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Beams on Deformable Foundation

Poutres sur fondation déformable

Balken auf verformbarer Unterlage

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1. Introduction

The beams on continuous and deformable support are usually calculated taking in consideration the proportionality between the deformation at each point and the pressure directly exerted on it. The corresponding theory is found in all treatises on the subject, there existing books exclusively dedicated to it [1] [2]. Its large use comes from the fact that, with it, we can obtain solutions represented by elementary functions (trigonometric, exponential and hyperbolic), listed in any engineering handbook[1]). There is, however, a great inconvenience in this theory (which we may call "classic") when referring to foundation beams, because in the soil the condition of independence between the settlement at a point and the pressures at the adjoining ones does not occur; although the law relating them is not yet well known (it does not refer to a perfectly elastic body, where Boussineq's theory would be applicable) it is possible to imagine some simple one (fig. 1b) closer to reality than the

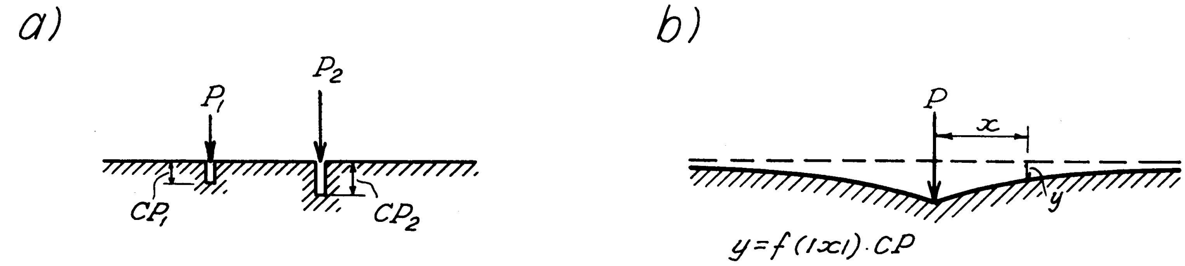

Fig. 1.

1) It is supposed, as in all this work, that the beam has constant section and that the foundation is homogeneously deformable.

classic theory (fig. 1a). In fact, there is a possibility of choosing a law under these conditions capable of leading to equations whose solutions may be expressed by means of the same elementary functions of the classic theory results.

2. Deformation

The law that characterizes the deformation y of a support in relation to the load transmitted to it is the following:

$$y = f(|x|)\,C\,P,$$

where P is the load applied on the support and distributed to a distance equal to the beam's width, C is a constant whose dimension is equal to a length divided by a force and equal to the deformation of the load application line when this one is unitary, and f is an adimensional function of the distance $|x|$ from that line to the considered point, under the beam's axis, so that $f(0) = 1$.

In order that the desired results may be obtained, that means, a solution represented by the mentioned elementary functions, the following choice must be made [2]):

$$f(|x|) = e^{-a|x|},$$

where a is a constant whose dimension is equal do the inverse of a length, characterizing the concentration of the influence of P along the axis (on the other hand C is a constant that characterizes the deformability of the soil, under the load). Experimentally the constants C and a are determined measuring the deformation caused by a certain load P [3]), in the position where it is applied (y_0) and at the distance d that has been chosen (y_d). We have then:

$$C = \frac{y_0}{P}, \qquad a = \frac{1}{d}\ln\frac{y_0}{y_d}.$$

Under these conditions we finally obtain:

$$y = P\,C\,e^{-a|x|}. \tag{1}$$

3. Reaction

When a prismatic beam whose length is l and the flexural rigidity EJ is under the action of the distributed load $p(x)$, the support will react with the

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[2]) As per HABEL [3], this law has been already used by WIEGHARDT.

[3]) The load P should be extended to a width equal to that of the beam and for a small length, but not too short in order to avoid a cut in the support material. The influence of this length l_0 is negligible, as it can be seen in the formulas included in a note at the foot of paragraph 9 (the value of C, for instance, obtained with the measure of y_0 would be influenced at a ratio of $1 + 0{,}5\,a l_0$).

force also distributed $q(x)$ plus the concentrated forces in the ends A and B (fig. 2a)[4]. These concentrated forces are due to the fact that the shearing forces at the beam's ends cannot be zero, because they are equal to the derivative of the bending moments, and those ones show a diagram with an angular point in those positions. The same happens, as it is known, in the ends of a beam's portions that are not deformable for being leaning against a fixed obstacle (fig. 2b).

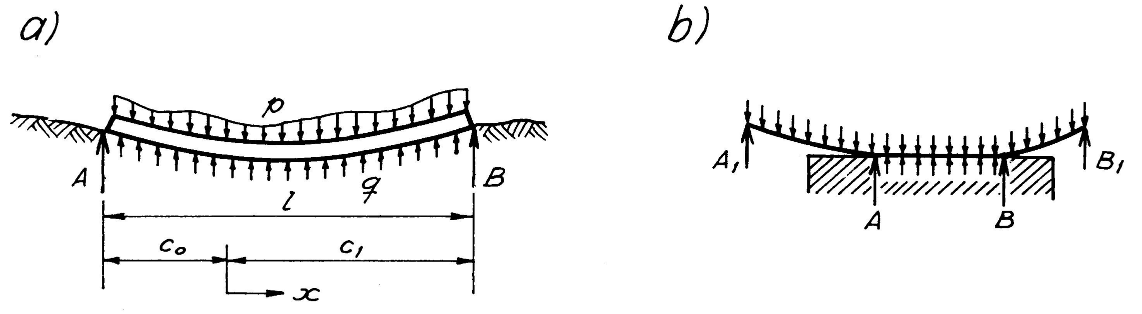

Fig. 2.

