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A Method of Analysis of a Solid Structure Composed of Plate Elements

Méthode de calcul des ouvrages continus formés de plaques

Zur Berechnung von Plattensystemen

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1. Introduction

In this report, the authors describe a method of analysis of a solid structure composed of plate elements, together with its application to practical design. The method is based essentially on an approximation for boundary conditions. Although the differential equation itself is satisfied at all interior points on the plate elements, the compatibility condition is satisfied only in a pointwise manner along the boundary lines. The condition may be transformed into the form in which the distribution of redundant forces on the boundary line is summed with respect to a certain number of blocks, the number and length of such blocks being chosen so as to bring the required accuracy of the solution into effect. The computational work involved in the mathematical procedure is considerably reduced by the proposed approximation.

Some examples of the application of the method for purposes of practical design are worked out with regard to a culvert-type structure. The results are compared with those obtained on the bases of the Rahmen-theory, when some interesting differences are found.

2. A Method of Relaxation Proposed for the Boundary Condition of the Plate Element

As shown in Fig. 1, a rectangular plate with the moment M_ξ distributed along the edge $(y=0)$ is taken as an example. The boundary condition of the plate is modified in the following manner. The distribution of the edge moment

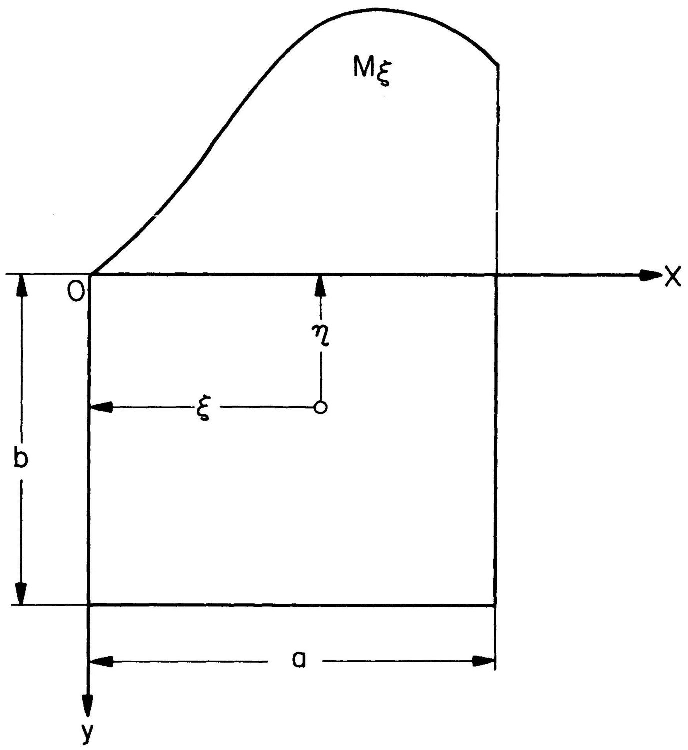


Fig. 1.

