Out-of-Plane Buckling of I-Section Rings

Flambage latéral des anneaux à section en double-té

Über das Kippen von Kreisringen mit I-Querschnitt

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Introduction

The creative and imaginative trend displayed by contemporary architecture has brought to the attention of structural engineers a number of problems which have not been common in the earlier context of structural engineering. This situation has been especially true in the design of roofs to enclose such large spaces as arenas, exhibition halls, gymnasia and fieldhouses.

When the plan of a building of this type is circular, a suspended roof may be found to be a particularly effective solution to the problem of providing a cover for the large area, and several such roofs have been constructed for circular buildings of these types. In these buildings, the suspended roof is supported by a system of radial cables which are attached to an outer boundary member. This member, in turn, may be supported by a set of columns equally spaced around the circumference of the buildings.

This ring must be designed to resist the inward pull of the cables as well as the circumferentially distributed gravity load of the roof, including possible snow load. The requirement that the working stresses do not exceed safe and acceptable values provides certain criteria for the choice of material of the ring, for its cross sectional area, for its section modulus with respect to bending in the vertical direction, and for an analogous torsional strength property. Under possibly extreme circumstances, such rings may buckle in one of several ways, and, when these circumstances prevail, the design criteria may be more closely related to the choice of suitable rigidities in the appropriate directions than to the properties mentioned above. It may be important, consequently, to determine the adequacy of the ring with regard to its ability to resist buckling in each of the several possible modes of collapse when subjected to the anticipated loads from the roof system.

When a ring is under radial compressive loads which produce a sufficiently large hoop compressive force, the possibility that the ring may buckle is obvious. It is somewhat less obvious that, under certain circumstances, the ring may buckle if subjected to outward-acting and suitably placed distributed loads of sufficiently large magnitude. Thus, if the roof is convex and is supported by arch ribs instead of cables so that the outer ring becomes a thrust ring and the hoop force is tensile, a critical magnitude of the tensile hoop force may also exist.

Aside from the possibility of local element buckling or crippling, a circular ring when subjected to hoop compression may buckle in its own plane or may buckle out of its plane. In the former case, the mode of buckling is characterized by the ring going out-of-round. The phenomenon of buckling of a ring in its own plane is fairly well understood and a generally adequate theory is available [1][1]) for the treatment of, at least, the basic problem of this kind. Application of the available theory for inplane buckling determines the moment of inertia of the cross section of the ring, about the centroidal axis normal to the plane of the ring, which is required to preclude buckling in this mode.

The present paper is concerned with the development of stability criteria for out-of-plane buckling of circular rings. In particular, uniform rings of I-section are considered and formulas are presented which give the critical magnitudes of hoop forces for rings forming a complete circle and supported at a number of equally spaced points. The cross sections to which the formulas apply directly should be symmetrical about the web but need not be symmetrical about an axis normal to the web. Thus, I-sections which are symmetrical about the web but have unequal flanges are included. The theory may be readily extended, however, to more general shapes. For simplicity, the present treatment emphasizes rings which are curved in the plane of the web. The equations and resulting formulas may be modified to fit the case of rings of unsymmetrical or symmetrical I-section which are curved in the planes of the flanges.

The problem of lateral buckling of initially straight beams of I-section was discussed by TIMOSHENKO [2]. TIMOSHENKO's results were extended to the case of a straight I-beam with unequal flanges by WEBER [3]. Subsequently, WAGNER [4] presented a more general theory for thin-walled open sections which was further improved by KAPPUS [5] and GOODIER [6].

The topic of out-of-plane buckling of arches having thin-walled open cross sections was discussed by FEDERHOFER [7], who used the energy method to determine the critical loads for segments of rings subjected to circumferential compression.

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[1]) Numbers in brackets refer to corresponding items in the Bibliography.

In the present paper, the matter of stability is approached through the derivation of a pertinent set of differential equations and emphasis is on the case of a complete ring. The equations are valid for both compressive and tensile hoop forces. The major formula, having been derived from a quadratic form and therefore yielding a pair of roots or solutions, normally will, in fact, provide critical values for both compression and tension for the same number of waves in the buckled mode.

Displacement-Moment Relations

The cross section of a typical ring is shown in Fig. 1. The point O is the centroid of the section. The distances from O to the centroids of the two flanges are b_1 and b_3. The distance from O to the centroid of the web, not shown in Fig. 1, is b_2. These distances, in subsequent equations, will be taken as positive if the centroid of the element lies inside of the centroid of the section, and negative if the centroid of the element lies outside of the centroid of the section. The distributed radial loads, p, are applied at a distance b_p from O, and it should be noted that the point of application of this loading in the cross section need not lie within the section itself. Thus, the loads may be introduced through brackets or lugs which are not a part of the actual section. The sign convention mentioned above for b_1, b_2 and b_3 applies also to b_p. The loads, p, are taken as positive when acting inwardly along or parallel to the respective radii.

A differential element of the ring is shown in Fig. 2. The central angle subtended by the undeformed element is $d\theta$. Points O and O' are the centroids

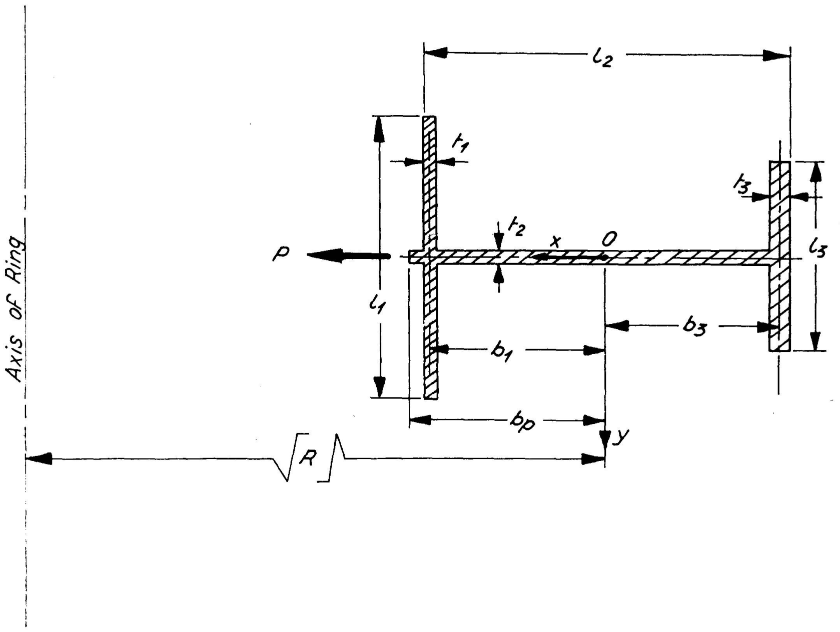

Fig. 1. Cross Section of Ring.

of the two end cross sections. In discussing the displacements and deformations of the ring, the assumption is made that the cross sections are not deformed in their own planes but may be deformed or warped in the directions normal to their planes. The displacement of O normal to the original plane of the ring is denoted by $v = v(\theta)$ and the rotation of the cross section about the tangential axis is denoted by $\phi = \phi(\theta)$. Positive v is taken downward, and ϕ is positive when the upper part of the section moves toward the center of the ring.