To this reaction, equal to the beam's action on the support, corresponds a support's deformation given by (1) for each load element $q\,dx$ and for the forces A and B. If the abscissas' origin is at the distance c_0 from the left end and c_1 from the right one (practically we should make $c_0=0$ or $c_0=l/2$, at the best convenience), the support's deformation is:

$$y = A\,C\,e^{-a(c_0+x)} + B\,C\,e^{-a(c_1-x)} + C\int_{-c_0}^{c_1} q(z)\,e^{-a|x-z|}\,dz. \qquad (2)$$

We have, also, the equation of straight beam's deflection curve:

$$\frac{d^4 y}{d x^4} = \frac{p-q}{EJ}. \qquad (3)$$

4. General Equation

Calling φ the inclination of the beam's axis, M the bending moment and Q the shear force, we obtain, by successive derivation of (2):

$$\begin{aligned}\frac{dy}{dx} = \varphi = &-a\,A\,C\,e^{-a(c_0+x)} + a\,B\,C\,e^{-a(c_1-x)} \\ &- a\,C\int_{-c_0}^{x} q(z)\,e^{-a(x-z)}\,dz + a\,C\int_{x}^{c_1} q(z)\,e^{-a(z-x)}\,dz,\end{aligned} \qquad (4)$$

[4]) The conclusions concerning the distributed load p are also applicable to the load P concentrated in the point of abscissa c, since the impulse function $\mathfrak{F}_c(x)$ is used, putting $p(x)=P\,\mathfrak{F}_c(x)$, as shown in paragraph 6.

$$\frac{d^2 y}{d x^2} = -\frac{M}{E J} = a^2 y - 2 a C q(x), \tag{5}$$

$$\frac{d^3 y}{d x^3} = -\frac{Q}{E J} = a^2 \varphi - 2 a C \frac{d q}{d x}, \tag{6}$$

$$\frac{d^4 y}{d x^4} = \frac{p-q}{E J} = a^4 y - 2 a^3 C q - 2 a C \frac{d^2 q}{d x^2}. \tag{7}$$

The double Eq. (7) is the one that characterizes the two unknown quantities q and y. In order to eliminate one of them, the last two members are derived four times and $d^4 y/d x^4$ is substituted by its value given by the two first members — thus giving:

$$\frac{d^6 q}{d x^6} + (a^2 - 2\psi)\frac{d^4 q}{d x^4} + 2\psi a^4 q = 2\psi\left(a^4 p - \frac{d^4 p}{d x^4}\right), \tag{8}$$

where
$$\psi = \frac{1}{4 a C E J} \tag{9}$$

is a constant (inverse of an area) characterized by the coefficients that define the deformation of the beam and supports.

Eq. (8) gives q, so solving completely the problem. It should be noted, however, that, among the six constants of integration, only four are acceptable, in accordance with the physical conditions of the problem; the two ones to be eliminated are those which multiply the particular solutions of (8) which does not satisfy the equality (7). From this equality still results the values of A and B, as it can be seen ahead.

5. Solution of the Differential Equation

As it is known [4], the general solution of the constant coefficients linear equation, of order m, is

$$q = \sum_{n=1}^{m} \frac{e^{u_n x}}{k_n} \int e^{-u_n x} X(x)\, d x, \tag{10}$$

where $X(x)$ is the second member of the equation and where the u_n are, for (8), the roots (here supposed to be different) of the equation

$$u^6 + (a^2 - 2\psi) u^4 + 2\psi a^4 = 0 \tag{11}$$

and the k_n have the values:

$$k_n = 6 u_n^5 + 4 (a^2 - 2\psi) u_n^3. \tag{12}$$

The m constants of the m integrals of (10) are the constants of integration. In the present case, we have:

$$u_1 = -u_3 = \sqrt{\psi - \sqrt{\psi^2 - 2\psi a^2}}, \qquad u_2 = -u_4 = \sqrt{\psi + \sqrt{\psi^2 - 2\psi a^2}} = \frac{a}{u_1}\sqrt{2\psi},$$
$$u_5 = -u_6 = a\sqrt{-1} \tag{13}$$

and
$$X(x) = 2\psi\left(a^4 p - \frac{d^4 p}{d x^4}\right). \tag{14}$$

With the four derivations made in order to come from (2) to (7), four integration constants arise and have to be put aside, among the six of the solution of (8). Two of them are substituted by the constants A and B of (2) to which convenient values will be attributed, and the other two are those corresponding to u_5 and u_6, leading to solutions of (8) which do not satisfy the Eq. (2).

When a particular solution $q_0(x)$ of the Eq. (8) is obtained, the general solution may be written under the form

$$q = C_1 e^{u_1 x} + C_2 e^{u_2 x} + C_3 e^{u_3 x} + C_4 e^{u_4 x} + q_0, \tag{15}$$

where the C_n are the constants of integration to be determined in accordance with the beam's end conditions.