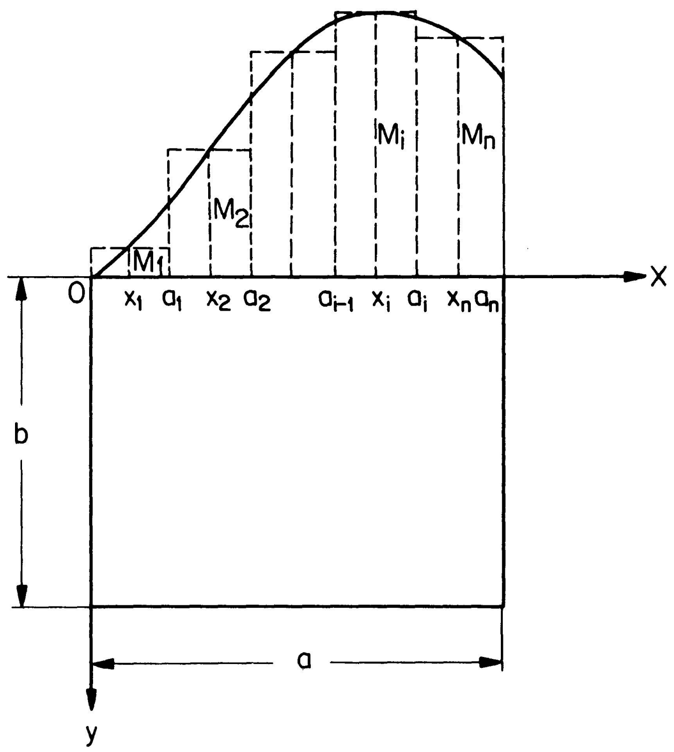


Fig. 2.

M_ξ is divided into a certain number of sections. The number and length of these sections must be determined in accordance with the required accuracy of the solution, and on the assumption that the moment distributed over every section is uniform, as shown by the dotted line in Fig. 2. Thus, an approximate solution for a plate shown in Fig. 1 may be obtained. By suitably increasing the number of sections, the solution may be made still more accurate.

If the deflection $\overline{W}$ of a plate under a unit concentrated load at a certain point (ξ, η) be expressed in the form

$$\overline{W} = F(\xi, \eta, x, y) \tag{1}$$

the deflection W of the plate shown in Fig. 2 can be given as the summation:

$$W = \sum_{i=1}^{n} M_i \int_{a_{i-1}}^{a_i} \left(\frac{\partial F}{\partial \eta}\right)_{\eta=0} d\xi, \tag{2}$$

where M_i is a uniform moment in the ith section distributed over a range $(a_{i-1} \sim a_i)$. By means of (1) the slope at a selected point $(x_i, 0)$ in the ith section is obtained in the following form:

$$(\theta)_{\substack{x=x_i\\y=0}} = \left(\frac{\partial W}{\partial y}\right)_{\substack{x=x_i\\y=0}}. \tag{3}$$

The compatibility condition is satisfied at those points $(x = x_1, x_2, \ldots x_i, \ldots \ldots x_n, y = 0)$ in pointwise manner along the boundary line $(y = 0)$. Thus, the condition may be transformed into a form convenient for numerical computation within certain limits of accuracy. Such a condition may be written as a system of simultaneous linear equations involving unknown moments $M_1, M_2, \ldots M_i, \ldots M_n$ which are distributed so as to satisfy the above approximate condition. Solving the equations, the partially distributed moments $M_1, M_2, \ldots M_i, \ldots, M_n$, together with the approximate variation of the redundant moment M_ξ, will be obtained. Applying this method, the computational work involved, having regard to the convergency of the solution, and the number of the simultaneous equations, may be considerably reduced. A similar procedure was developed and worked out by the same authors, with some numerical examples relating to the plate, in a previous paper [1] which confirms the satisfactory accuracy of the results.

3. A Method of Analysis of Solid Structures Consisting of Plate Elements

The concepts described in the previous section may be applied to the solution of solid structures composed of plate elements.

In the present paper, the symmetrical bending in a culvert type structure composed of rectangular plates is worked out.

The structure, together with its coordinate system of reference, is shown in Fig. 3a. The assumptions introduced in to the analysis, in addition to those usually understood in the ordinary theory of plates of medium thickness, are as follows.

1. The structure is monolithic and is formed of rectangular plates.
2. Each of the elementary plates may be regarded as a rectangular plate

with two opposite edges free and the other two edges under the influence of an unknown distributed moment.

3. The distribution of the moments applied along the joints of several elementary plates, in order to secure the continuity of adjacent parts, is expressed by means of a system of several blocks having uniform moments, as described in the previous section.

Fig. 3b shows the continuity of the plates at a certain point P_i in the ith section of the joint.

