Zeitschrift: IABSE publications = Mémoires AIPC = IVBH Abhandlungen
Band: 20 (1960)

Artikel: A general analysis of elasto-plastic three-dimensional frames

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DOI: https://doi.org/10.5169/seals-17551

A General Analysis of Elasto-Plastic Three-Dimensional Frames

Etude générale des cadres tridimensionnels élasto-plastiques

Eine allgemeine Untersuchung von elasto-plastischen 3-dimensionalen Rahmen

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Introduction

The purpose of this paper is to establish, by the Principle of Virtual Work, General Analytical Equations and to present a method of design for elasto-plastic three-dimensional frames, taking account of all internal resistances. Generally, a special and simpler procedure for uni-planar reinforced concrete frames subject to deformation due to bending only, is all that is required. However, cases do arise when the influence of torsion or axial force on deformation must be considered. It is important, too, to be able to determine when deformations due to causes other than bending are negligible; for instance, torsional strain induced by a secondary beam causing dangerous shear stresses in a main beam. It is probable that, in building frames generally, the influence of deformations apart from those due to bending is negligible, at any rate in regard to ultimate strength criteria, except, perhaps, for brittle conditions, such as may occur in beams not reinforced for shear. More tests are required, so that safe limiting values of the parameters governing shear and torsion strength and deformation for various shapes of section, and the spread of plasticity at releases, can be firmly established by statistical treatment of the inevitable scatter of the results. An important point emerges in favour of basing designs on ultimate criteria. The application of linear equations to homogeneous elastic three dimensional systems becomes very complex when cracking, as in reinforced concrete, affects the stiffness of sections. When an assumed ultimate distribution of bending moments etc. has been made, the neutral axis position and cracked zones of all sections can be determined, and hence the required stiffness factors.

Definition of Elasto-Plastic Members

For the purpose of this analysis, the essential characteristics of elasto-plastic frame members are that, when subject to bending, torque, shear or axial load of uniform distribution along a member, the load-deformation diagram for the member approximates to the idealised diagram in Fig. 1. Deformations are angular in the case of bending and torsion, uni-directional in the case of shear or axial load. Analytical calculations are based on such limits as L_1 and L_2 (ref. Fig. 1), so that calculated strength values are less than actual values, but calculated elastic deformation greater than actual values. Maximum permissible plastic deformation values are less than possible values. Occasionally, when studying research results, it may be necessary to use a limit such as L_3 over parts of a frame deforming elastically in a direction which produces negative values of calculated plastic deformation, as indicated by the sign and influence of the terms of the general equations.

Notation

Ref. Fig. 2 (a)

X_k^1 = unknown force acting in direction X at section K.
X_k^2 = unknown force acting in direction Y at section K.
X_k^3 = unknown force acting in direction Z at section K.
X_k^4 = unknown moment acting in plane OXY at section K.
X_k^5 = unknown moment acting in plane OXZ at section K.
X_k^6 = unknown moment acting in plane OYZ at section K.

Ref. Fig. 2 (b)

f_{ki}^{11} = deformation at section K in direction of X_k^1 due to $X_i^1 = 1$ acting.
⋮
f_{ki}^{nn} = deformation at section K in direction of X_k^n due to $X_i^n = 1$ acting.
⋮
f_{ki}^{66} = deformation at section K in direction of X_k^6 due to $X_i^6 = 1$ acting.

U_k^1 = deformation at section K in direction of X_k^1 due to external load acting.
⋮
U_k^n = deformation at section K in direction of X_k^n due to external load acting.
⋮
U_k^6 = deformation at section K in direction of X_k^6 due to external load acting.

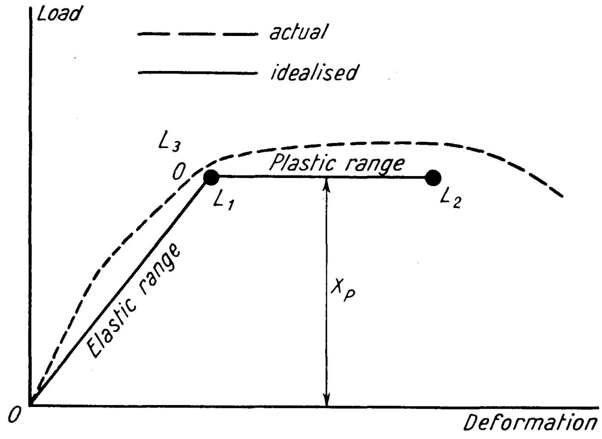

Fig. 1. Typical Load/Deformation Diagram.

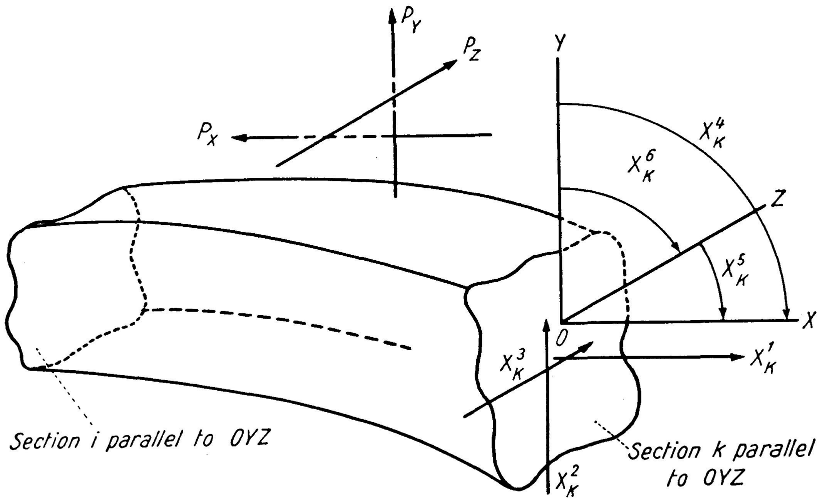

Fig. 2a. Typical Isolated Member.