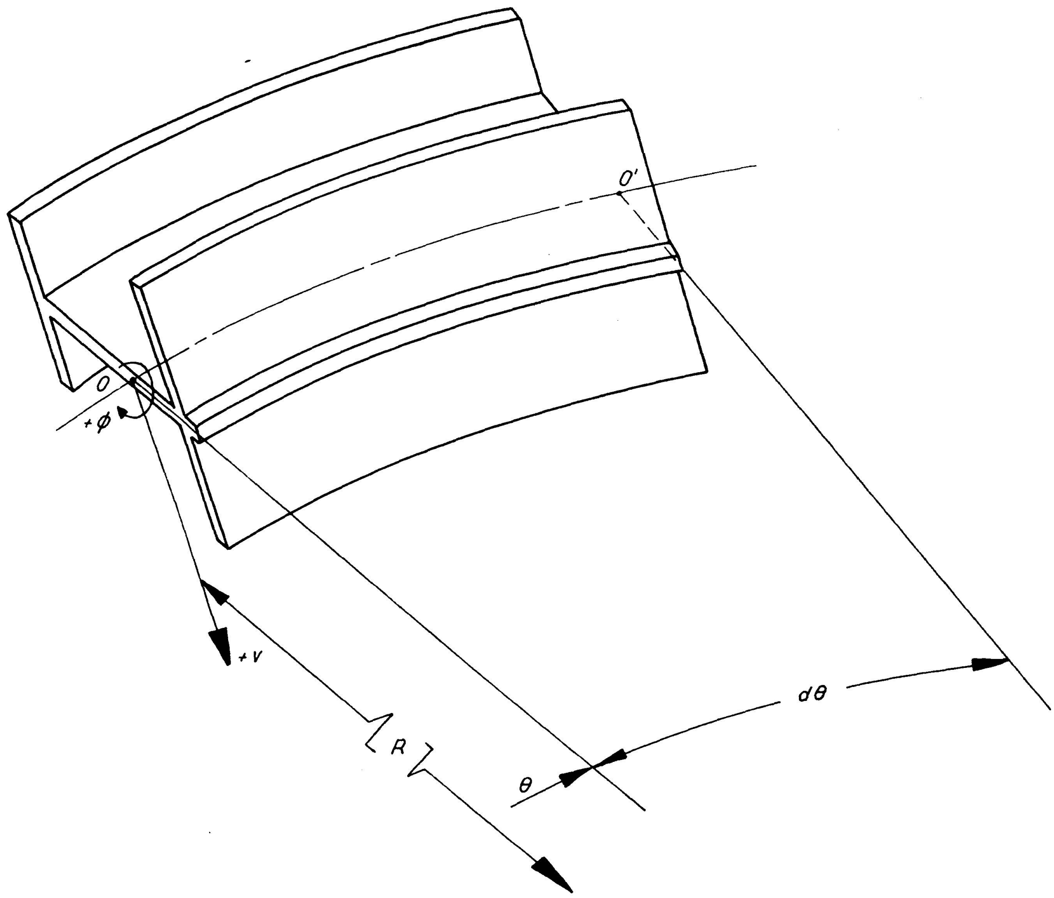

Fig. 2. Differential Element of Ring.

The section is considered as comprising three elements, namely, the two flanges and the web. The changes in curvature and twist of each of the elements are readily determined by vector methods. For clarity, the inner flange is considered.

If the circular axis passing through the centroid of the flange is given vertical displacements, v_1, varying in general along the circular axis, the slope of this axis in the tangential direction may be taken as

$$\psi_1 = \frac{d v_1}{R d\theta} \tag{1}$$

ignoring the slight difference between the radius of this flange and the radius of the centroid of the entire section. The tangential slope at a neighboring point will be

$$\psi_1' = \frac{d v_1}{R d\theta} + \frac{d}{R d\theta}\left(\frac{d v_1}{R d\theta}\right) R d\theta. \tag{2}$$

These slopes or rotations may be represented as vectors along the axes about which they take place, i. e., directed along the respective radial lines, and they may be added or subtracted as ordinary vectors. They are shown in Fig. 3

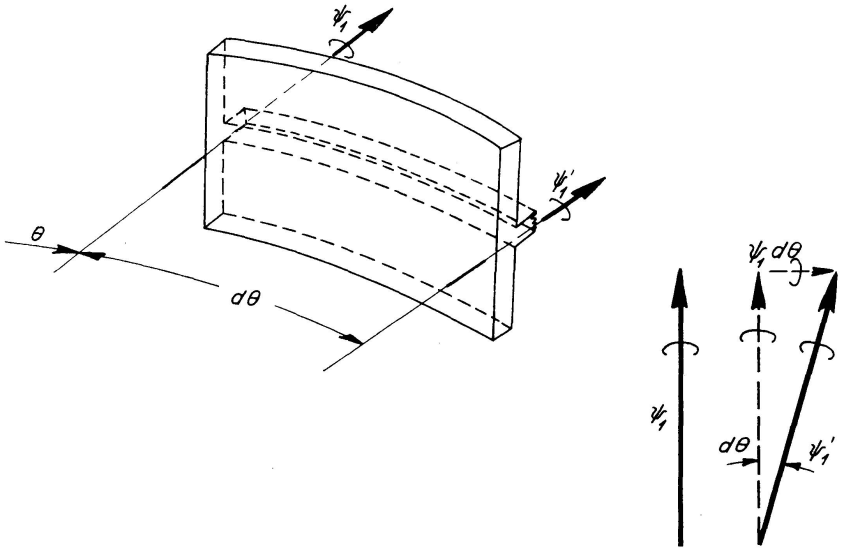

Fig. 3. Vector Analysis of Tangential Slopes of Inner Flange.