The last term of (2) when q is changed for q_0 has the value (A_0 and B_0 are constants)[5]):

$$C e^{-ax}\int_{-c_0}^{x} q_0(z)\, e^{az}\, dz + C e^{ax}\int_{x}^{c_1} q_0(z)\, e^{-az}\, dz = A_0 C e^{-a(c_0+x)} + B_0 C e^{-a(c_1-x)} + y_0 \tag{16}$$

allowing to write — putting (15) successively in (2) and (7):

$$y = \left(A + A_0 - \sum_1^4 \frac{C_n e^{-u_n c_0}}{u_n + a}\right) C e^{-a(c_0+x)} + \left(B + B_0 + \sum_1^4 \frac{C_n e^{u_n c_0}}{u_n - a}\right) C e^{-a(c_1-x)}$$
$$-2aC\sum_1^4 \frac{C_n}{u_n^2 - a^2} e^{u_n x} + y_0, \tag{17}$$

$$2\psi E J a^4 y = 2p\psi + (a^2 - 2\psi)\, q_0 + \frac{d^2 q_0}{d x^2} + \sum_1^4 (a^2 - 2\psi + u_n^2)\, C_n e^{u_n x}. \tag{18}$$

From the comparison between (17) and (18) it follows:

$$A = -A_0 + \sum_1^4 \frac{C_n e^{-u_n c_0}}{u_n + a}, \qquad B = -B_0 - \sum_1^4 \frac{C_n}{u_n - a} e^{u_n c_1}, \tag{19}$$

$$y_0 = \left[2p\psi + (a^2 - 2\psi)\, q_0 + \frac{d^2 q_0}{d x^2}\right]\frac{2C}{a^3} \tag{20}$$

and
$$-\frac{2aC}{u_n^2 - a^2} = \frac{a^2 - 2\psi + u_n^2}{2\psi E J a^4},$$

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[5]) Putting $\int q_0(z)\, e^{bz}\, dz = F(z,b)\, e^{bz} +$ const., we have:

$A_0 = -F(-c_0, a), \qquad B_0 = F(c_1, -a), \qquad y_0 = C\,[F(x,a) - F(x,-a)].$

which is the 4th degree equation in relation to u_n, whose roots have the values (13) already found.

For an easier use it is suggested the substitution of the constants of integration, making:

$$(C_1 e^{u_1 x} + C_3 e^{-u_1 x}) = (C_a \sinh u_1 x + C_b \cosh u_1 x)(u_1^2 - a^2),$$
$$(C_2 e^{u_2 x} + C_4 e^{-u_2 x}) = (C_c \sinh u_2 x + C_d \cosh u_2 x)(u_2^2 - a^2)$$

and so the (15) and the (19) will be written:

$$\begin{aligned} q = \; & q_0 + (u_1^2 - a^2)(C_a \sinh u_1 x + C_b \cosh u_1 x) \\ & + (u_2^2 - a^2)(C_c \sinh u_2 x + C_d \cosh u_2 x), \end{aligned} \tag{21}$$

$$\begin{aligned} A &= -A_0 + (u_1 - a)(-C_a \sinh u_1 c_0 + C_b \cosh u_2 c_0) \\ & \quad + (u_2 - a)(-C_c \sinh u_2 c_0 + C_d \cosh u_2 c_0), \\ B &= -B_0 - (u_1 + a)(C_a \sinh u_1 c_1 + C_b \cosh u_1 c_1) \\ & \quad - (u_2 + a)(C_c \sinh u_2 c_1 + C_d \cosh u_2 c_1) \end{aligned} \tag{22}$$

maintaining the (20) and obtaining from (17) and its derivatives:

$$y = y_0 - 2aC(C_a \sinh u_1 x + C_b \cosh u_1 x + C_c \sinh u_2 x + C_d \cosh u_2 x), \tag{23}$$

$$\begin{aligned} \varphi = \; & \frac{d y_0}{d x} = 2aC(u_1 C_a \cosh u_1 x + u_1 C_b \sinh u_1 x \\ & + u_2 C_c \cosh u_2 x + u_2 C_d \sinh u_2 x), \end{aligned} \tag{24}$$

$$\begin{aligned} M = & -EJ\frac{d^2 y_0}{d x^2} + \frac{1}{2\psi}(u_1^2 C_a \sinh u_1 x + u_1^2 C_b \cosh u_1 x \\ & + u_2^2 C_c \sinh u_2 x + u_2^2 C_d \cosh u_2 x), \end{aligned} \tag{25}$$

$$\begin{aligned} Q = & -EJ\frac{d^3 y_0}{d x^3} + \frac{1}{2\psi}(u_1^3 C_a \cosh u_1 x + u_1^3 C_b \sinh u_1 x \\ & + u_2^3 C_c \cosh u_2 x + u_2^3 C_d \sinh u_2 x). \end{aligned} \tag{26}$$

The Eqs. (20) to (26) solve the whole problem. For its practical application some considerations must be made about the particular solution q_0 (paragraph 6), about the boundary conditions (with which the constants of integration are determined, paragraph 7) and about the real and imaginary u_n (paragraph 8). Paragraph 9 refers to a beam extremely rigid (practically undeformable). In paragraphs 10 and 11 some examples of the applicability to particular cases are presented.

6. Particular Solution

The particular solution q_0 of (8) can be obtained by means of (10) and (14). In common cases, however, in which p is expressed under the form of polynominal of integer power of x (including the case of p constant) or Fourier series or represented by an impulse function[6]) (concentrated force) or binary function (external moment), q_0 can be found as shown ahead. It must be noted that the solution of these last two cases allows the working out of influence lines for external forces and moments.