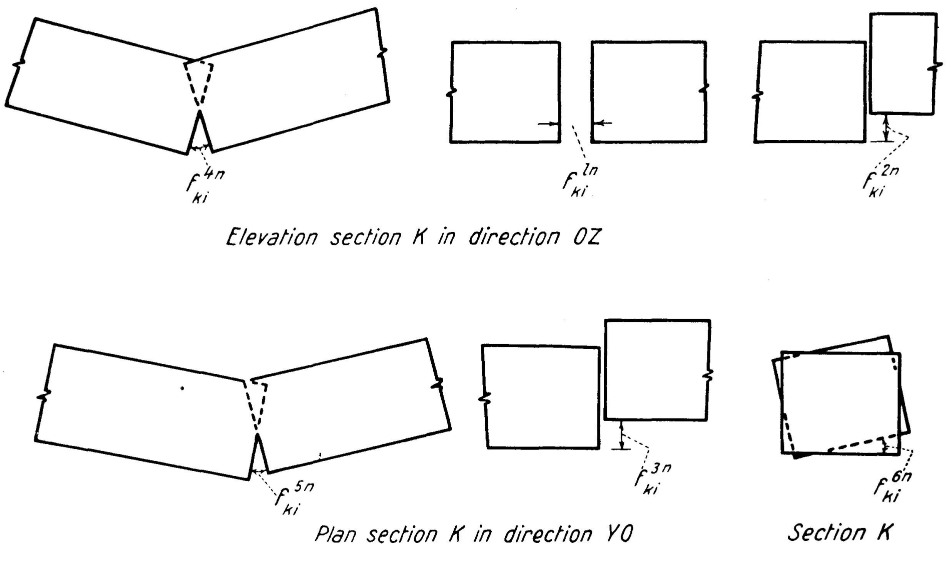

Fig. 2b. Typical Relative Deformations at Releases.

a_k^1 = movement at section K in direction of X_k^1 due to other influences[1]).

⋮

a_k^n = movement at section K in direction of X_k^n due to other influences.

⋮

a_k^6 = movement at section K in direction of X_k^6 due to other influences.

ω_k^n = ordinate at any section of frame of distribution diagram for $X_k^n = 1$.

ϵ^n = displacment in direction of application of ω_k^n over unit length of member caused by unknowns such as $\omega_k^n = 1$ acting.

Equations of Equilibrium

Each force of any system of forces acting on a body can be replaced by components in directions OX, OY, OZ parallel to three axes of reference mutually at right angles (ref. Fig. 2). Each set of components in each direction can be combined to give either a single resultant in that direction or equal and opposite resultants, i.e. a couple. The external forces or loads acting on a body can therefore be reduced to three forces, such as $P_X\, P_Y\, P_Z$, acting in directions OX, OY, and OZ, or pairs of equal and opposite forces, such as $P_X\, P_Y$ and P_Z.

Any member of a frame can be isolated from the frame by cuts such as those at sections i and k (Fig. 2). The cuts may be made parallel to the most convenient of the planes of reference OXY, OYZ, OZX. Section K parallel to plane OYZ may be considered as a typical section or cut. The cut releases all internal restraints which act across section K when the frame is loaded and there is full continuity. If, after making the cut, the internal restraints are replaced by equal and opposite external restraints or support reactions which are applied to the members on either side of the cut, the continuity of the member in effect will be restored. At a section such as K, all such restraints can be resolved into components and reduced to resultants which are sometimes referred to as stress resultants, such as:

X_k^1 = a force in direction X (axial force)
X_k^2 = a force in direction Y (shear)
X_k^3 = a force in direction Z (shear)
X_k^4 = a moment in plane OXY (moment)

---

[1]) Other influences include
(1) Plastic movement.
(2) Creep movement.
(3) Shrinkage.
(4) Support movement.
(5) Temperature movement.

X_k^5 = a moment in plane OXZ (moment)
X_k^6 = a moment in plane OYZ (torque)

The value and location of action of the above forces or moments will depend on the unit stress-values and their distribution throughout the section. Each cut, such as K, may therefore release six resultant restraints. It is convenient to assume that restraints can be released separately or in groups, if required, by inserting suitable imaginary devices. It may help to visualise such devices as follows:

1. for moments — a frictionless hinge acting in one plane;
2. for shear — a frictionless slide acting in one direction;
3. for axial forces — a frictionless slide acting in one direction;
4. for torsion — a frictionless swivel acting in one plane.

Generally, such imaginary devices may be referred to as "*Releases*" or more particularly, say, a shear release or, say, "X_k^n release", which means that the release, when inserted, reduces to zero the internal force or moment X_k^n. When internal restraints, such as X_k^n, are replaced by equal and opposite external forces or moments X_k^n applied to the members on either side of the release, the external force or moment X_k^n is referred to as a *Re-action* or *Support Re-action* when such forces or moments are considered as supports to a loaded member or body. When considered as unknown external forces or moments acting on members of the frame, when made statically determinate, by the insertion of a sufficient number of releases, such forces or moments are referred to as "*unknowns*".

A body or member, as in Fig. 2, is generally isolated from a frame by two cuts, such as at sections i and k, or one cut, if the frame member is a cantilever, or one cut, if the frame is an isolated ring. The member between the cuts may, in principle, have any shape. The following conditions of equilibrium must apply:

1. The sum of the resultant forces acting on the member in each of the three directions of reference must be zero.
2. The sum of the moments of the resultant forces acting on the member in each of the three directions of reference about each axis of reference must be zero (moments, either as loads or support reactions, being regarded as equal and opposite forces appropriately spaced apart).

The above conditions provide six equations for each isolated member, referred to as the "*Equations of Equilibrium*".

When all loads and reactions are in one plane, such as OYX, condition (1) gives two equations and condition (2) only one equation, making a total of three equations.

Degree of Statical Indeterminacy

A space frame may be divided by cuts into isolated members, such as in Fig. 2, their positions being selected as shown in the typical frame Fig. 3, so that each restraint which may act on the isolated member is released. Where several members intersect at a joint, a cut in one of the members can be omitted, since it is already isolated by the other cuts adjacent to the joint.