using the right hand screw sign convention. As may be seen in the figure, the difference between ψ_1' and ψ_1 for small $d\theta$ may be resolved into a radial component which, to first order in the differential is

$$\frac{d}{R d\theta}\left(\frac{d v_1}{R d\theta}\right) R d\theta = \frac{d^2 v_1}{R^2 d\theta^2} R d\theta \tag{3}$$

and a tangential component which, to the same order, is

$$\psi_1 d\theta = \frac{d v_1}{R d\theta} d\theta. \tag{4}$$

The first of these corresponds to a change in curvature such as would be associated with vertical bending of the element. Since the distance between the two sections is taken to be $R d\theta$ the change in curvature is

$$K_1 = \frac{d^2 v_1}{R^2 d\theta^2}. \tag{5}$$

The second or tangential component corresponds to a twist and the rate of twist due to the vertical displacement becomes

$$\beta_1 = \frac{1}{R} \frac{d v_1}{R d\theta}. \tag{6}$$

The angular displacement of the flange at θ in the plane of the cross section is

$$\chi_1 = \phi \tag{7}$$

and at a neighboring point,

$$\chi_1' = \phi + \frac{d\phi}{R\,d\theta} R\,d\theta. \tag{8}$$

These rotations may be represented as vectors directed along the tangential axes as shown in Fig. 4, again using the right hand rule.

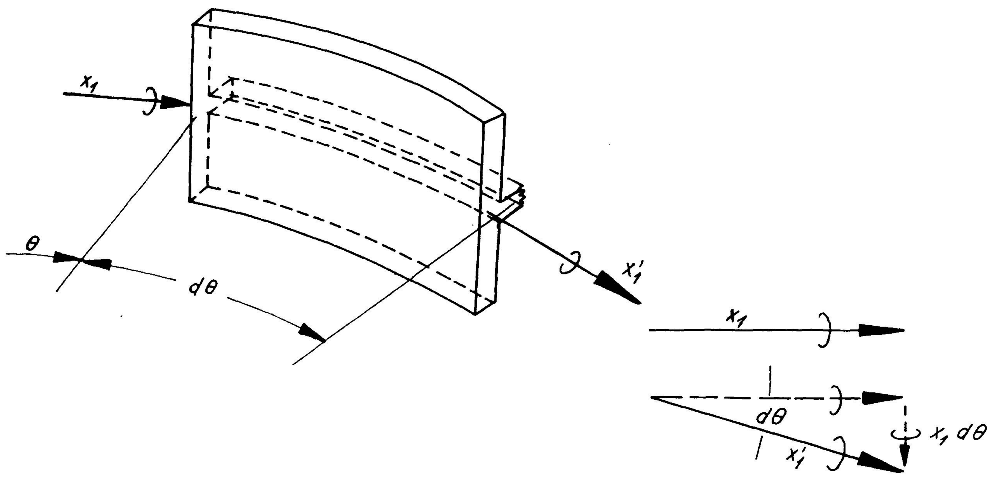

Fig. 4. Vector Analysis of Twisting Displacements of Inner Flange.

As in the previous case, the difference between the two vectors may be resolved into components parallel and normal to the tangential axis at θ. The tangential component of this difference represents an additional twist and, since the distance between the two sections is $R\,d\theta$, the associated rate of twist, to a first order in the differential, is

$$\beta_1 = \frac{d\phi}{R\,d\theta}. \tag{9}$$

The radial component of the difference in the vectors represents a contribution to vertical bending and, as may be seen from Fig. 4, has a magnitude

$$\chi_1 d\theta = \frac{\phi}{R} R\,d\theta. \tag{10}$$

The associated curvature is

$$K_1 = -\frac{\phi}{R}, \tag{11}$$

where the negative sign indicates that the curvature is in a sense opposite to that of the curvature given by Eq. (5).

In view of Eqs. (5), (6), (9) and (11), the rate of twist and the normal curvature due to vertical and angular displacements are

$$\beta_1 = \frac{1}{R}\left(\frac{d v_1}{R\, d\theta} + \frac{d\phi}{d\theta}\right), \tag{12}$$

$$K_1 = \frac{d^2 v_1}{R^2 d\theta^2} - \frac{\phi}{R}. \tag{13}$$

The bending moment in the inner flange about the radial axis, that is, the vertical bending moment, is

$$M_1 = -E I_1 K_1 \tag{14}$$

and, by substitution of Eq. (13), this becomes

$$M_1 = -E I_1\left(\frac{d^2 v_1}{R^2 d\theta^2} - \frac{\phi}{R}\right), \tag{15}$$

where I_1 is the moment of inertia of the cross section of the inner flange about the radial axis.

According to the theory of Saint Venant [2], the torsional moment in the inner flange may be taken as

$$T_1 = C_1 \beta_1, \tag{16}$$

where C_1 is the torsional rigidity of the flange, and becomes, through substitution of Eq. (12),

$$T_1 = \frac{C_1}{R}\left(\frac{d v_1}{R\, d\theta} + \frac{d\phi}{d\theta}\right). \tag{17}$$

If the flange is rectangular, its torsional rigidity may be taken as [2],

$$C_1 = \frac{G l_1 t_1^3}{3}\left(1 - 0.63\frac{t_1}{l_1}\right), \tag{18}$$

in which G is the modulus of rigidity or shearing modulus.

The bending and twisting moments in the web and in the outer flange are obtained by substituting the appropriate subscripts into Eqs. (15), (17) and (18).

An expression for the normal shear in each element will be required. This may be obtained through consideration of the equilibrium of a differential length of the element. The relevant tractions acting on the inner flange are shown in Fig. 5.

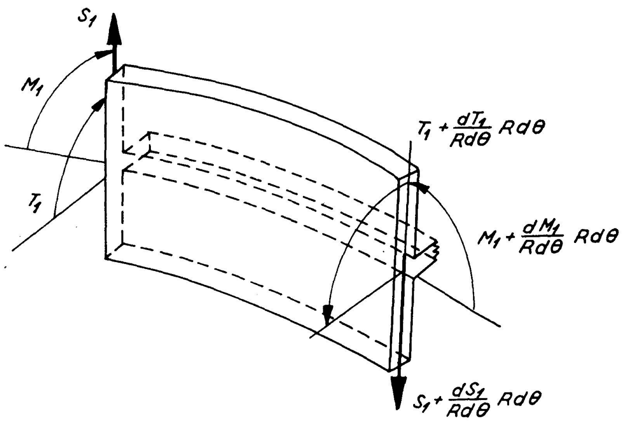

Fig. 5. Internal Tractions Acting on Inner Flange.