For
$$p = \sum_{n=0}^{m} a_n x^n \tag{27}$$

we have
$$q_0 = \sum_{n=0}^{m} b_n x^n \tag{28}$$

with

$$b_n = a_n, \qquad \text{if } n > m-4,$$

$$b_n = a_n + \frac{(n+4)!}{n!\,a^4}\left[\left(1-\frac{a^2}{2\psi}\right) b_{n+4} - a_{n+4}\right], \qquad \text{if } m-4 \geqq n \geqq m-5,$$

$$b_n = a_n + \frac{(n+4)!}{n!\,a^4}\cdot\left[\left(1-\frac{a^2}{2\psi}\right) b_{n+4} - \frac{(n+5)(n+6)}{2} b_{n+6} - a_{n+4}\right], \quad \text{if } n < m-5. \tag{29}$$

For
$$p = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} a'_n \sin\frac{n\pi x}{l} \tag{30}$$

we have
$$q_0 = a_0 + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b'_n \sin\frac{n\pi x}{l} \tag{31}$$

with (for b'_n, put a'_n instead of a_n):

$$b_n = 2\psi l^2 a_n \frac{a^4 l^4 - n^4\pi^4}{2\psi a^4 l^6 + (a^2 - 2\psi) n^4 \pi^4 l^2 - n^6 \pi^6}. \tag{32}$$

For (abscissas origin in the point of application of the concentrated load P):

$$p = P\,\mathfrak{J}_0(x), \tag{33}$$

we have (we may attribute to u any of the four first values of u_n of (13)):

$$q_0 = \frac{2P\psi}{u_2^2 - u_1^2}\,\mathfrak{R}_0(x)\left(\frac{u_1^2 - a^2}{u_1}\sinh u_1 x - \frac{u_2^2 - a^2}{u_2}\sinh u_2 x\right) \tag{34}$$

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[6]) The following designations and notations will be used here [5]:

Step function of argument c of the variable x, $\mathfrak{R}_c(x)$, is the function that is zero for $x<c$ and equal to the unit for $x>c$ (and finite for $x=c$);

Impulse function of argument c of the variable x, $\mathfrak{J}_c(x)$, is the function that is zero for $x \neq c$ and such that, for any b and d real and positive

$$\int_{c-b}^{c+d} \mathfrak{J}_c(x)\,dx = 1;$$

Binary function of argument c of the variable x, $\mathfrak{J}'_c(x)$, is the function that is zero for $x \neq c$ and such that, for any b and d real and positive

$$\int_{c-b}^{c+d} \mathfrak{J}'_c(x)\,dx = 0, \qquad \int_{c-b}^{c+d} -x\,\mathfrak{J}'_c(x)\,dx = 1.$$

with

$$A_0 = 0, \quad B_0 = \frac{2P\psi}{u_2^2 - u_1^2}\left(\frac{a}{u_1}\sinh u_1 c_1 - \frac{a}{u_2}\sinh u_2 c_1 + \cosh u_1 c_1 - \cosh u_2 c_1\right). \tag{35}$$

With this solution the influence lines corresponding to vertical loads can be constructed. Those corresponding to external moments are obtained in accordance with the known rule, by deriving the expressions of those referring to vertical loads. The same results should be obtained starting from

$$p = M_0 \mathfrak{J}_0'(x), \tag{36}$$

to which corresponds

$$q_0 = \frac{2P\psi}{u_2^2 - u_1^2}\mathfrak{R}_0(x)\left[(u_1^2 - a^2)\cosh u_1 x - (u_2^2 - a^2)\cosh u_2 x\right]. \tag{37}$$

7. Boundary Conditions

Each end of the beams can be free, simply supported or built-in. For each one of these cases there are two conditions to be fulfilled, existing, therefore, for the two extremities, four conditions, with which the four constants of integration are determined.

For the free ends we have:

$$\frac{d^2 y}{dx^2} = -\frac{M}{EJ} = 0, \qquad \frac{d^3 y}{dx^3} = -\frac{Q}{EJ} \tag{38}$$

with $Q = A$ or $Q = -B$, according to whether the free end is at the left or at the right (corresponding to the abolishing of the reaction $R_a = Q - A$ or $R_b = -Q - B$, that should exist if there existed a support).

For the simply supported end:

$$y = 0, \qquad \frac{d^2 y}{dx^2} = -\frac{M}{EJ} = 0. \tag{39}$$

For the perfectly built end:

$$y = 0, \qquad \frac{dy}{dx} = 0. \tag{40}$$

If the beam is symmetrical, with a symmetrical load, it is convenient to take the origin of the abscissa at the middle of the span ($c_0 = c_1 = l/2$), from where we have in the Eqs. (21) to (26):

$$C_a = C_c = 0, \tag{41}$$

reducing to two the unknown constants. If the load would be anti-symmetrical (in the symmetrical beam), we should have:

$$C_b = C_d = 0. \tag{42}$$

8. Real and Imaginary Values of u_n

Whenever the beams flexural rigidity EJ is less than $1/8a^3C$ (very flexible beams) we have $\psi > 2a^2$ and the u_n are all real, and so the Eqs. (20) to (26) can be used directly.

If the beam has small flexibility (a case of more common application), that is, if[7]):

$$EJ > \frac{1}{8a^3C} \tag{43}$$

we will have $\psi < 2a^2$ and the roots u_1 to u_4 of (11), given by (13), will be imaginary:

$$\begin{aligned} u_1 &= -u_3 = \sqrt{\psi - i\sqrt{2\psi a^2 - \psi^2}} = v_1 - i v_2, \\ u_2 &= -u_4 = \sqrt{\psi + i\sqrt{2\psi a^2 - \psi^2}} = v_1 + i v_2 \end{aligned} \tag{44}$$

with

$$\begin{aligned} v_1 = \sqrt{\tfrac{1}{2}(a\sqrt{2\psi} + \psi)}, \quad v_2 = \sqrt{\tfrac{1}{2}(a\sqrt{2\psi} - \psi)}, \quad v_3 = 2v_1v_2 = \sqrt{2\psi a^2 - \psi^2}, \\ v_4 = v_1^2 - 3v_2^2 = 2\psi - a\sqrt{2\psi}, \qquad v_5 = 3v_1^2 - v_2^2 = 2\psi + a\sqrt{2\psi}. \end{aligned} \tag{45}$$