It is assumed that each support is joined by an infinitely short connection to the frame. When the supports are rigidly fixed in position and direction, a cut in the support connection releases six possible restraints, as in typical section K (Fig. 2). In any space frame, let:

$\overline{M}$ be the number of isolated members;
$\overline{C}$ be the number of cuts required to isolate the members;
$\overline{R}$ be the number of real releases to be constructed in the frame or to exist at the supports.

Real releases, as distinct from imaginary releases, are devices such as hinges or sliding bearings, which may be included in the construction of the frame in order actually to release restraints which would otherwise act under load. Freedom of a support to move is equivalent to inserting one or more real releases according to the number of degrees of freedom. Thus, the releases, to be assumed as "Real Releases" in the support connection, must release the same restraints which movement of the support would release, if the connection were fully continuous.

For each isolated member, there are six equations of equilibrium, and for each cut six unknowns. A frame is therefore $6\overline{C}-6\overline{M}-\overline{R}$ times statically indeterminate, the degree of statical indeterminacy being the number of restraints remaining unknown after solving the equations of equilibrium.

Applying the above expression to the frame in Fig. 3:

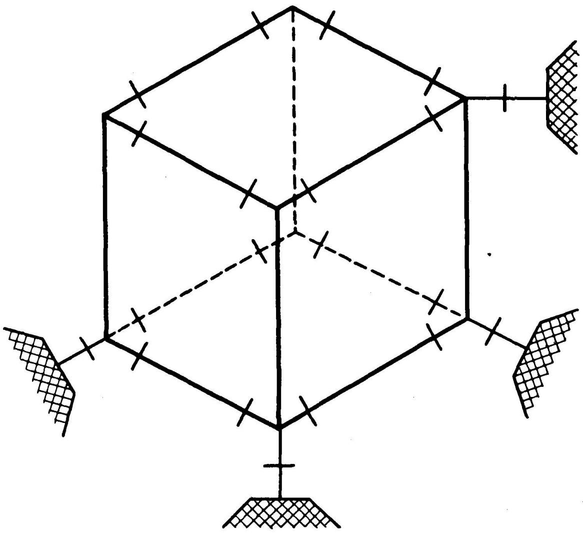

Fig. 3. Typical Space Frame.

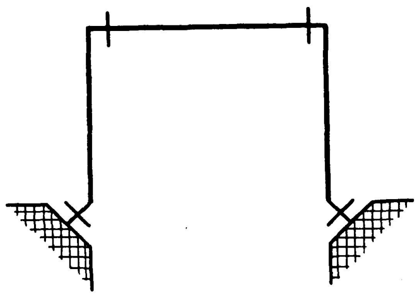

Fig. 4. Typical Space Frame.

$$\bar{C} = 20, \quad \bar{M} = 12, \quad \bar{R} = 0.$$

The frame is therefore 48 times statically indeterminate.

In the frame in Fig. 4, which is in one plane,

$$\bar{C} = 4, \quad \bar{M} = 3, \quad \bar{R} = 0.$$

The frame is therefore three times statically indeterminate.

Equations of Deformation

At a sufficient number of sections, such as K, in a frame which is $6\bar{C} - 6\bar{M} - \bar{R}$ times statically indeterminate, $6\bar{C} - 6\bar{M} - \bar{R}$ imaginary releases are inserted, so that the frame as a whole and in regard to each member remains stable, under any external load; i.e. so that the equations of equilibrium can be satisfied. The frame is then statically determinate, since each isolated member is acted on by only six unknown restraints, and there are six equations of equilibrium for each member. It is then possible to divide the loads acting on the frame into $(6\bar{C} - 6\bar{M} - \bar{R}) + 1$ cases, as follows:

1. The external load.
2. Remaining cases. Each of the selected unknowns, such as X_k^n, assumed equal to unity, acting in opposite directions on the members on either side of the imaginary releases.

The distribution of the internal restraints, caused by external load and each unknown such as $X_k^n = 1$, along each member can then be found and plotted along each member. From such diagrams, values of U_k^n and f_{ki}^{nn} can be found. Then, at any section K, to satisfy the condition of continuity in the frame, the resultant deformation due to all loads, unknowns and other influences in the direction of each restraint, is zero. Thus, the following general equation is established:

$$U_k^n + \sum X_k^n f_{ki}^{nn} + a_k^n = 0. \tag{1}$$

Evaluation of U_k^n and f_{ki}^{nn}

The internal restraints, such as ω_k^n, together with $X_k^n = 1$, form a balanced system of forces acting on certain members of the frame. If another influence, such as the external load or the unknown $X_i^n = 1$, acts and deforms the frame, the virtual work done by the internal restraints ω_n^k is equal to the virtual work done by $X_k^n = 1$, which $= 1 \cdot f_{ki}^{nn}$.

Each short length ds of the frame is deformed $\omega_i^n \epsilon_n \, ds$ when $X_i^n = 1$ acts. The virtual work, therefore, done over a short length ds by ω_k^n is $\omega_i^n \epsilon_n \omega_k^n ds$.

Then

$$f_{ki}^{nn} = \int \omega_i^n \epsilon_n \omega_k^n \, ds \tag{2}$$

deformations being integrated over any part of the frame where restraints due to X_k^n and X_i^n act and internal virtual work is done. The value of U_k^n can be evaluated in a similar way only substituting restraints due to external load for those due to $X_i^n = 1$.