Summing moments about a radial axis leads to

$$S_1 = \frac{d M_1}{R\, d\theta} + \frac{T_1}{R}. \tag{19}$$

Similar expressions for the normal shear in the web and in the outer flange are obtained by substituting the appropriate subscripts into Eq. (19).

Equilibrium Equations

Before proceeding to the writing of the pertinent equilibrium equations, it is desirable to evaluate the additional loading due to the combination of circumferential stress and deformation. Taking N to be the resultant hoop force, and assuming that, before buckling occurs, the circumferential stresses are uniformly distributed, these stresses are found to be

$$\sigma = \frac{N}{A}, \tag{20}$$

where A is the cross sectional area, and both N and σ are taken as positive when the ring is in compression.

Before the ring is deformed from a perfectly circular shape, the hoop stresses do not exert loads in a direction normal to the plane of the ring since there is no curvature in this direction. Furthermore, since the section is symmetrical about the plane of the web, the uniformly distributed hoop stresses combined with only the initial or natural curvature in the plane of the ring result in a balanced resultant radial force which produces no moment about the centroid.

When the ring is deformed, the hoop stresses acting along the additionally curved fibers will produce equivalent loads in the directions of curvature. With an origin at the centroid of the cross section and coordinate axes as shown in Fig. 1, the vertical and radial displacements, respectively, of any point may be written as

$$\begin{aligned} w &= v + x\phi, \\ u &= y\phi, \end{aligned} \tag{21}$$

where, as stated previously, v is the vertical displacement at the centroid. The curvature of a deformed fibre, measured in the tangential plane normal to the web, may be obtained by substituting w for v_1 in Eq. (13), and becomes

$$K_n = \frac{d^2 w}{R^2\, d\theta^2} - \frac{\phi}{R},$$

or

$$K_n = \frac{d^2 v}{R^2\, d\theta^2} + x\frac{d^2 \phi}{R^2\, d\theta^2} - \frac{\phi}{R}. \tag{22}$$

The change in curvature in the horizontal plane is

$$K_h = -y \frac{d^2\phi}{dx^2}. \tag{23}$$

As a consequence of these curvatures, the hoop stresses have the effect of producing equivalent loads distributed in the circumferential direction and varying in intensity over the cross section. The intensity of the upward-acting load per unit area of cross section (and per unit length of beam) is

$$q_n = \sigma K_n,$$

or

$$q_n = \sigma\left(\frac{d^2 v}{R^2 d\theta^2} + x\frac{d^2\phi}{R^2 d\theta^2} - \frac{\phi}{R}\right) \tag{24}$$

and the additional inward-acting load per unit area of cross section is

$$q_h = -\sigma K_h,$$

or

$$q_h = \sigma y \frac{d^2\phi}{R^2 d\theta^2}, \tag{25}$$

in which σ is related to the hoop force through Eq. (20).

The upward component of the resultant load per unit length of beam is

$$Q_n = \int q_n \, dA, \tag{26}$$

where the integration is over the entire cross section. Since the origin is taken at the centroid of the section, substitution of Eq. (24) leads to

$$Q_n = N\left(\frac{d^2 v}{R^2 d\theta^2} - \frac{\phi}{R}\right). \tag{27}$$

The radial component of the resultant load, in excess of the load in the unbuckled state, clearly vanishes since the loading varies linearly with the distance from the centroid.

The radial and normal loads have a resultant moment about the tangential axis at the centroid equal to

$$M_N = \int (x q_n + y q_h)\, dA$$

per unit length of beam. Upon substitution of Eqs. (24) and [25), this becomes

$$M_N = \sigma \int \left[x\left(\frac{d^2 v}{R^2 d\theta^2} + x\frac{d^2\phi}{R^2 d\theta^2} - \frac{\phi}{R}\right) + y^2 \frac{d^2\phi}{R^2 d\theta^2}\right] dA$$
$$= \sigma \frac{d^2\phi}{R^2 d\theta^2} \int (x^2 + y^2)\, dA$$

and therefore may be written as

$$M_N = \frac{N}{A} J \frac{d^2\phi}{R^2 d\theta^2}, \tag{28}$$

in which J is the polar moment of inertia of the cross section about its centroid. A positive value of M_N indicates that the moment acts in a clockwise direction about the tangential axis at the centroid of the section shown in Fig. 1.

In addition to the radial loading, p, and to the hoop force, which produces the resultant normal load and torsion given by Eqs. (27) and (28), a differential element of the complete section is subjected to three sets of tractions similar to those shown in Fig. 5 as acting on the inner flange.

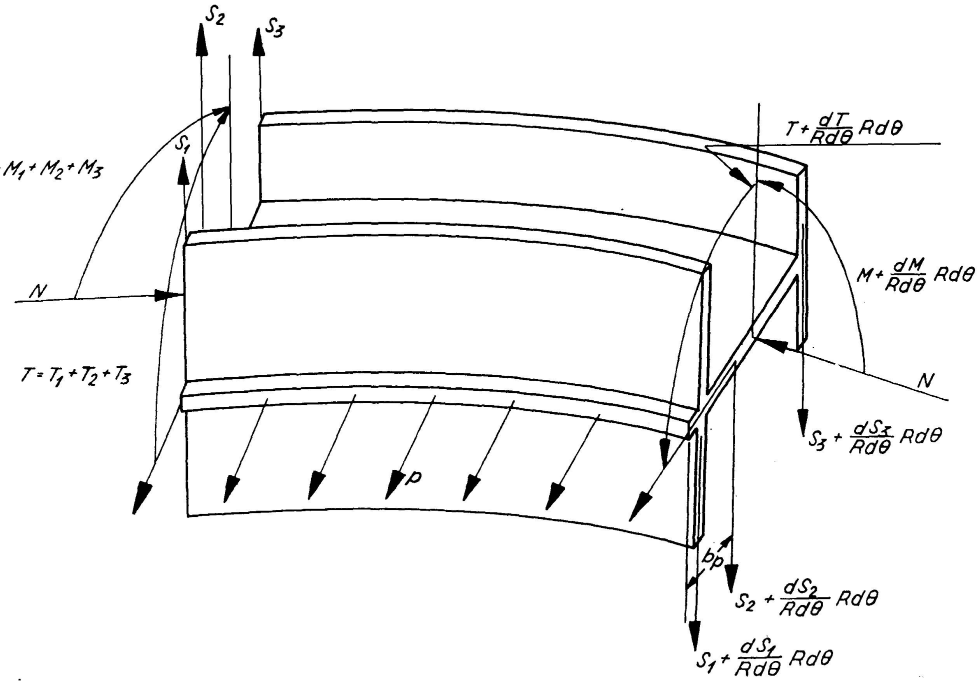

Fig. 6. Loads Acting on Differential Element.