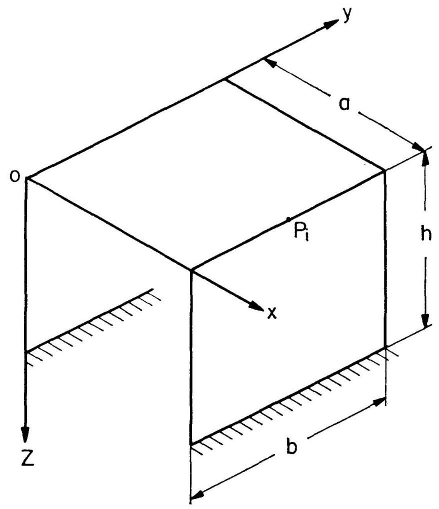


Fig. 3a.

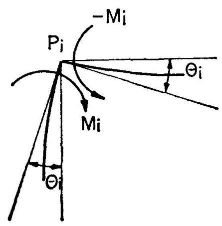


Fig. 3b.

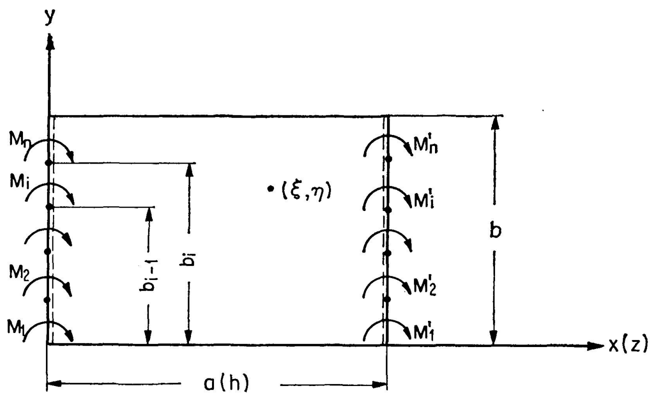


Fig. 4.

Referring to Fig. 4, the deflection W_M of each elementary plate, due to the joint moments $M_1, M_2, \ldots, M_i, \ldots, M_n$, $M'_1, M'_2, \ldots, M'_i, \ldots, M'_n$ may be obtained by the following operation

$$W_M = \sum_{i=1}^{n} M_i \int_{b_{i-1}}^{b_i} \left(\frac{\partial \overline{W}}{\partial \xi}\right)_{\xi=0} d\eta + \sum_{i=1}^{n} M'_i \int_{b_{i-1}}^{b_i} \left(\frac{\partial \overline{W}}{\partial \xi}\right)_{\xi=a} d\eta, \tag{4}$$

where $\overline{W}$ denotes the deflection of the same plate with two opposite free edges, the other two edges being simply supported and subjected to a unit load at a point (ξ, η), while M_i and M'_i are the uniform moments at the ith section distributed over a range $(b_{i-1} \sim b_i)$. By means of (4) and a well-known solution for $\overline{W}$, the deflection M_M may be given in the following form:

$$W_M = \frac{4}{D\,a\,b}\sum_{i=1}^{n}\sum_{m}\sum_{n}\frac{\alpha_m f_{mn}(y)}{(\alpha_m^2+\beta_n^2)^2\beta_n}(\cos\beta_n\eta_{i-1}-\cos\beta_n\eta_i)$$
$$\cdot\{M_i+(-1)^m M_i'\}\sin\alpha_m x, \tag{5}$$

where $\alpha_m = \dfrac{m\pi}{a}, \qquad \beta_n = \dfrac{n\pi}{b},$

$$f_{mn}(y) = \sin\beta_n y + \frac{\beta_n}{\alpha_m}\left(\frac{\beta_n^2}{\alpha_m^2}+2-\nu\right)(A_{mn}\cosh\alpha_m y + B_{mn}\alpha_m y\sinh\alpha_m y$$
$$+C_{mn}\sinh\alpha_m y + D_{mn}\alpha_m y\cosh\alpha_m y), \tag{6}$$