Eq. (2) may be applied with respect to each kind of unknown, e.g. when X_k^n and X_i^n are moments referring to ω_k^n as

$$M_k \text{ and } \omega_i^n \text{as } M_i. \quad \text{Since } \frac{M}{I} = \frac{E\,e}{n}, \quad \frac{M}{E\,I} = M\,\epsilon_n.$$

Where M = bending moment, I = moment of inertia of section, e = strain at compression edge, E = Young's modulus of material, n = depth of neutral axis

$$\therefore\ f_{ki}^{nn} = \int \frac{M_i M_k}{E\,I} d\,s. \tag{3}$$

Solution of Deformation Equations

Elastic case (1). All deformations equal to, or less than, L_1 value (Fig. 1)

From the general Eq. (1) $6\overline{C} - 6\overline{M} - \overline{R}$ linear equations are developed having $6\overline{C} - 6\overline{M} - \overline{R}$ unknowns and may be solved by relaxation or matrix methods. There is considerable scope for selecting releases which result in well-conditioned equations (ref. 1).

Ultimate or Plastic Case (2). All deformations at release sections lie between the values L_1 and L_2 (Fig. 1)

It is generally possible to design and analyse a frame by a process of trial and adjustment applied to the X values until Eq. (1) is satisfied at all release sections. The X values are no longer unknown, since they are equal to $X\,p$ (ref. Fig. 1) the plastic restraint value for the assumed section. There is, however, an unknown plastic deformation at each restraint section in a direction opposed to each restraint which may be referred to as a_k^p.

Eq. (1) then becomes:

$$U_k^n + \sum \overline{X}_k^n f_{ki}^{nn} + \overline{a}_k^n = -a_k^p. \tag{4}$$

Where $\overline{X}_k^n$ indicates a known plastic value of X_k^n and $a_k^n = \overline{a}_k^n + a_k^p$. All values on the left of the equation are, therefore, known and it is only necessary to check the value of a_k^p for the following reasons (1) unless the sign of a_k^p has the same sign as X_k^p, members may not remain elastic as assumed between the release sections.

(2) The value of a_k^p must be less than a permissible value related to the deformation range between the limits L_1 and L_2, and the spread of the plastic zone along the member.

In designing frames, it is generally not difficult to adjust $\bar{X}$ values to satisfy (1), if they are moments and axial forces particularly if release sections have been assumed at critical sections having maximum stress under elastic conditions. Rules for determining permissible values of a_k^p are best established from tests (refs. 2 and 3).

It may sometimes be necessary to reduce sections of members locally over a short length including a release section to satisfy requirement (1). For the practical purpose of obtaining uniformity, stronger sections may be used in the actual structure without risk of weakening the frame seriously, due to increasing deformations at other sections. The spread of plasticity which actually occurs beyond the release sections does not invalidate the equations, since such plastic deformation does not alter in sign when spread, and can be assumed to be concentrated at the release section.

Reinforced Concrete Frames — Notation

E = secant value of concrete or steel at limit L_1.
I = "Moment of Inertia" of "Equivalent Section".
M_i = bending moment ordinate on member due to $\bar{X}_1 = 1$ acting.
M_k = bending moment ordinate on member due to $\bar{X}_k = 1$ acting.
N_i = axial force on member due to $N_i = 1$ acting.
N_k = axial force on member due to $N_k = 1$ acting.
A = area of web of beam assumed to resist shear.
S = total shear acting on a section.
S_i = shear acting at a section due to $\bar{X}_i = 1$ acting.
S_k = shear acting at a section due to $\bar{X}_k = 1$ acting.
L_p = "equivalent plastic length" on one side of a release section.
ϵ_p = deformation per unit length between limits L_1 and L_2 at a release section.
$L_p\,\epsilon_p$ = total deformation between limits L_1 and L_2 on one side of a release section, assumed to be concentrated at the section.
L' = distance from a release section to the point at which deformations just reach limit L_1.
d = depth of section.
e_p = strain of concrete between L_1 and L_2.
$n_u d$ = depth of neutral axis for bending strains at L_2.
e_d = difference of strains across a section with the neutral axis outside the section at limit L_2 in bending.

Evaluation of $\int \omega_i^n \varepsilon_n \omega_k^n \, ds$

The distribution of ω_i and ω_k may be indicated in diagrams as shown in Fig. 5. It is important to use such diagrams in order to ensure that ω_i and ω_k

cause deformations of the same category in the same member. The indices cannot always be used as a guide as on plane frames subject to bending only. The value of ϵ_n at each section of the frame will depend on the category of