The introduction of these values in formulas (21) to (26) permits the elimination of the imaginary ones and to write all of them under the form

$$\begin{aligned} X = K_0 + K_5 (K_1 \sinh v_1 x \sin v_2 x + K_2 \cosh v_1 x \cos v_2 x \\ + K_3 \sinh v_1 x \cos v_2 x + K_4 \cosh v_1 x \sin v_2 x), \end{aligned} \tag{46}$$

where K_0 depends on the particular solution q_0 (or y_0 of (20)), K_5 of the known C, a or ψ, and the other K, of the new constants of integration K_a, K_b, K_c, K_d in accordance with the following tabulation (including also the reaction $R_a = Q - A$ and $R_b = -Q - B$ of the non settling supports of the extremities).

For symmetrical loads in symmetrical beams, we have, with $c_0 = c_1 = l/2$:

$$K_c = K_d = 0 \tag{47}$$

and for anti-symmetrical loads:

$$K_a = K_b = 0. \tag{48}$$

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[7]) The limit case $EJ = 1/8a^3C$ is not considered. It would lead to equal roots ($u_1 = u_2 = u_3 = u_4$), with which, instead of (15), we would obtain the solution:

$$q = (c_1 + c_2 x) e^{u_1 x} + (c_3 + c_4 x) e^{-u_1 x} + q_0$$

from which come the expressions of the other unknown quantities (y, φ, etc.). Also for infinite EJ, that is, $\psi = 0$, the Eq. (11) should have equal roots, since $u_1 = u_2 = u_3 = u_4 = 0$. This case is dealt with in paragraph 9.

X	K_0	K_1	K_2	K_3	K_4	K_5
q	q_0	$(\psi-a^2)\,K_a-v_3\,K_b$	$(\psi-a^2)\,K_b+v_3\,K_a$	$(\psi-a^2)\,K_c+v_3\,K_d$	$(\psi-a^2)\,K_d-v_3\,K_c$	1
A (with $x=-c_0$)	$-A_0$	$v_1\,K_d-v_2\,K_c-a\,K_a$	$v_1\,K_c+v_2\,K_d-a\,K_b$	$v_1\,K_b+v_2\,K_a-a\,K_c$	$v_1\,K_a-v_2\,K_b-a\,K_d$	1
B (with $x=c_1$)	$-B_0$	$-v_1\,K_d+v_2\,K_c-a\,K_a$	$-v_1\,K_c-v_2\,K_d-a\,K_b$	$-v_1\,K_b-v_2\,K_a-a\,K_c$	$-v_1\,K_a+v_2\,K_b-a\,K_d$	1
y	y_0	K_a	K_b	K_c	K_d	$-2\,a\,C$
φ	$\dfrac{d\,y_0}{d\,x}$	$v_1\,K_d-v_2\,K_c$	$v_1\,K_c+v_2\,K_d$	$v_1\,K_b+v_2\,K_a$	$v_1\,K_a-v_2\,K_b$	$-2\,a\,C$
M	$-E\,J\dfrac{d^2\,y_0}{d\,x^2}$	$\psi\,K_a-v_3\,K_b$	$\psi\,K_b+v_3\,K_a$	$\psi\,K_c+v_3\,K_d$	$\psi\,K_d-v_3\,K_c$	$\dfrac{1}{2\,\psi}$
Q	$-E\,J\dfrac{d^3\,y_0}{d\,x^3}$	$v_1\,v_4\,K_d-v_2\,v_5\,K_c$	$v_2\,v_5\,K_d+v_1\,v_4\,K_c$	$v_2\,v_5\,K_a+v_1\,v_4\,K_b$	$v_1\,v_4\,K_a-v_2\,v_5\,K_b$	$\dfrac{1}{2\,\psi}$
R_a (with $x=-c_0$)	$A_0-E\,J\dfrac{d^3\,y_0}{d\,x^3}$	$v_1K_d+v_2\,K_c-K_a\sqrt{2\,\psi}$	$v_1\,K_c-v_2\,K_d-K_b\sqrt{2\,\psi}$	$v_1\,K_b-v_2\,K_a-K_c\sqrt{2\,\psi}$	$v_2\,K_b+v_1K_a-K_d\sqrt{2\,\psi}$	$\dfrac{-a}{\sqrt{2\,\psi}}$
R_b (with $x=c_1$)	$B_0+E\,J\dfrac{d^3\,y_0}{d\,x^3}$	$v_1K_d+v_2\,K_c+K_a\sqrt{2\,\psi}$	$v_1\,K_c-v_2\,K_d+K_b\sqrt{2\,\psi}$	$v_1\,K_b-v_2\,K_a+K_c\sqrt{2\,\psi}$	$v_2\,K_b+v_1K_a+K_d\sqrt{2\,\psi}$	$\dfrac{a}{\sqrt{2\,\psi}}$

When the u are imaginary, the particular solution (34) and the corresponding constants A_0 and B_0 may be written:

$$q_0 = -\frac{P}{v_3}\,\Re_0(x)\,(v_2 v_5 \sinh v_1 x \cos v_2 x + v_1 v_4 \cosh v_1 x \sin v_2 x)\,, \tag{49}$$

$$A_0 = 0, \quad B_0 = \frac{P\sqrt{2\psi}}{v_3}(v_2 \sinh v_1 c_1 \cos v_2 c_1 - v_1 \cosh v_1 c_1 \sin v_2 c_1 - \sinh v_1 c_1 \sin v_2 c_1)\,, \tag{50}$$

from where, introducting (33) and (49) in (20):

$$y_0 = -\frac{2CP\sqrt{2\psi}}{v_3}\,\Re_0(x)\,(v_2 \sinh v_1 x \cos v_2 x - v_1 \cosh v_1 x \sin v_2 x) \tag{51}$$

and from (5):

$$M_0 = -\frac{P}{2}\,\Re_0(x)\left(\frac{1}{v_1}\sinh v_1 x \cos v_2 x + \frac{1}{v_2}\cosh v_1 x \sin v_2 x\right). \tag{52}$$