Assuming that the direction of the radial loads is unchanged during buckling, and summing forces in the direction normal to the web yields

$$-S_1-S_2-S_3+\left(S_1+\frac{d\,S_1}{R\,d\,\theta}\,R\,d\,\theta\right)+\left(S_2+\frac{d\,S_2}{R\,d\,\theta}\,R\,d\,\theta\right)$$
$$+\left(S_3+\frac{d\,S_3}{R\,d\,\theta}\,R\,d\,\theta\right)-N\left(\frac{d^2\,v}{R^2\,d\,\theta^2}-\frac{\phi}{R}\right)R\,d\,\theta-p\,\phi\,R\,d\,\theta=0,$$

in which the last term is due to the component of the radial loads normal to the web.

This reduces to

$$\frac{d}{R\,d\,\theta}(S_1+S_2+S_3)-N\left(\frac{d^2\,v}{R^2\,d\,\theta^2}-\frac{\phi}{R}\right)-\frac{N}{R}\phi=0$$

and finally to

$$\frac{d}{R\,d\,\theta}(S_1+S_2+S_3)-N\frac{d^2\,v}{R^2\,d\,\theta^2}=0. \qquad (29)$$

Summing moments about the tangential axis at the centroid yields, upon simplifying,

$$\frac{d}{R\,d\theta}(T_1+T_2+T_3)-\frac{1}{R}(M_1+M_2+M_3) + \frac{d}{R\,d\theta}(b_1 S_1+b_2 S_2+b_3 S_3)-N\frac{J}{A}\frac{d^2\phi}{R^2\,d\theta^2}-b_p\,p\,\phi = 0. \tag{30}$$

Substitution of Eq. (19) together with similar expressions for S_2 and S_3 into Eqs. (29) and (30) gives

$$\frac{d^2}{R^2\,d\theta^2}(M_1+M_2+M_3)+\frac{d}{R^2\,d\theta}(T_1+T_2+T_3)-N\frac{d^2 v}{R^2\,d\theta^2} = 0, \tag{31}$$

$$\frac{d}{R\,d\theta}(T_1+T_2+T_3)-\frac{1}{R}(M_1+M_2+M_3)+\frac{d^2}{R^2\,d\theta^2}(b_1 M_1+b_2 M_2+b_3 M_3) + \frac{d}{R^2\,d\theta}(b_1 T_1+b_2 T_2+b_3 T_3)-N\frac{J}{A}\frac{d^2\phi}{R^2\,d\theta^2}-N\frac{b_p}{R}\phi = 0. \tag{32}$$

It is convenient to introduce the abbreviations,

$$\begin{aligned} B_1 &= E I_1+E I_2+E I_3, \\ B_2 &= E I_1 b_1+E I_2 b_2+E I_3 b_3, \\ B_3 &= E I_1 b_1^2+E I_2 b_2^2+E I_3 b_3^2, \\ B_4 &= C_1+C_2+C_3, \\ B_5 &= C_1 b_1+C_2 b_2+C_3 b_3, \\ B_6 &= C_1 b_1^2+C_2 b_2^2+C_3 b_3^2. \end{aligned} \tag{33}$$

Observing, also, that

$$\begin{aligned} v_1 &= v+b_1\phi, \\ v_2 &= v+b_2\phi, \\ v_3 &= v+b_3\phi \end{aligned} \tag{34}$$

and substituting the expressions for the internal moments and shears transforms Eqs. (31) and (32) into

$$-B_1\left(\frac{d^4 v}{R^4\,d\theta^4}-\frac{d^2\phi}{R^3\,d\theta^2}\right)-B_2\frac{d^4\phi}{R^4\,d\theta^4}+\frac{B_4}{R}\left(\frac{d^2 v}{R^3\,d\theta^2}+\frac{d^2\phi}{R^2\,d\theta^2}\right) + \frac{B_5}{R^2}\frac{d^2\phi}{R^2\,d\theta^2}-N\frac{d^2 v}{R^2\,d\theta^2} = 0 \tag{35}$$

and
$$\frac{B_4}{R}\left(\frac{d^2 v}{R^2\,d\theta^2}+\frac{d^2\phi}{R\,d\theta^2}\right)+\frac{B_5}{R}\frac{d^2\phi}{R^2\,d\theta^2}+\frac{B_1}{R}\left(\frac{d^2 v}{R^2\,d\theta^2}-\frac{\phi}{R}\right) + \frac{B_2}{R}\frac{d^2\phi}{R^2\,d\theta^2}-B_2\left(\frac{d^4 v}{R^4\,d\theta^4}-\frac{d^2\phi}{R^3\,d\theta^2}\right)-B_3\frac{d^4\phi}{R^4\,d\theta^4} + \frac{B_5}{R}\left(\frac{d^2 v}{R^3\,d\theta^2}+\frac{d^2\phi}{R^2\,d\theta^2}\right)+\frac{B_6}{R}\frac{d^2\phi}{R^3\,d\theta^2}-N\left(\frac{J}{A}\frac{d^2\phi}{R^2\,d\theta^2}+\frac{b_p}{R}\phi\right) = 0. \tag{36}$$

This pair of equations together with expressions for the internal tractions in terms of the displacements govern the process of out-of-plane buckling of ideal circular rings of I-section for the assumed loading condition.

Solution of Equations

For an ideal complete ring with appropriate support conditions, namely, equally spaced simple supports, a solution of the governing equations may be taken with sufficient generality in the form,

$$\begin{aligned} v &= V \cos n\theta, \\ \phi &= \Phi \cos n\theta, \end{aligned} \tag{37}$$

in which V and Φ are the amplitudes, undetermined as yet, and n is an integer representing number of circumferential waves in the buckled ring.