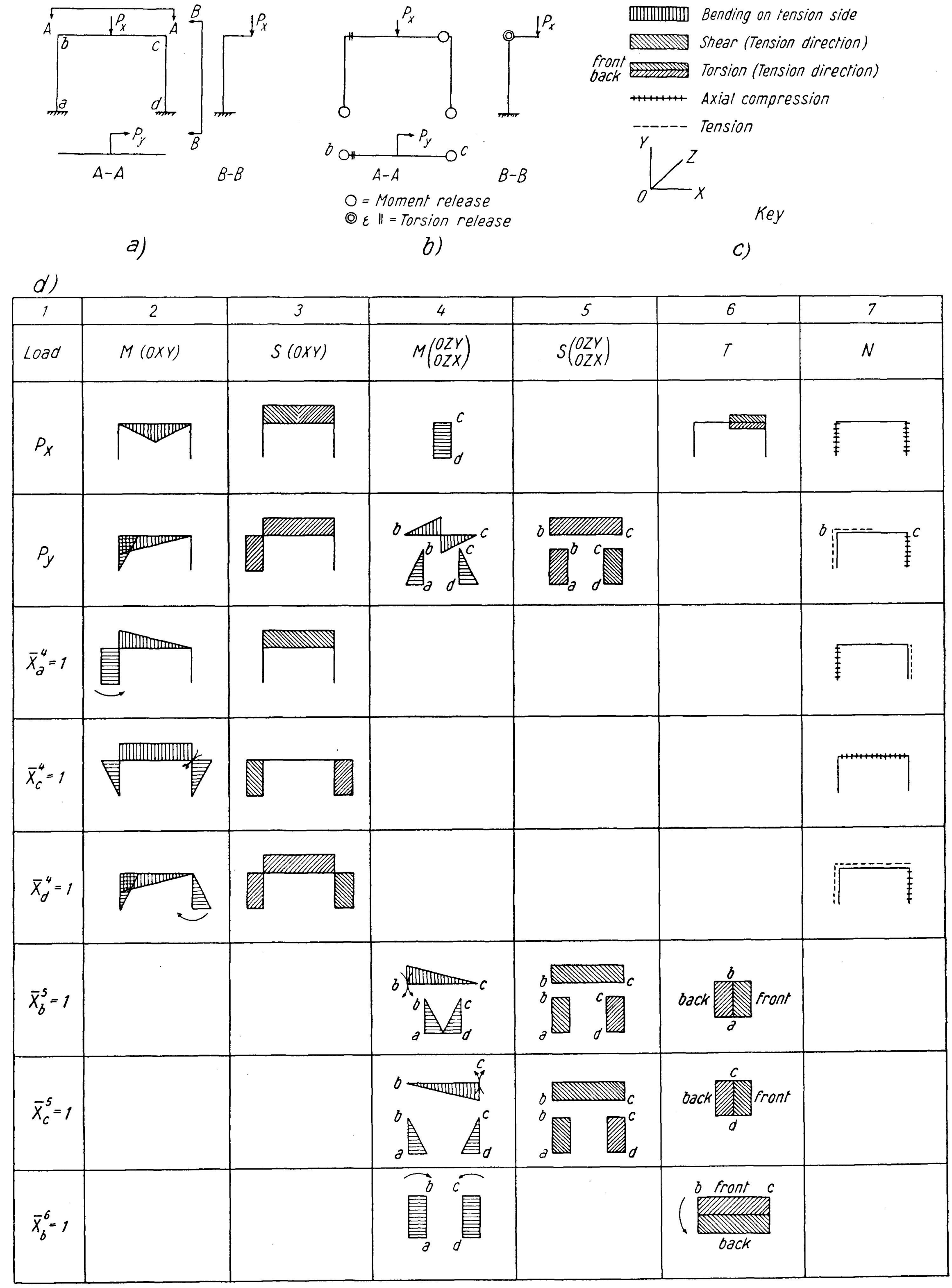

Fig. 5. Table for Integration and Specific Deformation Categories.

the ω terms, i. e. whether they are bending, shear, torsion or direct stress. The combined effect of these influences operating between 0 and limit L_1 over each part of the frame must also be considered in regard to the effect of cracking on the stiffness of the section.

Bending

The following assumptions are made (ref. Fig. 6):

1. The distribution of strain across a section is linear.
2. The distribution of compressive stress in the concrete is parabolic.
3. Concrete resists no tension.
4. Limit L_1 is reached when, according to the neutral axis position, either the steel begins to yield or the concrete develops a strain of 0.002. An arbitrarily selected yield point must be assumed in the case of cold worked and high tensile steels (ref. Fig. 7).

When the $\bar{X}$ values have been assumed, the resultant stresses at any section due to combined bending and axial force and their distribution may be found, and hence the resultant neutral axis and compression zone defined. For the purpose of determining the I value of the section, an equivalent area

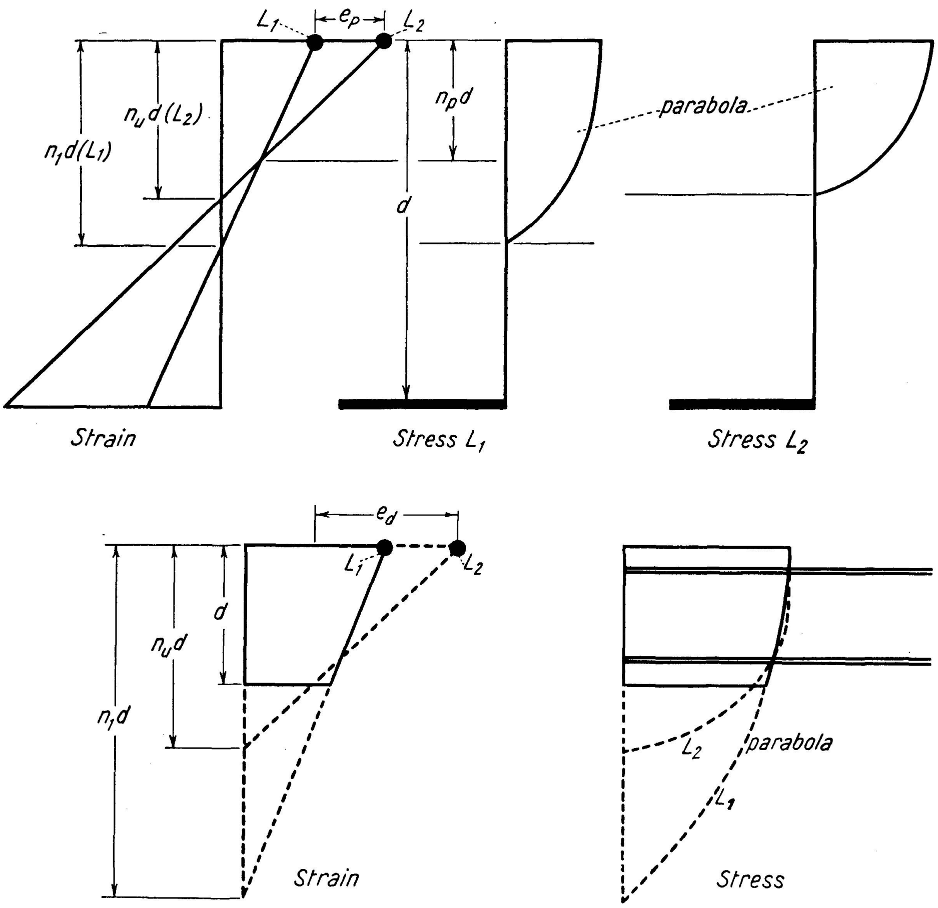

Fig. 6. Strain and Stress Distributions.

may then be drawn (see Fig. 8a). The full line is derived from the dotted line by increasing each horizontal ordinate in the ratio $\frac{ac}{ab}$ (Fig. 8c). The equivalent steel area is found by multiplying the actual area by $m-1$, where $m = \frac{E \text{ for steel at } L_1}{E \text{ for concrete at } L_1}$. The equivalent area may then be used as though the section were homogeneous in order to find the value of I for bending in each of the directions of reference. ϵ_n then $=\frac{1}{EI}$ with the appropriate value of I in each direction of reference and E the secant value for concrete at L_1.