9. Non Deformable Beam

When the beam is non deformable, $EJ = \infty$ or $\psi = 0$, the deflection curve is a straight line:

$$y = k_0 + k_1 x\,. \tag{53}$$

There will be no reaction of the deformed support if one of the beam extremities is built-in or if both of them are supported. Two further cases may be considered: the case of free extremities, with k_0 and k_1 to be determined, and the case of the beam with an end simply supported (supposing to be the corresponding to $x=0$) and the other one free, being $k_0=0$, and remaining to determine k_1 and the reaction R_a of the support.

In this case, the Eq. (2) is enough to solve the problem. Its solution is:

$$q = \frac{a}{2C}(k_0 + k_1 x) \tag{54}$$

with
$$A = \frac{1}{2C}\left(k_0 - k_1 c_0 - \frac{k_1}{a}\right), \qquad B = \frac{1}{2C}\left(k_0 + k_1 c_1 + \frac{k_1}{a}\right). \tag{55}$$

The constants k_0, k_1 and R_a are determined with the equilibrium conditions: If the resultant P of the p is at the distance e from the left end of the beam, for which is made $x=0$ (that is, $c_0=0$ and $c_1=l$), we have:

$$R_a + A + B + \int_0^l q\,dx = P\,, \qquad Bl + \int_0^l q\,x\,dx = Pe \tag{56}$$

from where, if the ends are free[8]):

$$R_a = 0, \qquad k_0 = \frac{2CP}{2+al} - \frac{k_1 l}{2}, \qquad k_1 = \frac{12aCP(2e-l)}{(a^2l^2+6al+12)}, \qquad A = \frac{ak_0-k_1}{2aC}, \qquad B = \frac{ak_0+k_1(1+al)}{2aC} \tag{57}$$

and, if a support exists:

$$k_0 = 0, \qquad k_1 = \frac{6aCPe}{l(a^2l^2+3al+3)}, \qquad R_a = P - \frac{k_1 l}{4C}(al+2), \qquad A = -\frac{k_1}{2aC}, \qquad B = \frac{1+al}{2aC}k_1. \tag{58}$$

10. Example of a Beam with Free Ends

It is taken into consideration a prismatic beam with free ends and an uniformly distributed load p. For p constant,

$$q_0 = p \tag{59}$$

is a particular solution of (8) as it can be understood from (27) to (29) with $n=0$ and $a_0=b_0=p$. From this we have (20):

$$y_0 = \frac{2C}{a}p \tag{60}$$

and, from (16) (see corresponding footnote where $F=p/b$):

$$A_0 = -\frac{p}{a} = B_0. \tag{61}$$

The determination of the deflection curve is requested, as well as the reactions of the support and the bending moments of the beam of length $l=2\,m$, with $EJ=10^{12}/72\,\text{kg}\cdot\text{cm}^2$ in support (soil) of $C=0{,}01\,\text{cm}\cdot\text{kg}^{-1}$ and $a=0{,}09\,\text{cm}^{-1}$, from where (9) $\psi=2\times10^{-8}\,\text{cm}^{-2}$. As we have $EJ>1/8\,a^3C$, the formulas of paragraph 8 are applicable, where, with $C_0=c_1=l/2=c$:

$$v_1 \cong v_2 = 0{,}003\,\text{cm}^{-1}, \qquad v_3 = 18\cdot10^{-6},$$
$$v_1c = v_2c = 0{,}3, \qquad \tanh v_1c = 0{,}29131, \qquad \tan v_2c = 0{,}30934.$$

In view of the symmetry we have (47) $K_c=K_d=0$ and, from $M=0$ and $R_b=0$ for $x=c$, using (46) and the corresponding table:

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[8]) If the load is symmetrical:

$$k_1=0, \qquad y_0=k_0=\frac{2CP}{2+al}, \qquad q=\frac{aP}{2+al}, \qquad A=B=\frac{P}{2+al}.$$

$$K_a(\psi \tanh v_1 \tan v_2 c + v_3) - K_b(v_3 \tanh v_1 \tan v_2 c - \psi) = 0, \tag{62}$$

$$\begin{aligned} K_a\left(\tanh v_1 c \tan v_2 c + \frac{v_1 \tan v_2 c - v_2 \tanh v_1 c}{\sqrt{2\psi}}\right) \\ + K_b\left(1 + \frac{v_1 \tanh v_1 c + v_2 \tan v_2 c}{\sqrt{2\psi}}\right) = \frac{p}{a^2 \cosh v_1 c \cos v_2 c} \end{aligned} \tag{63}$$

or, with the numerical data (p in kg/cm and K in kg·cm):

$$18{,}00\,K_a - 1{,}602\,K_b = 0$$

$$0{,}3606\,K_a + 10{,}010\,K_b = 123{,}6\,p$$

from where:

$$K_a = 1{,}095\,p, \qquad K_b = 12{,}31\,p.$$

From (46) the equations sought of y (in cm), q (in kg/cm, as p) and M (in kg·cm), are obtainable as shown on fig. 3 (note that, by the classic theory,

