frac{B_5}{B_4}$
$\frac{M_\theta}{Eh}$	$R(B_6K_2-B_7K_1)$	$-R(B_6K_1+B_7K_2)$	$\frac{B_6K_2-B_7K_1}{B_6K_1+B_7K_2}$
$\frac{Q_\varphi}{Eh}$	$\alpha(K_2-K_1)$	$-\alpha(K_2+K_1)$	$\frac{K_2-K_1}{K_2+K_1}$
$\frac{Q_\theta}{Eh}$	$\mp n K_2 \sec\varphi_c$	$\pm n K_1 \sec\varphi_c$	$\frac{K_2}{K_1}$

$$\frac{N_\theta}{Eh}=\frac{w}{R}-\frac{N_\varphi}{Eh},\qquad B_6=-B_1+1+\nu,\qquad B_7=-B_2+\frac{1-\nu^2}{2\alpha^2}$$

$$B_1=1-\frac{1-\nu}{2\alpha}\cot\varphi_c,\qquad B_2=\left(\alpha\cot\varphi_c+1-\frac{n^2}{\sin^2\varphi_c}\right)\frac{1-\nu}{2\alpha^2}$$

which enable us to choose four of the magnitudes (displacements or forces) related to the edge (u_c, v_c, w_c, χ_c, η_c, ξ_c, M, N, T, H, V, S) as, for instance:

a) Dome with clamped edge (Fig. 7a): $u_c=v_c=w_c=0$, $\chi_c=0$.

b) Dome with simply supported edge (set on a horizontal plane (Fig. 7b): $M=0$, $S=H=0$, $\eta_c=0$.

c) The same (set on a conical surface with generatrices normal to the sphere) (Fig. 7c): $M=0$, $T=S=0$, $v_c=0$.

d) Dome supported by suspensions with radial movement (Fig. 7d): $M=0$, $H=0$, $u_c=\eta_c=0$.

e) Dome on hinged immovable support (Fig. 7e): $M=0$, $\xi_c=u_c=\eta_c=0$.

f) Dome with clamped edge on a deformable ring (Fig. 7f):
$u_c = u_{ring}$, $\xi_c = \xi_{ring}$, $\chi_c = \chi_{ring}$, $\eta_c = 0$.

It should be noted that the forces on the edge $\overline{M}$, $\overline{N}$, $\overline{T}$, $\overline{S}$, $\overline{H}$ or $\overline{V}$, when $n > 1$, are separately in equilibrium, for $\overline{M} = M \cos n\theta$ [7]), so that we have:

$$(\int \overline{M} \sin\theta \, d\theta)^2 + (\int \overline{M} \cos\theta \, d\theta)^2 = 0$$

and, similarly, for $\overline{S}$, $\overline{H}$ and $\overline{V}$, and, therefore, for $\overline{T}$ and $\overline{N}$, which are obtained from $\overline{H}$ and $\overline{V}$ (fig. 8).

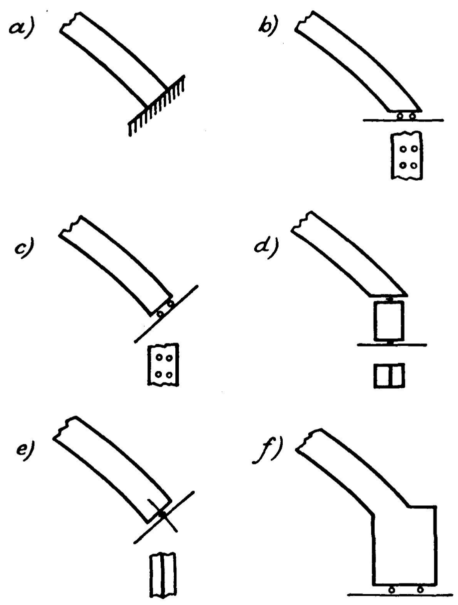

Fig. 7.

For $n = 0$, with the trigonometrical functions used in (2.4) and (2.5,) this equilibrium does not occur for $\overline{V}$ (and, therefore, for $\overline{N}$ and $\overline{T}$), which represents the vertical reaction, uniformly distributed along the edge and must, obviously, be equal to the vertical component of the resultant of the external forces. Should the trigonometrical functions of (2.4) and (2.5) be changed, that equilibrium would cease to exist for $\overline{S}$, since its moment must balance that of the external forces which tend to rotate the dome around its axis (in such cases the dome cannot be simply supported). The lack of the constant A_1, which does not exist for $n = 0$ (§ 3) is supplied by the condition of symmetry or asymmetry of the two cases mentioned ($\overline{u} = 0$, $\overline{S} = 0$, in the first case, and $\overline{v} = \overline{w} = 0$, $\overline{N} = \overline{T} = 0$, $\overline{M} = 0$ in the second case).

For $n = 1$, $A_1 = 0$ also, but there is, then, an additional constant A_3 which takes its place, as we have seen in § 4. In this case also, the forces on the edge

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[7]) The integrals $\int_0^{2\pi} {\sin \atop \cos} n\theta {\cos \atop \sin} \theta \, d\theta$ and $\int_0^{2\pi} {\cos \atop \sin} n\theta \, d\theta$ are zero for $n > 1$.

are not separately balanced (for instance: the $\overline{V}$ forces produce a moment and the $\overline{H}$ forces do not have a null resultant). So the dome cannot be simply supported, on a horizontal plane.

Once the integration constants are obtained, all three states of membrane, of pure bending and of disturbance of the edge and the corresponding forces, become known ((3.4), (3.5), (4.3) and (5.5) with the data of the table of § 5). It is, then, only necessary to add the ordinates of the respective diagrams (Figs. 4 to 6) to obtain the final diagrams, which will be used in designing the dome.