With the substitution of Eq. (37) and cancellation of certain common factors, Eqs. (35) and (36) become

$$\begin{aligned} &(A_1 - N)V + A_2 \Phi = 0, \\ &A_2 V + \left[A_3 - \left(\frac{J}{A} - \frac{R\,b_p}{n^2}\right)N\right]\Phi = 0, \end{aligned} \tag{38}$$

in which

$$\begin{aligned} A_1 &= \frac{n^2}{R^2} B_1 + \frac{1}{R^2} B_4, \\ A_2 &= \frac{1}{R} B_1 + \frac{n^2}{R^2} B_2 + \frac{1}{R} B_4 + \frac{1}{R^2} B_5, \\ A_3 &= \frac{1}{n^2} B_1 + \frac{2}{R} B_2 + \frac{n^2}{R^2} B_3 + B_4 + \frac{2}{R} B_5 + \frac{1}{R^2} B_6. \end{aligned} \tag{39}$$

Eqs. (38) constitute a pair of linear homogeneous equations in the amplitudes of the displacements. These equations have a non-trivial solution if the determinant, the elements of which are the coefficients of the amplitudes, vanishes. Expansion of this determinant yields the characteristic equation

$$\left(\frac{J}{A} - \frac{R\,b_p}{n^2}\right)N^2 - \left[A_1\left(\frac{J}{A} - \frac{R\,b_p}{n^2}\right) + A_3\right]N + A_1 A_3 - A_2^2 = 0, \tag{40}$$

which is a quadratic equation in N. The two roots of this equation are readily found by means of the quadratic formula and are

$$N = \frac{b \pm \sqrt{b^2 - 4\,a\,c}}{2\,a}, \tag{41}$$

in which

$$a = \frac{J}{A} - \frac{R}{n^2} b_p, \qquad b = A_1 a + A_3, \qquad c = A_1 A_3 - A_2^2. \tag{42}$$

One may note that Eq. (41) provides two values for the hoop force, N, which will maintain equilibrium of the buckled ring. Ordinarily one of these values will be positive, corresponding to hoop compression, and one will be negative, corresponding to hoop tension. The two values are associated with different combinations of V and Φ and the relative magnitudes of these amplitudes may be determined by substituting the appropriate value of N into either of Eqs. (38).

The case of $n = 0$, corresponding to an axisymmetric mode of buckling, is singular and warrants special consideration. In this case the displacements and the bending moment become independent of the angular coordinate, θ, and the internal torsional moment vanishes. Only the second of Eqs. (38) is relevant. Upon multiplying through by n^2 and setting n equal to zero, this equation reduces to

$$N = -\frac{B_1}{R b_p}. \tag{43}$$

This equation shows that, insofar as axisymmetric buckling is involved, a tensile critical hoop force can exist only if the denominator is positive, i.e., if the radial loads are applied on a circle lying within the circle defined by the centroids of the sections; and that a compressive critical hoop force can exist only if the radial loads are applied on a circle lying outside of the centroids.

Determination of Critical Load

In the case of a complete ring which is simply supported at equally spaced points and subjected only to radial loads, the critical magnitude of the hoop force may be determined through the use of Eqs. (41) and (43). For a given ring only n, the number of full cosine (or sine) waves in the buckled form of the ring, is not known *a priori*. The proper number of waves is the number, among the admissible numbers, which is associated with the lowest of the values given by Eqs. (41) and (43).

The proper value of n is most easily obtained by the following procedure. One determines the set of geometrically admissible values for n, substitutes each of these into Eqs. (39) and (41), and thus computes a set of corresponding values for N. One also computes the value for the symmetrical mode by means of Eqs. (43). The smallest value of the set of computed N's is the critical hoop force at which the ring will buckle.

The admissible values of n in a given case depend upon the number of supports and especially whether this number is odd or even. Clearly, n must be an integer since any non-integral number would imply a discontinuity which is untenable in a complete ring.

If m is the number of equal spaces and is odd, it is easily seen that the minimum admissible non-zero number of full waves is also m. Other admissible values of n are integral multiples of m. The nodes may be located at support points or at equally spaced positions located arbitrarily with respect to the supports.

If m is even, it is easily seen that the minimum non-zero number of full waves is $m/2$. Other admissible values are m, $3m/2$, $2m$ and so on. In the cases of fractional multiples, the support locations must be at nodes of the buckled configuration. In the cases of integral multiples, the nodes need not be located at the supports but may be located arbitrarily with respect to the supports.

An interesting and possibly significant exception to the foregoing is the case of a ring on three supports. With appropriate tilts about two diameters, any buckled ring may retain contact with any three arbitrarily specified support points. It follows that a ring supported at three points may buckle into any number of waves, for example, two, and this possibility exists whether the supports are spaced equally or not.

The case of a ring on four supports also deserves special consideration. In this case the ring may buckle in a sinusoidal mode having any number of nodes. This possibility exists when the supports are located symmetrically about at least one diameter, but do not lie on an axis of symmetry. Furthermore, it is easily seen that the supports need not be equally spaced so long as the axis of symmetry coincides with the diameter about which the buckled ring is symmetrical.

Eqs. (31) and (32) have been derived with the assumption that the direction of the radial loading, p, remains parallel to the plane of the undeformed ring. In some technically important cases the direction of the loads will change during the buckling process. For example, in the case of a suspended roof and with n greater than zero, all loads, p, may be required to pass through a point defined vertically in space at the center of the circle. In this case, the relative angle between the web axis of the cross section and the load p becomes approximately $\phi + (v + b_p \phi)/R$. As a result, the coefficient of N in Eq. (31) should be replaced by

$$- \left(\frac{d^2 v}{R^2 d\theta^2} + \frac{v + b_p \phi}{R^2} \right)$$

and the coefficient of N in Eq. (32) should be replaced by

$$- \left[\frac{J}{A} \frac{d^2 \phi}{R^2 d\theta^2} + \frac{b_p}{R} \left(\phi + \frac{v + b_p \phi}{R^2} \right) \right]$$

with corresponding changes in Eqs. (40) and (41). It should be noticed that

Eq. (43), which gives the critical load for axisymmetrical buckling, ordinarily will not require similar modification. This follows from the fact that the entire suspended structure is free to move up and down as an undeformed body unless an actual support is provided, say, in the form of a post at the center of the structure.

While the basic differential equations, Eqs. (35) and (36), are quite general, the solution culminating in Eq. (41) does not apply to the general case of elastic and unequally spaced supports, or to the case of initial in addition to those in the plane of the ring. The extension of the theory to include these cases will be considered in a future paper.