$$A_{mn} = -\frac{2B_{mn}}{1-\nu},$$
$$B_{mn} = \frac{\{(-1)^n\operatorname{cosech}\alpha_m b - \coth\alpha_m b\}\{3+\nu-(-1)^n(1-\nu)\alpha_m b\operatorname{cosech}\alpha_m b\}}{(3+\nu)^2-\{(1-\nu)\alpha_m b\operatorname{cosech}\alpha_m b\}^2},$$
$$C_{mn} = \frac{1}{1-\nu}\{(1+\nu)D_{mn}-1\}, \tag{7}$$
$$D_{mn} = \frac{3+\nu-(-1)^n(1-\nu)\operatorname{cosech}\alpha_m b}{(3+\nu)^2-\{(1-\nu)\alpha_m b\operatorname{cosech}\alpha_m b\}^2},$$

D = flexural rigidity of a plate,
ν = Poisson's ratio.

By means of (5) the slopes of the plate at selected points $(x=0, y=y_i)$ and $(x=a,\ y=y_i)$ are given as follows

$$(\theta_M)_{\substack{x=0\\y=y_i}} = \left(\frac{\partial W_M}{\partial x}\right)_{\substack{x=0\\y=y_i}} =$$
$$\frac{4}{D\,a\,b}\sum_{i=1}\sum_{m}\sum_{n}\frac{\alpha_m^2 f_{mn}(y_i)}{(\alpha_m^2+\beta_n^2)^2\beta_n}(\cos\beta_n y_{i-1}-\cos\beta_n y_i)\{M_i+(-1)^m M_i'\}, \tag{8}$$

$$(\theta_M)_{\substack{x=a\\y=y_i}} = \left(\frac{\partial W_M}{\partial y}\right)_{\substack{x=a\\y=y_i}} =$$
$$\frac{4}{D\,a\,b}\sum_{i=1}\sum_{m}\sum_{n}\frac{\alpha_m^2 f_{mn}(y_i)}{(\alpha_m^2+\beta_n^2)^2\beta_n}(\cos\beta_n y_{i-1}-\cos\beta_n y_i)\{(-1)^m M_i+M_i'\}. \tag{9}$$

Superimposing thereupon the slope of the plate due to the given load, the general expression for the slope of the plate is given by

$$\theta = \theta_M + \theta_L, \tag{10}$$

where θ_L denotes the slope of the same plate with two opposite free edges, the other two edges being simply supported under the given load.

The continuity equation at a point taken in the joint of the elementary plates may be obtained by equating the slopes at this point of the plates on both sides of the joint. The equations satisfying n selected points along the joints make up a system of simultaneous linear equations with n unknown moments. Furthermore, the preceding expressions (8) and (9) may also be expressed in a form more convenient for numerical calculation by introducing the summation formulas.

4. Examples: Symmetrical Bending of Culvert Type Structures

The problem is to deal with the symmetrical bending of a culvert-type structure composed of square plates with a symmetrical load.

As shown in Fig. 5, each of the joints is divided into 5 equal sections, and the midpoint of each section is chosen as the compatible point.

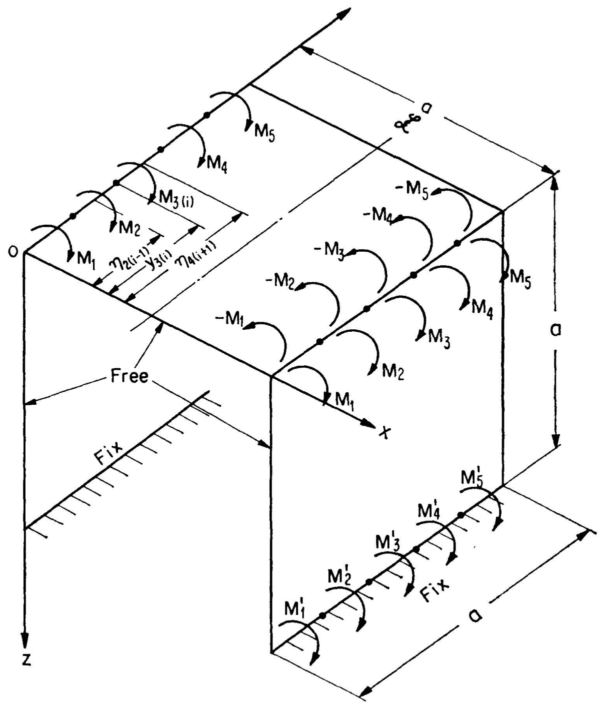


Fig. 5.