To make the determination of the constants easier, when $n > 0$ [8]), complete expressions for the forces and displacements of the edge are given below, four of which must be chosen in accordance with the imposed conditions (as illustrated at the beginning of this § 6) to constitute the equations which enable us to find the four constants:

$$A_1^* = \frac{A_1}{E h}\tan^n\frac{\varphi_c}{2}, \quad A_2^* = \frac{A_2}{R}\tan^n\frac{\varphi_c}{2}, \quad K_1, \quad K_2, \quad A_3^* = \frac{A_3}{R}. \tag{6.1}$$

In such cases, the terms which contain A_1^*, when $n=1$ and those which contain A_3^*, when $n>1$ must be eliminated. The constants B_1 and B_2 have been defined at the end of the table in § 5, and:

$$B_0 = \frac{1-\nu}{2\alpha^2}\,\frac{n^2}{\sin^2\varphi_c}, \qquad B_3 = \frac{k\,n\,(n^2-1)}{(1+\nu)}, \qquad B_8 = \cot\varphi_c + \frac{1-\nu}{2\alpha}. \tag{6.2}$$

The indices 0 represent, for $\varphi=\varphi_c$, the items of the magnitudes related to the state of membrane, under the same conditions in which it was used in (3.5) (examples are to be found in §§ 8 and 9).

$$\frac{M}{E h R} = -A_2^* B_3 \operatorname{cosec}^2\varphi_c - K_1 B_1 - K_2 B_2, \tag{6.3}$$

$$\frac{N}{E h} = \frac{N_{\varphi 0}}{E h} + A_1^* \operatorname{cosec}^2\varphi_c + K_1(\alpha\cot\varphi_c - n^2\operatorname{cosec}^2\varphi_c) + K_2\,\alpha\cot\varphi_c, \tag{6.4}$$

$$\frac{T}{E h} = A_2^*\, n\, B_3 \operatorname{cosec}^3\varphi_c - K_1\,\alpha\,(1-B_0) - K_2(\alpha + \alpha B_0 - B_0\cot\varphi_c), \tag{6.5}$$

$$\mp\frac{S}{E h} = \mp\frac{N_{\varphi\theta 0}}{E h} + \frac{A_1^* - A_2^* B_3}{\sin^2\varphi_c} + K_1 n\frac{\cot\varphi_c - \alpha}{\sin\varphi_c} - \frac{K_2\,\alpha\, n}{\sin\varphi_c}, \tag{6.6}$$

$$\begin{aligned}\frac{H}{E h} = {} & -\frac{N_{\varphi 0}}{E h}\cos\varphi_c - A_1^*\frac{\cot\varphi_c}{\sin\varphi_c} + \frac{A_2^*\, n\, B_3}{\sin^2\varphi_c} + K_1\frac{n^2 B_8 - \alpha}{\sin\varphi_c} \\ & - K_2\left(\frac{\alpha}{\sin\varphi_c} + B_0\,\alpha\sin\varphi_c - B_0\cos\varphi_c\right),\end{aligned} \tag{6.7}$$

[8]) For $n=0$ see § 7, where the simplifications which occur when $n=1$ are also to be found.

$$\begin{aligned}\frac{V}{E\,h} &= -\frac{N_{\varphi 0}}{E\,h}\sin\varphi_c - \frac{A_1^*}{\sin\varphi_c} - A_2^* n\, B_3 \frac{\cot\varphi_c}{\sin^2\varphi_c} + K_1 \frac{n^2}{\sin\varphi_c} B_1 + \\ &\quad + K_2\, B_0\,(\alpha - \cot\varphi_c)\cos\varphi_c,\end{aligned} \tag{6.8}$$

$$\begin{aligned}\mp\frac{u_c}{R} &= \mp\frac{u_0}{R} + A_1^*(1+\nu)\frac{n+\cos\varphi_c}{(n^2-1)\sin\varphi_c} - A_2^*\sin\varphi_c - K_1(1+\nu)\, n \operatorname{cosec}\varphi_c \\ &\quad + A_3^*(1+\cos\varphi_c),\end{aligned} \tag{6.9}$$

$$\begin{aligned}\frac{v_c}{R} &= \frac{v_0}{R} + A_1^*(1+\nu)\frac{n^2 + n\cos\varphi_c - \sin^2\varphi_c}{n\,(n^2-1)\sin\varphi_c} + A_2^*\sin\varphi_c - K_1\alpha(1+\nu) \\ &\quad - K_2\alpha(1+\nu) + A_3^*(1+\cos\varphi_c),\end{aligned} \tag{6.10}$$

$$\frac{w_c}{R} = \frac{w_0}{R} + A_1^*(1+\nu)\frac{n+\cos\varphi_c}{n\,(n^2-1)} - A_2^*(n+\cos\varphi_c) + K_2\, 2\,\alpha^2 + A_3^*\sin\varphi_c, \tag{6.11}$$

$$\chi_c = \chi_0 - A_2^* \frac{n\,(n+\cos\varphi_c)}{\sin\varphi_c} - K_1\, 2\,\alpha^3 + K_2\, 2\,\alpha^3 - A_3^*, \tag{6.12}$$

$$\begin{aligned}\frac{\xi_c}{R} &= \frac{\xi_0}{R} + A_1^*(1+\nu)\frac{1+n\cos\varphi_c}{(n^2-1)\sin\varphi_c} - A_2^* n\sin\varphi_c - K_1\alpha(1+\nu)\cos\varphi_c \\ &\quad + K_2[2\,\alpha^2\sin\varphi_c - \alpha(1+\nu)\cos\varphi_c] + A_3^*(1+\cos\varphi_c),\end{aligned} \tag{6.13}$$

$$\begin{aligned}\frac{\eta_c}{R} &= \frac{\eta_0}{R} - A_1^*\frac{1-\nu}{n} - A_2^*(1+n\cos\varphi_c) + K_1\alpha(1+\nu)\sin\varphi_c \\ &\quad + K_2[2\,\alpha^2\cos\varphi_c + \alpha(1+\nu)\sin\varphi_c] - A_3^*\sin\varphi_c.\end{aligned} \tag{6.14}$$