Illustrative Example

As an illustrative example, critical magnitudes of hoop force of the ring shown in Fig. 7 are calculated for several sinusoidal buckling modes. Inward

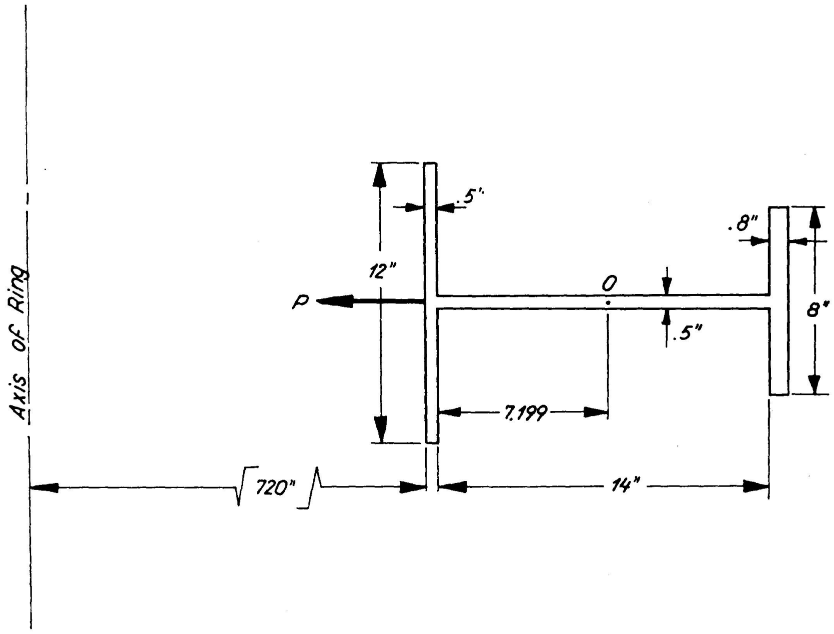

Fig. 7. Cross Section of Ring for Illustrative Example.

acting radial loads are applied at the middle line of the inner surface and it is assumed that the direction of these loads does not change during buckling.

Relevant properties of the section are,

$$
\begin{aligned}
A &= 19.4 \text{ sq. in.} & R &= 727.7 \text{ in.}\\
Iyy &= 779.87 \text{ in.}^4 & b_p &= 7.699 \text{ in.}\\
J &= 886.15 \text{ in.}^4 & E &= 30\times 10^6 \text{ psi}\\
J/A &= 45.6778 \text{ in.}^2 & G &= 11.5\times 10^6 \text{ psi}
\end{aligned}
$$

Additional necessary quantities are calculated below.

Table I. Properties of Ring

Element	I	$\frac{C}{G}$	b	Ib	Ib^2	$\frac{C}{G}b$	$\frac{C}{G}b^2$
1	72.000	0.4868	7.499	536.33	3995.1	3.6262	27.012
2	0.146	0.5625	0.199	0.03	0.0	0.1119	0.022
3	34.133	1.3023	-7.201	-245.79	1769.9	-9.3779	65.530
Total	106.279	2.3516		290.67	5765.0	-5.6398	94.564

$$B_1 = E\,(106.279) = 3.1884 \times 10^9 \qquad B_4 = G\,(2.3516) = 27.043 \times 10^6$$
$$B_2 = E\,(290.67) = 8.7201 \times 10^9 \qquad B_5 = G\,(-5.6398) = -64.858 \times 10^6$$
$$B_3 = E\,(5765.0) = 172.95 \times 10^9 \qquad B_6 = G\,(94.564) = 1087.5 \times 10^6$$

One now computes the quantities, A_1, A_2, A_3, a, b and c by means of Eqs. (39) and (42), and finally, computes the value of N by means of Eq. (41). The results are summarized in Table II.

Table II. Critical Hoop Forces for Various Modes

n	Admissible No. of Supports	N, Critical Load, lbs.	
		Compressive	Tensile
0	Any	—	571 200
1	2, 3	340	576 900
2	2, 3, 4	606	603 200
3	2, 3, 4, 6	3 663	656 800
4	2, 3, 4, 8	12 200	754 400
5	2, 3, 4, 5, 10	31 230	924 400
6	2, 3, 4, 6, 12	58 180	1 218 500
7	2, 3, 4, 7, 14	99 150	1 725 200
8	2, 3, 4, 8, 16	157 340	2 670 500
9	2, 3, 4, 6, 9, 18	217 110	4 695 400
10	2, 3, 4, 5, 10, 20	292 700	10 840 500

The second column of the table lists the number of equally spaced simple supports which are compatible with the indicated modes of buckling. It appears reasonably clear that the use of larger values of n will give correspondingly larger computed values for N in both tension and compression.

The magnitudes of the critical load for out-of-plane buckling may be compared with the magnitude for buckling in the plane of the ring. For the latter case, the critical hoop force is given by the formula [1],

$$N = \frac{3\,E\,I_{yy}}{R^2}. \tag{44}$$

Since I_{yy}, the moment of inertia about the axis normal to the web, is 779.87 in.^{4}, Eq. (44) yields approximately 135,000 pounds for the magnitude of the compressive hoop force for buckling in the plane of the ring. Reference to Table II shows that this value falls between the values for out-of-plane buckling corresponding to modes having seven and eight full waves. Consequently, the sole use of Eq. (44) as a design criterion would be unconservative, and perhaps greatly so, if the ring had fewer than fourteen equally spaced simply supports.

The critical loads shown in Table II have been computed, for simplicity, using the indicated elastic values for the two moduli. Consequently, certain of the results are implausibly high and exceed the yield strength of presently available materials. These excessively high values are reduced to realistic values by intróducing an appropriate modulus reduction factor. The correct critical load, stress and moduli are readily found by a trial-and-error procedure starting with, say, the elastic results. One now chooses a stress, determines the corresponding moduli from the proper stress-strain curves, and reduces the computed hoop force in the ratio of the reduced modulus to the elastic modulus. The stress corresponding to the reduced hoop force is now compared with the previously assumed stress. If these values coincide, the reduced hoop force is the critical force. If not, a new trial value of stress is chosen and the comparison process is repeated.

Since the radial loads act inwardly, the tensile critical loads listed in Table II are irrelevant and may be disregarded. If, however, the ring were being designed to take the thrusts of a large number of arches as for a ribbed dome, the tensile values would be significant. This would also be true if the ring were being designed to support a shell dome, particularly if the dome were somewhat less than hemispherical.