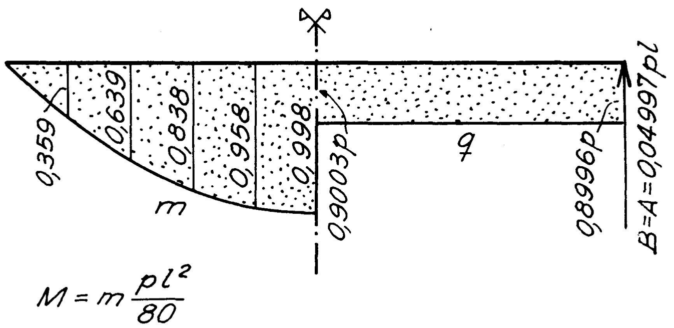

Fig. 3.

we should have y= constant, $q=p$, $A=B=0$ and $M=0$), with x in meters:

$$y = 18\,p\,(123{,}46 - 1{,}095 \sinh 0{,}3x \sin 0{,}3x - 12{,}31 \cosh 0{,}3x \cos 0{,}3x) \cdot 10^{-4}$$

$$q = p\,(10^4 - 90{,}91 \sinh 0{,}3x \sin 0{,}3x - 996{,}9 \cosh 0{,}3x \cos 0{,}3x) \cdot 10^{-4}$$

$$A = B = 0{,}09994\,p\,c$$

$$M = 25\,p\,(-221{,}6 \sinh 0{,}3x \sin 0{,}3x + 19{,}96 \cosh 0{,}3x \cos 0{,}3x).$$

11. Example of a Beam with a Built-in End and Concentrated Load

Let us consider the same beam of the preceding example, but with a built-in end (the left one) and with concentrated load at the distance $c_0 = 160$ cm from this extremity ($c_1 = -c_0 = 40$ cm).

Besides the constants already calculated in the preceding item, we have still, from (45):

$$v_4 = -17{,}96 \times 10^{-6}\,\text{cm}^{-2}, \qquad v_5 = 18{,}04 \times 10^{-6}\,\text{cm}^{-2}.$$

From (49) to (52), with the unities kg and cm, except for x whose unity is the meter, as in the preceding example, it comes:

$$q_0 = 0{,}003\, P\, \Re_0(x)\,(\cosh 0{,}3x \sin 0{,}3x - \sinh 0{,}3x \cos 0{,}3x), \qquad A_0 = 0,$$

$$B_0 = \frac{P}{15000}(3 \sinh 0{,}12 \cos 0{,}12 - 3 \cosh 0{,}12 \sin 0{,}12 - 1000 \sinh 0{,}12 \sin 0{,}12) = -0{,}00096\, P,$$

$$y_0 = \frac{P}{1500} \Re_0(x)\,(\cosh 0{,}3x \sin 0{,}3x - \sinh 0{,}3x \cos 0{,}3x),$$

$$M_0 = -\frac{500}{3} P\, \Re_0(x)\,(\cosh 0{,}3x \sin 0{,}3x + \sinh 0{,}3x \cos 0{,}3x),$$