In the coefficients of K_1 and K_2 of the former expressions, $1/\alpha^2$ was disregarded as compared with unity, since the respective values were obtained from the table given in § 5, where such an approximation was used. A greater simplification could be obtained if the terms that contain $1/\alpha$ were also disregarded, compared with those that do not contain it, provided these latter are of about the same magnitude as that of the factor of $1/\alpha$ [9]). We must take into account the fact that, in most cases, B_0 is negligible in comparison with unity. When we have to solve a particular case numerically, as in the example given in § 8, there is no advantage in using the simplifications, because the work involved in solving the equations will be the same, since the numerical coefficients of the unknowns are available.

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[9]) Thus, it is possible that n^2/α may not be negligible as compared with unity or $\sin\varphi_c$ in comparison with $\alpha\cos\varphi_c$, since φ_c can tend to $\pi/2$, with $\cos\varphi_c \to 0$ and $\sin\varphi_c \to 1$. It should also be noted that constants to be determined may not have the same order of magnitude; that is what happens in the last example of the following footnote, in which K_1 is of the same order of magnitude as K_2/α and, therefore, K_2 is not negligible in comparison with αK_1.

7. Simplifications when $n=0$ or $n=1$

For $n=0$, when we use the trigonometrical functions of the expressions (2.4) and the following equations, we have the well known case of axially symmetrical loading, which can also be solved by means of the expressions (6.3) to (6.14), neglecting the terms which contain A_1^* and A_3^* and by establishing only three boundary conditions, since, then, $\bar{S}\equiv 0$ and $\bar{u}\equiv 0$. These conditions can be reduced to two, if we consider that $\bar{V}$ is known, for, its resultant is equal to that of $\bar{p}_v$ and $\bar{p}_w$, that is:

$$V=\frac{R}{\sin\varphi_c}\int_0^{\varphi_c}(p_v\sin\varphi-p_w\cos\varphi)\sin\varphi\,d\varphi=V_0 \tag{7.1}$$

and that $\bar{\eta}$ represents the movement of translation of the dome as a rigid body, characterized, then, by the choice of A_2^*. This constant has no influence on the other magnitudes (M, H, χ_c, ξ_c), whose expressions are the same, (6.3, (6.7), (6.12) and (6.13), when we eliminate the terms which contain A_1^*, A_3^* and A_2^* (this latter because it contains the factor $n=0$) and make $n=0$ in the others (therefore $B_0=B_3=0$):

$$\frac{M}{EhR}=-K_1+\frac{1-\nu}{2\alpha}\cot\varphi_c(K_1-K_2)-\frac{1-\nu}{2\alpha^2}K_2, \tag{7.2}$$

$$\frac{H}{Eh}=-\frac{N_{\varphi 0}}{Eh}\cos\varphi_c-\frac{\alpha}{\sin\varphi_c}(K_1+K_2), \tag{7.3}$$

$$\chi_c=\chi_0+2\alpha^3(K_2-K_1), \tag{7.4}$$

$$\frac{\xi_c}{R}=\frac{\xi_0}{R}+2\alpha^2\sin\varphi_c K_2-\alpha(1+\nu)\cos\varphi_c(K_1+K_2). \tag{7.5}$$

When these equations represent the membrane without load ($N_{\varphi 0}=0$, $\zeta=0$, $\chi_0=0$) and we make $M=1$, $H=0$ or $M=0$, $H=1$, by eliminating K_1 and K_2 and neglecting $1/\alpha$ in comparison with unity[10]), we find the well known coefficients of influence of GECKELER's solution:

$$\begin{aligned}(\xi_c)_{M=1}&=\frac{2\alpha^2}{Eh}\sin\varphi_c, & (\xi_c)_{H=1}&=-\frac{2\alpha}{Eh}R\sin^2\varphi_c,\\ (\chi_c)_{M=1}&=\frac{4\alpha^3}{EhR}, & (\chi_c)_{H=1}&=-\frac{2\alpha^2}{Eh}\sin\varphi_c.\end{aligned} \tag{7.6}$$

If we change the trigonometrical functions in the expressions (2.4) and the subsequent equations, we have as single magnitude which is not zero in

[10]) It should be noted that, for $H=0$, $K_1=-K_2$, and we could, in (7.2) make, $M/EhR=-K_1$, since α is very large (GECKELER). For $M=0$, we should have, for the same reasons, K_1 negligible as compared with K_2 and $K_1+K_2\approx K_2-K_1$. A more clearly approximate solution is obtained from (7.2) to (7.5) when these simplifications are not made.

the first members of expressions (6.3) to (6.14) (in addition to u_c/R if the dome is not completely prevented from turning):

$$S = -\frac{M_h}{2\pi R^2 \sin^2\varphi_c} = \frac{-R}{\sin^2\varphi_c}\int_0^{\varphi_c} p_u \sin^2\varphi \, d\varphi, \tag{7.7}$$

where M_h is the moment of the p_u.