Then for bending $$\int \omega_i^n \epsilon_n \omega_k^n \, ds = \int \frac{M_i M_k}{EI} ds. \qquad (5)$$

The value of EI is approximately constant form 0 to L_1, because E reduces as the stress distribution bulges. The value at L_1 may therefore be assumed for ultimate deformation calculations.

Axial Force

The cross section of a member may be assumed to be the equivalent area A_1 (Fig. 8a).

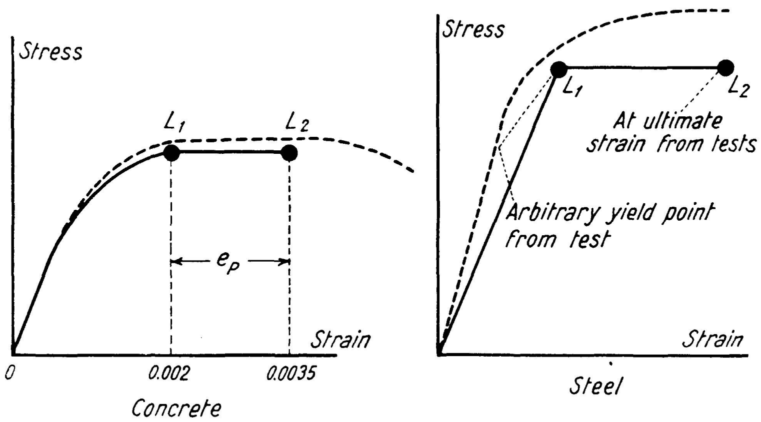

Fig. 7. Stress-Strain Relations.

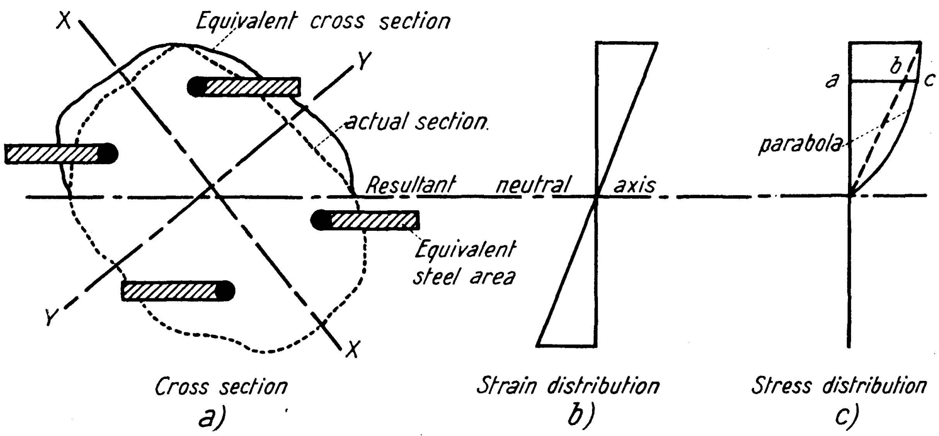

Fig. 8. Typical Combined Bending and Compression Stress Distributions.

Then $$\int \omega_i^n \epsilon_n \omega_k^n \, d\, s = \int \frac{N_i N_k}{E A_1} d\, s. \tag{6}$$

E being the secant value for concrete at L_1.

Shear

Since the tensile strength of concrete is uncertain and failure by tension is brittle, unreinforced sections should be avoided. When shear reinforcement is provided, it is assumed to act as the diagonal or vertical members of a series of pin-jointed frames (Fig. 9b) and c)), in which diagonal compression

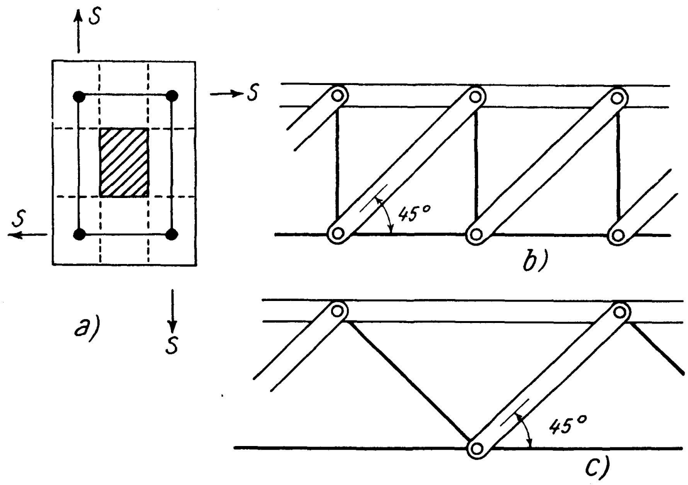

Fig. 9. Typical Truss Systems for Shear Resistance.

is resisted by concrete members. It is assumed that the frames can be superimposed regardless of their relative stiffness, and that the concrete in diagonal compression governs the stiffness in shear. When the correct amount of shear reinforcement is provided, so that at L_1 the maximum diagonal compression in the concrete just reaches ultimate unit strength, the average diagonal strain in the concrete, assuming a parabolic distribution, will be about $\frac{S}{1.5 A E}$, where E is the secant value for concrete at L_1 when the strain is 0.002. The shear deformation per unit length will then be about $\frac{2S}{1.5 A E}$.

Then $$\int \omega_i \epsilon_n \omega_k d\, s = \int \frac{4\, S_i\, S_k}{3 A E}. \tag{7}$$

Torsion

As in shear, the strength of concrete in tension being uncertain, failure being brittle, unreinforced sections for torsion should be avoided, except where pre-stressing provides an adequate margin of safety. Information regarding the strength and deformability of such sections, and the influence of bending compression and pre-stress is given in reference 4. L_1 and L_2 must be assumed to coincide, since plastic deformation is negligible.