Discussion

If the supports are not equally spaced, the ring ordinarily will buckle in a mode which is not sinusoidal. It is also conceivable that, even though the supports are equally spaced, the ring might buckle in a mode which is not sinusoidal. One may speculate that this is a likely possibility if the number of equally spaced supports is large or moderately so. Under these circumstances, a buckle might have an essentially local character. Thus, one might assume a mode which is almost a half-sine wave in one interval between two supports and with similar alternating forms but having continually attenuated amplitudes in successive intervals in both directions from the large buckle. Although Eqs. (35) and (36) are still applicable, the rigorous mathematical solution is quite difficult and tedious. One would have to write the general solution of this eighth order system for each interval with eight arbitrary constants or parameters. Continuity and compatibility conditions are then written to join the successive segments of the ring and will form a set of linear, homogeneous, algebraic equations. The requirement that the determinant of the coefficients of the arbitrary constants vanishes will provide the characteristic equation, and the lowest root of this polynominal equation will correspond to the critical load.

In view of the difficulty associated with obtaining a rigorous solution for this case, an approximate and generally conservative solution may be proposed. Such a practical solution is obtained by taking the mode in the form

$$v = V \sin \frac{\pi\theta}{\alpha}, \qquad \phi = \Phi \sin \frac{\pi\theta}{\alpha} \tag{45}$$

instead of using Eqs. (37). Here α is the angle subtended by the arc or interval between two adjacent supports and θ is measured from one of these supports toward the other. The consequence of this assumption is simply to replace n in the following equations and in the calculations by the term π/α. This assumed form of solution disregards the constraint imposed upon the interval in which the major buckling occurs by the somewhat attenuated buckles of the successive intervals.

When only buckling in a regular mode such as implied by Eqs. (37) is involved, the great increase which may be obtained in going from an even number of equally spaced supports to an odd number deserves notice and comment. For example, if the ring is supported at four points, it may buckle in an unsymmetrical mode having as few as two full waves. As shown in Table II for $n = 2$, the critical compressive load is 606 pounds. If the ring were supported at five points, it could buckle in a sinusoidal mode with a minimum of five full waves. The critical compressive load for $n = 5$, is 31,230 pounds. Thus, by adding one support, a very great increase in the critical load has been obtained, at least theoretically.

The appreciable increase in the magnitude of the critical load which is obtained by going from four supports to five is not wholly unexpected, although the amount of the increase in this particular problem is somewhat surprising. However, an even more striking situation is disclosed if one compares the critical loads for the cases of five and six supports. In the case of six equally spaced supports, buckling modes can occur with as few as three full waves. For $n = 3$, Table II shows the compressive buckling load to be 3,663 pounds. Assuming that the ring on five supports would buckle in a regular sinusoidal mode, one sees that an increase of about 750 per cent would be obtained by reducing the number of supports from six to five. This comparison strongly suggests that the designer should contemplate the use of an odd number of supports if these are to be equally spaced. While these conclusions have been based upon the assumption of buckling in a regular sinusoidal mode when the number of supports is odd as well as even, it is entirely likely that this condition will be very nearly attained if the number of odd supports is not large.

One need only consider the fact that, in the case of an odd number of supports, a mode having essentially one-half wave between supports must have an attenuating form so that compatibility can be satisfied in the complete ring. In the case of a small number of supports, it is doubtful that sufficient attenuation could occur in the limited number of successive spans on either side of the major buckle and still retain the nearly sinusoidal mode with one-half wave per span which is associated with this minimum critical load.

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Summary

A theory is presented for out-of-plane buckling of circular rings of I-section and the case of buckling of a complete ring under the action of a distributed radial loading is considered. Formulas are derived which give the critical magnitudes of the hoop forces associated with the out-of-plane buckling modes. The problem is of interest, for example, in connection with the design of the compression ring at the outer boundary of a suspended roof, and occasionally in connection with the design of the thrust ring at the base of a ribbed dome.

For simplicity, the theory is presented for sections which are symmetrical about a radial axis but which need not possess a normal axis of symmetry. Although the formulas presented are valid for singly and doubly-symmetric sections, the theory may be extended to unsymmetrical sections.

In addition to the possibility of out-of-plane buckling of a ring when subjected to a loading which produces a compressive hoop force, it is shown that a ring may also buckle, under certain conditions, when the hoop force is tensile.

Résumé

L'auteur étudie le flambement latéral des anneaux circulaires à section en double-té et examine le cas d'un anneau fermé, soumis à une charge radiale répartie. Il établit des formules donnant les valeurs critiques des efforts circulaires, associées aux divers modes d'instabilité latérale. Le problème se pose, par exemple, pour l'étude de la ceinture comprimée des toitures suspendues et, parfois, pour l'anneau inférieur des coupoles nervurées.

Pour simplifier l'exposé, l'auteur n'envisage que des sections possédant au moins un axe de symétrie, contenu dans le plan de l'anneau. Les formules présentées sont valables pour des sections à symétrie simple ou double; la théorie peut cependant être étendue à des sections quelconques.

L'auteur montre que, sous certaines conditions, un anneau peut flamber latéralement, non seulement lorsqu'il est comprimé, mais même lorsqu'il est soumis à des charges produisant une traction circulaire.

Zusammenfassung

Es wird eine Theorie für das Kippen von Kreisringen mit I-Querschnitt entwickelt und der Fall des Kippens eines geschlossenen Ringes unter einer verteilten Radialbelastung untersucht. Es werden Formeln hergeleitet, welche die kritischen Ringkräfte bei verschiedenen Arten von Kippen angeben. Das Problem ist z. B. im Zusammenhang mit der Ausbildung des Druckringes am Rand eines Hängedaches und gelegentlich im Zusammenhang mit der Ausbildung des Fußringes einer gerippten Kuppel von Interesse.

Der Einfachheit halber werden nur einfach-symmetrische Querschnitte untersucht, wobei die Symmetrieachse mit der Ringebene zusammenfällt. Obwohl die angeführten Formeln nur für einfach- und doppeltsymmetrische Querschnitte Geltung haben, so kann die Theorie ebenfalls auf unsymmetrische Profile ausgedehnt werden.

Es zeigt sich, daß ein Ring nicht nur unter Ringdruckkräften, sondern auch unter gewissen Umständen, bei äußeren Belastungen mit entsprechenden Ringzugkräften auskippen kann.