For $n=1$ the constants to be determined are K_1, K_2, A_2^* and A_3^*. The last two of these constants characterize the movement of the dome as a rigid body, as described in § 4; the first two, alone, remain to be determined by two boundary conditions. Indeed, the four conditions are reduced to two, since among the magnitudes M, V, H and S there are two which are not arbitrary, for, not being independently balanced, there are among them the obligatory relationships (Fig. 8)[11]):

$$\int_0^{2\pi} H\cos\theta\cos\theta\, r\, d\theta \pm \int_0^{2\pi} S\sin\theta\sin\theta\, r\, d\theta + F_{ext} = 0,$$

$$\int_0^{2\pi} M\cos\theta\cos\theta\, r\, d\theta + \int_0^{2\pi} V\cos\theta\, r\cos\theta\, r\, d\theta + M_{ext} = 0.$$

In these expressions, $r = R\sin\varphi_c$ is the radius of the supporting circle, F_{ext} and M_{ext} are the resultant and the moment (about the diameter of the supporting circle) of the external active forces (p_u, p_v and p_w). Hence:

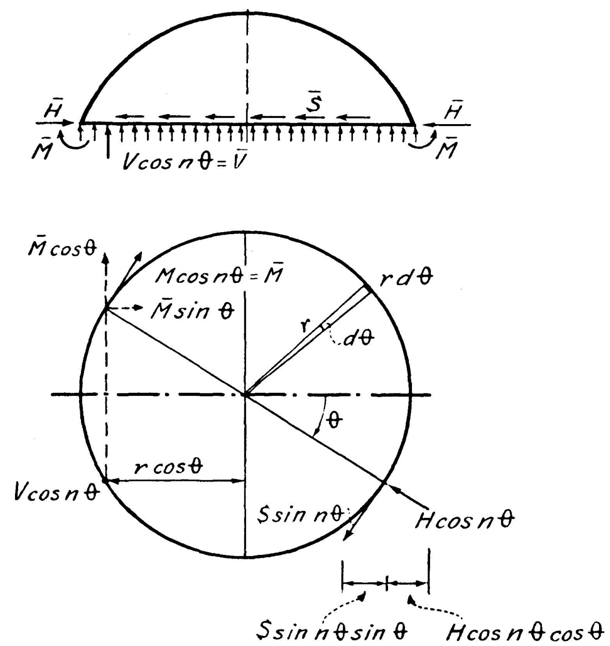

Fig. 8.

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11) The change in the trigonometrical functions of (2.4) and the following expressions has no influence on the results, for: $\int_0^{2\pi} \cos^2\theta\, d\theta = \int_0^{2\pi} \sin^2\theta\, d\theta = \pi$.

$$\pi r (H \pm S) = -F_{ext}, \qquad \pi r (M + V R \sin\varphi_c) = -M_{ext} \tag{7.8}$$

considering that

$$F_{ext} = \pi r (N_{\varphi 0} \cos\varphi_c \mp N_{\varphi\theta 0}), \qquad M_{ext} = \pi r^2 N_{\varphi 0} \sin\varphi_c, \tag{7.9}$$

and so:

$$\mp S = H + \frac{F_{ext}}{\pi r}, \qquad V = -\left(\frac{M_{ext}}{\pi r^2} + \frac{M}{r}\right). \tag{7.10}$$

H and M remain to be determined. In regard to the displacements, if we eliminate the movement of the dome as a rigid body, taking the horizontal plane of the edge and the vertical planes defined by $\theta = 0$ and $\theta = 0{,}5\,\pi$ as fixed, we have $\eta_c = u_c = 0$, and χ_c and ξ_c remain to be determined. Two of the four magnitudes H, M, χ_c and ξ_c will be given by boundary conditions and the other two will result from (6.3), (6.7), (6.12) and (6.13). Those two of these equations which correspond to the magnitude given by the boundary condition and the expressions (6.9) and (6.14), with the first member null, will determine the constants A_2^*, A_3^*, K_1 and K_2. All these expressions will have the term which contains A_1^* eliminated. By eliminating A_2^* and A_3^* with (6.9) and (6.14), the other four equations may be written (by neglecting $1/2\,\alpha^2$ in comparison with unity):

$$\frac{M}{E h R} = -K_1 B_1 - K_2 B_2, \tag{7.11}$$

$$\frac{H}{E h} = -\frac{N_{\varphi 0}}{E h} \cos\varphi_c - \frac{\alpha - \cot\varphi_c}{\sin\varphi_c} K_1 - K_2 \frac{\alpha}{\sin\varphi_c}, \tag{7.12}$$

$$\chi_c = \chi_0 - \frac{\eta_0}{R \sin\varphi_c} - 2\,\alpha^3 K_1 + 2\,\alpha^2 K_2 (\alpha - \cot\varphi_c), \tag{7.13}$$

$$\frac{\xi_c}{R} = \frac{\xi_0 \pm u_0}{R} - K_1 (1+\nu)\left(\alpha \cos\varphi_c - \frac{1}{\sin\varphi_c}\right) + K_2 [2\,\alpha^2 \sin\varphi_c - \alpha (1+\nu) \cos\varphi_c]. \tag{7.14}$$

It is interesting to observe that, to a first approximation (similar to GECKELER's), these equalities lead to the same coefficients of influence (7.6), as the axially symmetrical loading.