When reinforced diagonally to resist torsion tension, a rectangular beam may be treated as a box section (ref. Fig. 9a)) with each side designed to resist the combined shear due to torsion and normal shear. The shear deformation in each side per unit length will then be about $\frac{2S}{1.5AE}$ as in the case of normal shear, but where S is the shear due to torsion only. It is assumed that when each side is designed for resultant shear under combined shear and torsion, that resultant shear will develop at limit L_1 for the combined shears in each face. Test results are required to show to what extent modification of this assumption may be required for sections of small width in relation to depth. The virtual work done by forces ω_i when forces ω_k act, may be considered for each side of the box section separately, and the total obtained by addition, so that $\int \omega_i \epsilon_n \omega_k \, d s = \int \frac{4 S_i S_k}{3AE}$ for each side of the box section, S_i and S_k being shear due to torsion only. The value of a_k^p due to torsion is the sum of all such integrals worked out for each side of the box section.

Permissible Values of a_k^p

(see Note below on research)

Generally, permissible values of

$$a_k^p = L_p \epsilon_p . \tag{8}$$

Values of L_p (see Fig. 10)

In the case of bending or combined bending and compression, the value of L_p on one side of the critical section is about $L'/2$ or $d/2$, whichever is the greater. In the plastic zone, shear is not fully effective, and the stress at the critical section tends to spread and be uniform on either side at a distance of

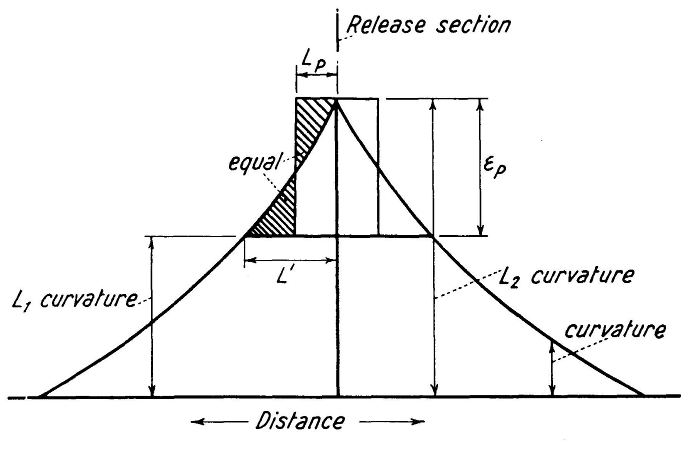

Fig. 10. Typical Curvature Distribution along Member.

about $d/2$. In compression, an approximately semi-circular plastic zone develops, and in tension ductility in the steel cracking and bond slip of the concrete cause local truss action.

The value of L' depends on the gradient of the actual load/deformation curve between L_1 and L_2, and may be assumed as a safe value to be equal to the length over which the bending compressive stress in the concrete reduces by 10%, according to the idealised assumptions.

In the case of shear or torsion, $2\,L_p$ may be assumed $= d$.

Values of ϵ_p

Bending. Cracked sections $\epsilon_p = \frac{e_p}{n_u d}$ within safe limits (see Fig. 6).

Uncracked sections $\epsilon_p = \frac{e_d}{d}$ (approximately).

Safe limiting values, $e_p = e_d = 0.0015$ unbound concrete,

$e_p = e_d = 0.01$ well-bound concrete.

Value of $n_u d$ is based on linear strain distribution and concrete strain at $L_2 = 0.0035$.

Axial Force. $\epsilon_p = e_p$ safe limiting value $= 0.0015$.

Shear $\epsilon_p = \frac{4}{3AE}$ safe limit value of E = secant value for concrete at $L_1 \cdot \frac{0.0035 - 0.002}{0.002}$.

Torsion ϵ_p as for shear on each side of box section.

Influence of Pre-stress

When a frame has been made statically indeterminate by the insertion of releases, the pre-stressing forces act in the same way as external loads applied to the sections. The influence of pre-stressing may therefore be investigated by introducing the pre-stress forces as additional external loads. This has been discussed and the influence of cracking in reference 2.

Future Research

Many load/deformation curves in regard to bending in one plane have now been obtained from tests, and safe-limiting values of parameters governing L_1 and L_2 deduced. More work, however, requires to be done in regard to bending in two planes and in regard to shear and torsion, in order to base safe limiting values of all the parameters governing deformations on a wide variety of tests. The influence of combined shear and bending on ultimate strength and deformation also requires careful study. The parameter values given above may be recommended in regard to bending in one plane, but for other cases should be considered to be approximate only.

Examples Indicating Procedure of Calculating a_k^p Values

Many examples have already been published (see refs. 2 and 3) for plane frames ignoring the influence on deformation of shear torsion and axial force. In practice, it is usually sufficiently accurate to consider only the influence of deformation due to bending. The following example indicates how deformation due to all possible causes may be considered. It also shows how, very complex the rigorous treatment of a space frame may become, and the desirability of being able to show that deformations, except those due to bending, may be neglected.