$$Q_0 = -P\, \Re_0(x) \cosh 0{,}3x \cos 0{,}3x.$$

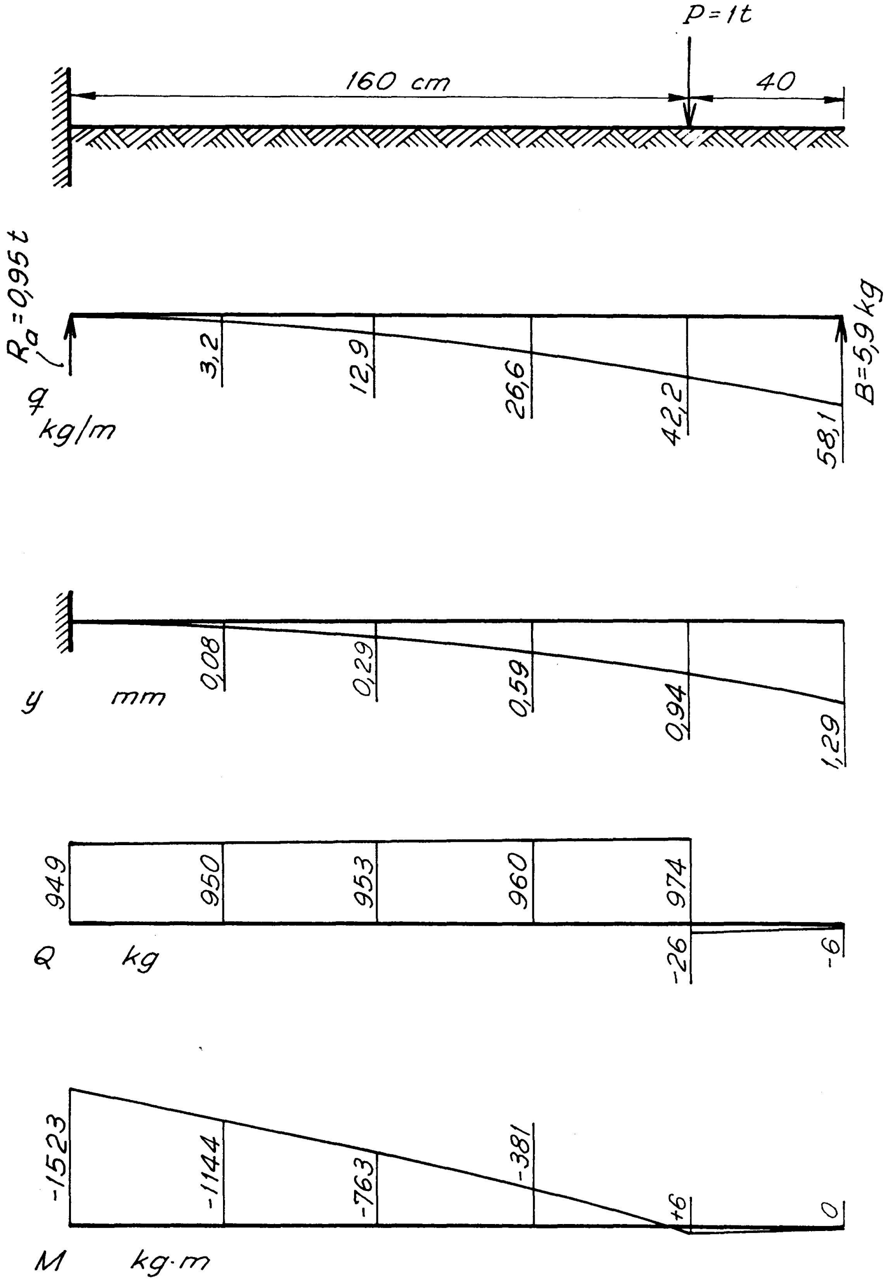

Fig. 4.

The constants K_a, K_b, K_c and K_d are determined from the conditions of being zero y and φ, for $x = -c_0 = -1{,}6$ m and R_b and M, for $x = c_1 = 0{,}4$ m:

$$0{,}2303\,K_a + 0{,}9911\,K_b - 0{,}4423\,K_c - 0{,}5159\,K_d = 0\,,$$
$$-0{,}9582\,K_a + 0{,}0736\,K_b + 0{,}7608\,K_c + 1{,}2214\,K_d = 0\,,$$
$$1{,}0000\,K_a - 0{,}0133\,K_b - 0{,}1205\,K_c + 0{,}1195\,K_d = 0{,}08889\,P\,,$$
$$0{,}0065\,K_a + 0{,}9200\,K_b + 3{,}0671\,K_c - 2{,}9327\,K_d = -2{,}2244\,P\,,$$

from where:

$$K_a = 0{,}0014, \qquad K_b = -0{,}0521\,,$$
$$K_c = -0{,}4422, \qquad K_d = 0{,}2797\,,$$

values that, when applied to the formulas of y, q, Q and M, lead to the results of fig. 4.

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Summary

The beams on continuous and deformable support are usually calculated taking in consideration the proportionality between the deformation in each point and the pressure directly exerted in it. The corresponding theory is found in all treatises on the subject. Its large use comes from the fact that, with it, we can obtain solutions represented by elementary functions (trigonometric, exponential and hyperbolic), listed in any engineering handbook. There is, however, a great inconvenience in this theory when referring to foundation beams, because in the soil the condition of independence between the settlement in a point and the pressures in the adjoining ones does not occur. Although the law relationing them is not yet well known, it is possible to imagine some simple one less far from the reality than the conventional theory. This paper shows that there is a possibility of choosing a law under these conditions capable of leading to equations whose solutions may be expressed by means of the same elementary functions of the conventional theory results.

Résumé

Les poutres reposant sur une fondation déformable continue sont calculées habituellement en admettant que la déformation en chaque point est proportionnelle à la pression qui s'y exerce directement. La théorie fondée sur cette hypothèse se trouve dans tous les traités se rapportant à ce sujet. Son emploi, très répandu, est dû au fait qu'elle permet d'obtenir des solutions représentées par des fonctions élémentaires (trigonométriques, exponentielles et hyperboliques) que l'on trouve dans tous les formulaires. Cette théorie présente cependant de grands inconvénients pour les poutres de fondation parce que, dans le sol, la condition d'indépendance entre la déformation en un point et les pressions s'exerçant sur les points voisins n'est pas réalisée. Bien que la loi qui les lie soit actuellement mal connue, il est possible d'imaginer une loi simple moins éloignée de la réalité que la théorie classique. La présente communication montre qu'il est possible de choisir une loi qui, dans ces conditions, puisse conduire à des équations dont les solutions soient exprimées par les mêmes fonctions élémentaires que celles de la théorie classique.

Zusammenfassung

Balken auf durchgehender und verformbarer Unterlage werden in der Regel unter Berücksichtigung der Proportionalität zwischen der Verformung an jedem Punkt und der direkt darauf ausgeübten Pressung berechnet. Die entsprechende Theorie findet man in allen Abhandlungen zu diesem Thema. Ihre weitverbreitete Anwendung beruht auf der Tatsache, daß sich damit Lösungen mit (trigonometrischen, exponentiellen und hyperbolischen) Elementarfunktionen erreichen lassen, die in jedem technischen Handbuch zu finden sind. Dieser Theorie eignen jedoch dann beträchtliche Unzulänglichkeiten, wenn es sich um Fundationsbalken handelt, weil im Boden eine Unabhängigkeit zwischen der Einsenkung an einem Punkt und den Pressungen an benachbarten Punkten nicht besteht. Obwohl die Gesetzmäßigkeiten ihrer Beziehungen bisher noch nicht eindeutig bekannt sind, ist ein einfaches Gesetz denkbar, das sich weniger von der Wirklichkeit entfernt als die herkömmliche Theorie. Der Verfasser zeigt in seiner Arbeit die Möglichkeit der Wahl eines Gesetzes, das unter den vorliegenden Umständen zu Gleichungen führen kann, deren Lösung mittels der gleichen Elementarfunktionen ausgedrückt werden können, wie diejenigen der klassischen Theorie.