8. Dome Under Wind Loading

As a first example, let us take the spherical dome on radially displaceable suspensions (that is, $M = 0$ and $H = 0$) stressed by wind, whose pressure is normal to the surface[12]):

[12]) This law is frequently used (see, for instance FLÜGGE [7]) and coincides with that which would be verified in an immersed dam, having the form of a spherical calotte with a horizontal axis, after making allowance for the uniform pressure equivalent to that which occurs at the height of this axis.

$$\bar{p}_w = -p \sin\varphi \cos\theta\,, \qquad \bar{p}_u = \bar{p}_v = 0\,. \tag{8.1}$$

Let us determine the expressions of the forces for the particular case in which $\varphi_c = 60°$, $\nu = {}^1/_6$ and $R/h = 64$ (that is, $\alpha = 10{,}4$, from (2.3)):

To (8.1) there correspond $n = 1$ and

$$p_w = p \sin\varphi\,, \qquad p_u = p_v = 0\,. \tag{8.2}$$

These expressions when used in (3.3) and (3.2), give us:

$$N_{\varphi 0} = -\frac{pR}{3}(2+\cos\varphi)\cot\varphi \tan^2\frac{\varphi}{2}, \tag{8.3}$$

$$N_{\varphi\theta 0} = -\frac{pR}{3}(2+\cos\varphi)\operatorname{cosec}\varphi \tan^2\frac{\varphi}{2}, \tag{8.4}$$

$$N_{\theta 0} = -\frac{pR}{3}(3+4\cos\varphi+2\cos^2\varphi)\operatorname{cosec}\varphi \tan^2\frac{\varphi}{2}. \tag{8.5}$$

From (7.11) and (7.12), with $M = 0$ and $H = 0$, and $N_{\varphi 0}$ from (8.3) with $\varphi = \varphi_c$, the numerical data of the proposition applied:

$$0{,}9769\,K_1 + 0{,}0218\,K_2 = 0\,,$$
$$11{,}342\,K_1 + 12{,}009\,K_2 = 0{,}08019\frac{pR}{Eh} = 5{,}1320\frac{p}{E}$$

and we have the values of K_1 and K_2 which, when introduced into the formulæ of the table given in § 5, enable us to obtain the equations of all forces, which must be added to (8.3), (8.4) and (8.5), resulting in (ϕ_c and ϕ_s from (5.3)):

$$K_1 = -0{,}0097\frac{p}{E}, \qquad K_2 = 0{,}4365\frac{p}{E},$$
$$N_\varphi = N_{\varphi 0} + ph\,(2{,}576\,\phi_c - 2{,}097\,\phi_s)\,,$$
$$N_{\varphi\theta} = N_{\varphi\theta 0} + ph\,(5{,}132\,\phi_c - 5{,}067\,\phi_s)\,,$$
$$N_\theta = N_{\theta 0} + ph \cdot 91{,}85\,\phi_c\,,$$
$$M_\varphi = phR \cdot 0{,}4266\,\phi_s\,,$$
$$M_{\varphi\theta} = -pRh\,(0{,}0198\,\phi_s + 0{,}0195\,\phi_c)\,,$$
$$M_\theta = pRh\,(0{,}9359\,\phi_s + 0{,}0093\,\phi_c)\,,$$
$$Q_\varphi = ph\,(4{,}640\,\phi_s - 4{,}439\,\phi_c)\,,$$
$$Q_\theta = -ph\,(0{,}8730\,\phi_s + 0{,}0194\,\phi_c)\,.$$

These expressions can assume the form (5.6) for the employment of (5.8) and (5.9) or of the graph given in Fig. 6.

As support reactions we have ((6.8), (6.6) and (2.8)), in addition to $\bar{M} = 0$ and $\bar{H} = 0$:

$$\bar{V} = 0{,}139\,pR\cos\theta\,,$$
$$\bar{S} = -0{,}240\,pR\sin\theta\,,$$

these values being coincident with those that we could obtain by means of (7.9) and (7.10), with $H = 0$ and $M = 0$.

9. Dome Under Non-Uniform Heating

For a second example we shall take the case of a dome having the form of a spherical calotte with clamped edge ($\chi_c = 0$, $\xi_c = 0$) and under the effect

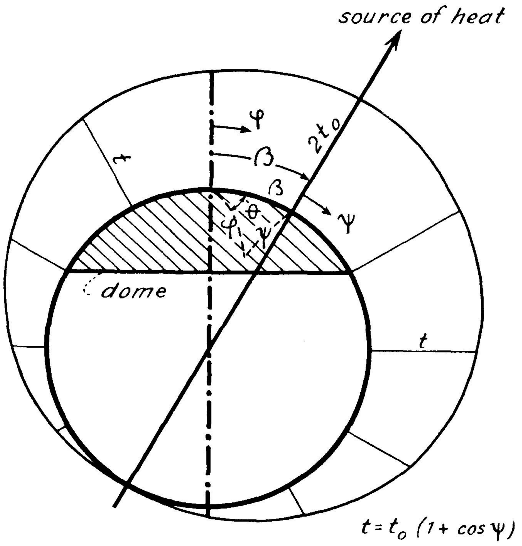

Fig. 9.

of heating which occurs, according to the law indicated in Fig. 9, axially symmetrically in relation to the axis that would reach the focus of heat, if it were prolonged. The measurement of θ will be made from a plane which contains that focus, denoting by β the angle that the straight line which unites the centre of the sphere with the focus, makes with the vertical. The modulus of elasticity E, the Poisson's ratio ν, the coefficient of thermal expansion α_t of the material are given, as well as the greatest difference $2t_0$ of temperature that could exist between two antipodal points, if the sphere were complete (Fig. 9). It is intended to draw the diagrams of forces for the case in which $E = 300\,\mathrm{t/cm^2}$, $h = 10\,\mathrm{cm}$, $\nu = {}^1/_6$, $\alpha_t = 10^{-5}/{}^\circ\mathrm{C}$ and $t_0 = 10^\circ\mathrm{C}$ with $\varphi_c = 60^\circ$, $\beta = 30^\circ$ and $R/h = 100$ (that is, $\alpha \approx 13$).

The law of variation of t given in Fig. 9 ($t/t_0 = 1 + \cos\psi$) can be expressed as a function of φ and θ, in the following way:

$$t = t_0 (1 + \cos\varphi \cos\beta + \sin\varphi \sin\beta \cos\theta) \tag{9.1}$$

as it is inferred from the well known cosine law of spherical trigonometry (Fig. 9).