Fig. 5 a) shows a portal frame with fixed supports carrying loads P_x and P_y on a bracket. In considering deformations in the frame, the moments and shears transmitted from the bracket may be assumed to be applied directly to the frame at the joint. The frame is six times statically indeterminate. Releases may be inserted as shown at (b). Axes of reference and a key to the distribution diagrams of ω values are shown at (c). At (d) column (1) gives the loads and $\overline{X}$ values assumed to act on the statically determinate frame. Thus $\overline{X}_a^4$ indicates an external moment equal to the plastic moment of resistance applied at section a in the plane OYX (ref. Fig. 2 a)). Other $\overline{X}$ values are similarly indexed. The direction and section of application of the moments is indicated in the diagrams by arrows. The direction of view of the frame diagrams is as shown at (a), Columns (2) to (7) show diagrammatically the distribution of ω values due to influences of column (1) along the various members, separated into moments and shears, according to their plane of action, and into torsions (T) and axial forces (N). Each member can only resist moments or shear in one of two planes. Thus, the horizontal rows show under categories the distributions of ω values due to each $\overline{X}$ or load. In any member the internal stresses due to any $\overline{X} = 1$ acting, can only do virtual work when other influences produce in that member stresses and deformations of the same category. Thus, $\int \omega_i \omega_k ds$ can only have a value when both ω_i and ω_k act on the same member and occur in the same vertical column. The ω values caused by load or $\overline{X}$ influences can easily be indicated numerically on the diagrams in any particular case, and the $\int \omega_i \omega_k ds$ values evaluated, using the appropriate value of ϵ for each member according to the category of the ω values. By plotting the ω diagrams as indicated in the key, the sign of $\int \omega_i \omega_k ds$ may be determined, the value being positive when ω_i and ω_k are not opposed and negative when opposed. Values of a_k^p (Eq. (3)) can then be checked for each release and adjustments made to $\overline{X}$ values, if necessary, to ensure that a_k^p is positive in value and not excessive. $\overline{X}$ values should not be given (when making adjustments) negative values, but the diagrams should be re-drawn, showing the correct direction of action.

References

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Summary

The paper uses the Principle of Virtual Work to extend the application of the Müller-Breslau general elastic equations as used for elastic frames to three-dimensional elasto-plastic frames, with allowance for deformations and restraints due to bending and shear in two planes, torsion and axial forces. The elasto-plastic characteristics of members of the frame are idealised and assumed to be within safe limits defined by a load-deformation relation, which is linear from 0 up to a limit L_1 and then constant in restraint up to an ultimate deformation limit L_2. L_1 is defined by the yield characteristics, and L_2 by the ultimate strength and deformation characteristics of critical sections. Frames must be designed to satisfy the general equations, so that deformations under assumed ultimate load are between limits L_1 and L_2 at selected critical sections. The assumption of ultimate strength at these sections enables the extent of cracking to be known at all sections and hence the stiffness. It is recognised that only occasionally is it likely to be necessary to allow for deformations due to shear and axial forces, but it is important to know, or to be able to determine, when these influences cannot be ignored without loss of security. For instance, the influence of restraint from secondary beams through torsion on the shear in the main beams may at times be critical.

Résumé

L'auteur étend, à l'aide du principe des travaux virtuels, l'application des équations générales de l'élasticité de Müller-Breslau, telles qu'elles ont été établies pour les cadres porteurs élastiques, aux cadres tridimensionnels élasto-plastiques; les déformations et les contraintes résultant de la flexion et des efforts tranchants dans deux plans sont prises en considération, ainsi que la torsion et les efforts normaux. Pour les caractéristiques élasto-plastiques des éléments des cadres, il est admis théoriquement qu'elles sont définies dans

des limites sûres par une relation contrainte-allongement, qui présente une allure linéaire de 0 à une valeur limite L_1, pour passer ensuite sous contrainte constante à la limite d'allongement à la rupture L_2. L_1 est donné par la limite élastique et L_2 par les valeurs de la contrainte de rupture et de l'allongement à la rupture des sections critiques. Le projet du cadre doit satisfaire à ces équations générales dans des conditions telles que les déformations sous la charge de rupture admise, dans des sections critiques choisies, restent comprises entre les limites L_1 et L_2. Le fait d'admettre la charge de rupture dans ces sections permet de déterminer l'importance de la formation des fissures dans toutes les sections et par suite également la rigidité. Il est reconnu qu'il n'est qu'occasionnellement nécessaire de faire intervenir les déformations résultant des efforts tranchants et des efforts normaux; il n'est pas moins essentiel de savoir, ou d'être en mesure de déterminer quand il n'est plus possible de négliger ces influences sans nuir à la sécurité. A titre d'exemple, l'auteur indique que l'influence qu'exerce l'encastrement des poutres secondaires sur les contraintes de cisaillement des poutres principales, par suite de la torsion, est souvent dangereuse.

Zusammenfassung

In dieser Veröffentlichung wird mit Hilfe des Prinzips der virtuellen Arbeit die Anwendung der allgemeinen Elastizitätsgleichungen von Müller-Breslau, wie sie für elastische Rahmentragwerke aufgestellt wurden, auf dreidimensionale, elasto-plastische Rahmen erreicht, wobei die Formänderungen und die Spannungen infolge Biegung und Querkräfte in zwei Ebenen, Torsion und Normalkräfte, berücksichtigt werden. Für die elasto-plastischen Eigenschaften der Rahmenglieder wird idealisierend angenommen, daß sie in sicheren Grenzen durch eine Spannungs-Dehnungsbeziehung definiert sind, welche von 0 bis zu einem Grenzwert L_1 linear verläuft, um dann bei konstanter Spannung bis zur Bruchdehnungsgrenze L_2 weiterzugehen. L_1 wird durch den Fließzustand und L_2 durch die Bruchlast- und Bruchdehnungswerte der kritischen Querschnitte gegeben. Der Entwurf des Rahmens muß die allgemeinen Gleichungen so weit erfüllen, daß die Formänderungen unter angenommener Bruchlast in besonderen kritischen Schnitten zwischen den Grenzen L_1 und L_2 bleiben. Die Annahme der Bruchlast in diesen Querschnitten gestattet das Ausmaß der Rissebildung in allen Schnitten und damit auch die Steifigkeit zu erfassen. Daß es nur gelegentlich notwendig werden kann die Formänderungen infolge Quer- und Normalkräfte zu berücksichtigen, ist anerkannt, aber es ist immerhin wichtig, zu wissen oder bestimmen zu können, wann diese Einflüsse ohne Sicherheitseinbuße nicht mehr vernachlässigt werden können. Zum Beispiel kann der Einfluß der Einspannung von sekundären Trägern durch Torsionswirkung auf die Schubspannung in Hauptträgern manchmal gefährlich werden.