The deformations that would take place if no contact existed among the adjacent elements, would be, in all directions:

$$\bar{\epsilon}_t = \alpha_t t, \qquad \gamma = 0. \tag{9.2}$$

In the formulæ that relate N_φ and N_θ to ϵ_u and ϵ_v (2.7) these deformations must be substituted respectively by $(\epsilon_u - \epsilon_t)$ and $(\epsilon_v - \epsilon_t)$. When these formulæ are introduced into the equilibrium equations of the membrane, without external loading action, we find that, with the variation law chosen for t (9.1), the expansion of the sphere occurs without the existence of forces:

$$\bar{N}_\varphi = \bar{N}_\theta = \bar{N}_{\varphi\theta} = 0. \tag{9.3}$$

Thus, we have:

$$\bar{\gamma} = 0, \qquad \bar{\epsilon}_u = \bar{\epsilon}_v = \bar{\epsilon}_t = \alpha_t t_0 (1 + \cos\varphi\cos\beta + \sin\varphi\sin\beta\cos\theta). \tag{9.4}$$

The study of the displacements can be made by considering separately the items corresponding to $n=0$ and $n=1$, that is, respectively:

$$\begin{aligned} \epsilon_u &= \epsilon_v = \alpha_t t_0 (1 + \cos\varphi\cos\beta), & \gamma &= 0, \\ \epsilon_u &= \epsilon_v = \alpha_t t_0 \sin\varphi\sin\beta, & \gamma &= 0, \end{aligned} \tag{9.5}$$

to which there correspond, in (3.13) and (3.14), null integrals ($\phi=0$, $N_\varphi - N_\theta = 0$), and there remains only the movement of the dome as a rigid body (§ 4) and the expansion that occurs because $\epsilon_v \neq 0$ in (3.11), that is:

$$\begin{aligned} &\text{for } n = 0: & w &= R\alpha_t t_0 (1 + \cos\beta\cos\varphi), \\ &\text{for } n = 1: & w &= R\alpha_t t_0 \sin\beta\sin\varphi. \end{aligned} \tag{9.6}$$

To the above equations there correspond the displacements of the edge:

$$\begin{aligned} &\text{for } n=0: & \xi_0 &= R\alpha_t t_0 \sin\varphi_c (1 + \cos\beta\cos\varphi_c), \\ & & \chi_0 &= -\alpha_t t_0 \cos\beta\sin\varphi_c, \end{aligned} \tag{9.7}$$

$$\begin{aligned} &\text{for } n=1: & \xi_0 &= R\alpha_t t_0 \sin\beta\sin^2\varphi_c, \quad \chi_0 = \alpha_t t_0 \sin\beta\cos\varphi_c, \\ & & \eta_0 &= R\alpha_t t_0 \sin\beta\sin\varphi_c\cos\varphi_c, \quad u_0 = 0. \end{aligned} \tag{9.8}$$

When these values are introduced, with $N_{\varphi 0}=0$, into (7.2) to (7.5) and (7.11) to (7.14), we obtain the solutions[13]), under the condition of clamped edge ($\xi_c=0$, $\chi_c=0$):

for $n=0$:

$$\begin{aligned} K_2 &= \frac{-\xi_0}{2\alpha^2 R\sin\varphi_c}, & K_1 &= K_2 + \frac{\chi_0}{2\alpha^3} \approx K_2, \\ H &= \frac{E h \xi_0}{R\alpha\sin^2\varphi_c}, & M &= \frac{E h \xi_0}{2\alpha^2\sin\varphi_c}, \end{aligned} \tag{9.9}$$

for $n=1$ (with $\eta_0 = R\chi_0\sin\varphi_c$ and $u_0=0$):

$$\begin{aligned} K_2 &= \frac{-\xi_0}{2\alpha^2 R\sin\varphi_c}, & K_1 &= K_2\left(1 - \frac{\cot\varphi_c}{\alpha}\right) \approx K_2, \\ H &= \frac{E h \xi_0}{R\alpha\sin^2\varphi_c}, & M &= \frac{E h \xi_0}{2\alpha^2\sin\varphi_c}. \end{aligned} \tag{9.10}$$

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[13]) With the same simplifications used to find (7.6), of the type of those described in the corresponding footnote.

Hence, on the whole[14]):

$$\begin{aligned} \overline{H} &= \frac{E\,h\,\alpha_t t_0}{\alpha \sin\varphi_c}(1+\cos\beta\cos\varphi_c+\sin\beta\sin\varphi_c\cos\theta)\,, \\ \overline{M} &= \frac{E\,h\,R\,\alpha_t t_0}{2\,\alpha^2}(1+\cos\beta\cos\varphi_c+\sin\beta\sin\varphi_c\cos\theta)\,. \end{aligned} \tag{9.11}$$

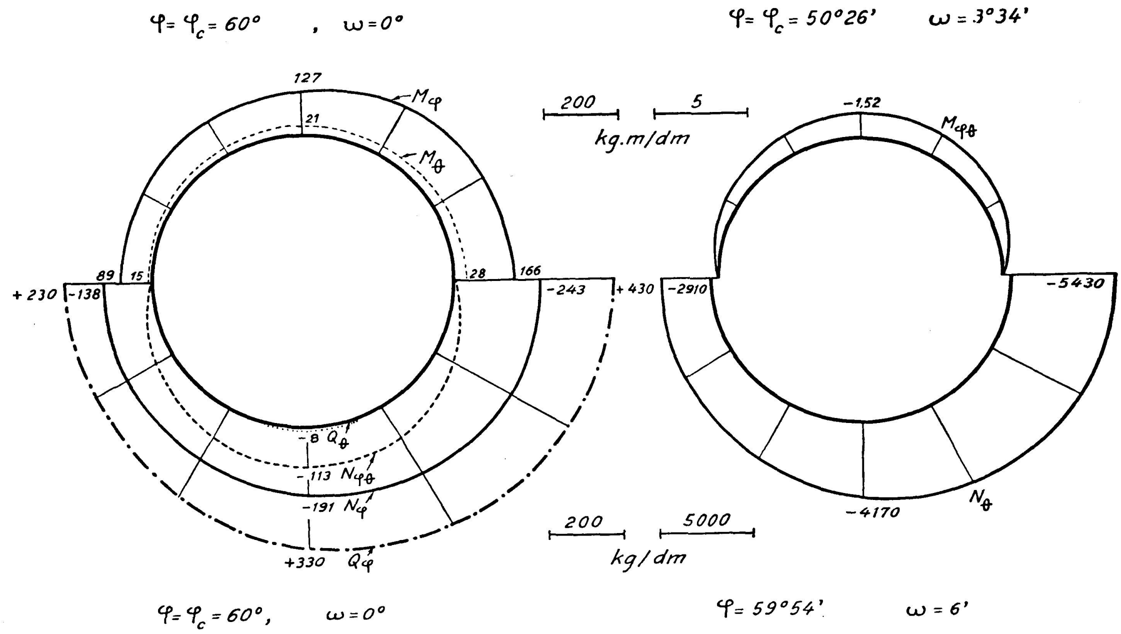

Fig. 10.

For the particular case corresponding to the numerical data suggested at the beginning of this § 9, the solution found leads to the internal forces given in Fig. 10, related to the parallel (ω), where the respective maximum absolute value occurs. The formulæ that determine these forces are those of the table given in § 5, since there are no forces in states of membrane and pure bending, where we make, for $n=0$:

$$K_1 \approx K_2 = \frac{-\alpha_t t_0}{2\,\alpha^2}(1+\cos\beta\cos\varphi_c) = -4{,}24\times 10^{-7},$$

$$E\,h\,K_1 = E\,h\,K_2 = -127 \text{ kg/m},$$

and for $n=1$:

$$K_1 \approx K_2 = \frac{-\alpha_t t_0}{2\,\alpha^2}\sin\beta\sin\varphi_c = -1{,}28\times 10^{-7},$$

$$E\,h\,K_1 = E\,h\,K_2 = -38{,}4 \text{ kg/m}.$$

[14]) It should be noted that the expressions in brackets, in (9.11), will be equivalent to $(1+\cos\psi_c)$, if we denote by ψ_c the angle ψ corresponding to the point of the edge under consideration (fig. 9).

10. Spherical Calotte with Compressed Edge

As a last example, we shall determine the variations of the radii of the parallels ($\bar{\xi}$) of the spherical calotte with $\varphi_c = 45°$, $\alpha = 10$ and $\nu = 0{,}25$, radially loaded, along the edge, by a distributed force (Fig. 11):

$$\bar{H} = \frac{P}{2}(1 + \cos 2\theta). \tag{10.1}$$

As this is the only force that acts on the edge of the dome, we shall have $\bar{M} = 0$, $\bar{S} = 0$ and $\bar{V} = 0$, in addition to $N_{\varphi 0} = N_{\varphi\theta 0} = N_{\theta 0} = 0$. By taking separately the items corresponding to $n = 0$ ($H = 0{,}5\,P$) and to $n = 2$ (also $H = 0{,}5\,P$),

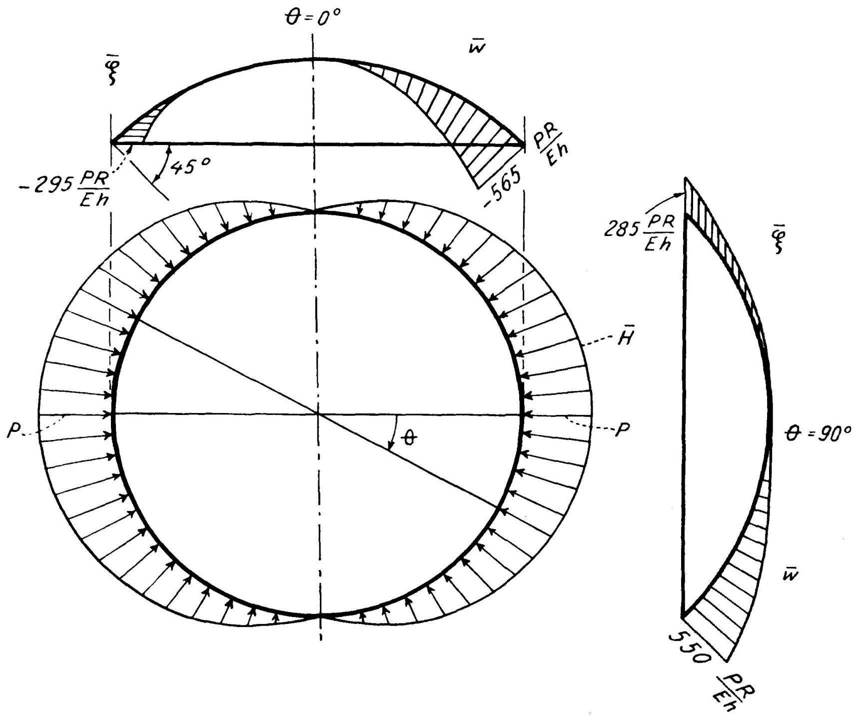

Fig. 11.

with $\varphi_c = 0{,}25\,\pi$, and applying (7.11) and (7.12,) in the first case, and (6.3), (6.6), (6.7) and (6.8) in the second case, we shall have:

for $n = 0$: $$\frac{77}{80}K_1 + \frac{33}{800}K_2 = 0, \qquad 9K_1 + 10K_2 = \frac{-P}{2Eh\sqrt{2}}, \tag{10.2}$$

hence, $$K_1 = 0{,}001576\,P/Eh, \qquad K_2 = -0{,}03677\,P/Eh, \tag{10.3}$$

for $n = 2$:
$$\begin{aligned} &2A_2^* B_3 + \frac{77}{80}K_1 + \frac{9}{800}K_2 = 0, \\ &2A_1^* - 2A_2^* B_3 - 18\sqrt{2}K_1 - 20\sqrt{2}K_2 = 0, \\ &\sqrt{2}A_1^* - 4A_2^* B_3 + 5{,}85\sqrt{2}K_1 + 10{,}135\sqrt{2}K_2 = \frac{-P}{2Eh}, \\ &\sqrt{2}A_1^* + 4A_2^* B_3 - 3{,}85\sqrt{2}K_1 - 0{,}135\sqrt{2}K_2 = 0, \end{aligned} \tag{10.4}$$

hence $K_1 = -0{,}04883\,P/E\,h, \qquad K_2 = +0{,}02457\,P/E\,h, \qquad (10.5)$

$A_2^*\,B_3 = 0{,}02336\,P/E\,h, \qquad A_1^* = -0{,}2507\,P/E\,h \qquad (10.6)$

and, using (6.1) and (6.2):

$$A_1 = -1{,}461\,P, \qquad A_2 = 1210{,}2\,P\,R/E\,h. \qquad (10.7)$$

There is, therefore, a state of membrane, with $n=2$, characterized by the constant A_1 to be introduced into (3.4) (forces) and into (3.16) and (3.17) (displacements); a state of pure bending, also with $n=2$, defined by the constant A_2 introduced into (4.3) (forces) and into (4.2), (4.4) and (4.5) (displacements); and a state of disturbance of the edge, partly axially symmetrical ($n=0$) and partly with $n=2$, characterized by the constants K_1 and K_2 respectively of (10.2) and (10.5), to be applied in the formulæ of § 5.

The displacements $\bar{\xi}$ to be determined, can be obtained through the formulæ, mentioned previously, for v and w, which enable us to write (with $\omega = 45° - \varphi$):

$$\frac{E\,h}{P\,R}\bar{v} = -0{,}479\,e^{-10\omega}\sin 10\,\omega + 0{,}440\,e^{-10\omega}\cos 10\,\omega + [0{,}917\,e^{-10\omega}\sin 10\,\omega + 0{,}303\,e^{-10\omega}\cos 10\,\omega - 0{,}304\,(3 + 2\cos\varphi + \cos^2\varphi)\,\mathrm{cosec}\,\varphi\tan^2(\varphi/2) + 1210{,}2\sin\varphi\tan^2(\varphi/2)]\cos 2\,\theta.$$

$$\frac{E\,h}{R\,P}\bar{w} = 0{,}315\,e^{-10\omega}\sin 10\,\omega - 7{,}354\,e^{-10\omega}\cos 10\,\omega - [9{,}766\,e^{-10\omega}\sin 10\,\omega - 4{,}914\,e^{-10\omega}\cos 10\,\omega + 1210{,}5\,(2 + \cos\varphi)\tan^2(\varphi/2)]\cos 2\,\theta.$$

These expressions, when introduced into (2.12), lead to the required values of $\bar{\xi}$ which are shown in Fig. 11, together with those of $\bar{w}$, for $\theta = 0°$ and $\theta = 90°$. In the scale of the drawing the oscillating part of the diagram cannot be indicated. This part corresponds to the state of disturbance of the edge, but there is marked predominance of the state of pure bending, as it is inferred from the value of A_2 in comparison with those of A_1, K_1 and K_2. It will not be the same for the moments, since $A_2^*\,B_3$ is of the same order of magnitude as K_1 and K_2.

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Summary

An approximate theory, easy to employ and of sufficient precision, has been developed for domes having the form of a not very flat spherical calotte, under any asymmetrical loading, provided that it can be represented by rapidly convergent Fourier's series.

The three effects of this loading are studied separately: the effect of membrane, of pure bending and of disturbance of the edge. They are subsequently superimposed and equations are derived which enable us to determine the integration constants.

Application of this theory is made to domes exposed to wind, to those which are subjected to non-uniform heating and to shells compressed along the edge.

Résumé

Une théorie approchée, facile à utiliser et de précision suffisante a été développée pour le calcul des coupoles chargées asymétriquement, ayant la forme de calottes sphériques pas trop plates, en admettant que la charge puisse être développée en séries de Fourier qui convergent rapidement.

Les trois effets de cette charge: efforts dus à l'état de membrane, à la flexion pure et aux perturbations des bords, sont étudiés séparément. On tire de leur superposition subséquente des équations qui permettent la détermination des constantes d'intégration.

Cette théorie est ensuite appliquée au calcul de coupoles soumises au vent, à un chauffage non-uniforme, ainsi qu'à une coque comprimée le long de ses bords.

Zusammenfassung

Für unsymmetrisch belastete Kuppeln, welche die Form einer nicht sehr flachen Kugelhaube aufweisen, wurde eine leicht anzuwendende und genügend genaue Näherungstheorie entwickelt, unter der Bedingung, daß die Belastung durch rasch konvergierende Fourierreihen dargestellt werden kann.

Die drei Folgen dieser Belastung: Membran-, reiner Biegungs- und Randstörungszustand, werden getrennt untersucht. Aus ihrer nachfolgenden Superposition werden Gleichungen abgeleitet, welche die Bestimmung der Integrationskonstanten erlauben.

Diese Theorie wird auf den Fall einer windbelasteten und einer ungleichmäßig beheizten Kuppel sowie einer längs den Rändern gedrückten Schale angewendet.