The slopes of the vertical plate due to the joint moments M and M' at the compatible points ($z=0$, $y=y_i$ and $z=a$, $y=y_i$) are obtained from the expressions (8) and (9) by means of summation formulas:

$$(\theta_M)_{\substack{z=0\\ y=y_i}} = \frac{a}{D\pi^2}\sum_{i=0} M_i(\phi_{\eta_{i-1}}-\phi_{\eta_{i+1}})+M'_i(\phi'_{\eta_{i-1}}-\phi'_{\eta_{i+1}}), \tag{11}$$

where $M_1=M_5, \quad M_2=M_4, \quad M'_1=M'_5, \quad M'_2=M'_4,$

$$\begin{aligned}
\phi &= \phi_1+\phi_2+\phi_3,\\
\phi' &= \phi'_1+\phi'_2,\\
\phi_1 &= \psi_\lambda\left(\lambda=\frac{y_i+\eta}{a}\right)+\psi_\lambda\left(\lambda=\frac{y_i-\eta}{a}\right),\\
\psi_\lambda &= \frac{\pi\lambda}{2}\left[1-\log\frac{\pi\lambda}{2}-\cdots\frac{2^{2n}(2^{2n-1}-1)B_n}{n(n+1)!}\left(\frac{\pi\lambda}{2}\right)^{2n}-\cdots\right],
\end{aligned} \tag{12}$$

B_n: Bernoulli's number, $n=1,3,5,\dots,$

$$\phi_2 = \sum_{\substack{n \\ 1,3,5\ldots}} \left(\coth n\pi - \frac{\pi}{n}\,\mathrm{cosech}^2\, n\pi - 1\right) \cos\alpha_n \eta \sin\alpha_n y_i\,,$$

$$\phi_3 = \sum_{\substack{m \\ 1,2,3\ldots}} \rho_m\,,$$

$$\rho_m = \frac{(\nu-1)\pi}{4m}\left[\cosh\alpha_m\eta\,\mathrm{sech}^2\frac{m\pi}{2} + \frac{2\eta}{a}(K_m - e^{-\alpha_m\eta}) - \frac{2}{m\pi}\left(1+\frac{2}{1-\nu}\right)(K'_m + e^{-\alpha_m\eta})\right] X_m(y_i)\,,$$

$$K_m = \left(\tanh\frac{m\pi}{2} - 1\right)\sinh\alpha_m\eta\,, \tag{12}$$

$$K'_m = \left(\tanh\frac{m\pi}{2} - 1\right)\cosh\alpha_m\eta\,,$$

$$X_m(y_i) = A_{mn\,(n=1,3,5\ldots)}\cosh\alpha_m y_i + B_{mn\,(n=1,3,5\ldots)}\alpha_m y_i \sinh\alpha_m y_i + C_{mn\,(n=1,3,5\ldots)}\sinh\alpha_m y_i + D_{mn\,(n=1,3,5\ldots)}\alpha_m y_i\cosh\alpha_m y_i\,,$$

$$\phi'_1 = \sum_{\substack{n \\ 1,3,5\ldots}} \left(\frac{1}{n} - \pi\coth n\pi\right)\mathrm{cosech}\, n\pi\cos\alpha_n\eta\sin\alpha_n y_i\,,$$

$$\phi'_2 = \sum_{\substack{m \\ 1,2,3\ldots}} (-1)^m \rho_m\,,$$

$$(\theta_M)_{\substack{z=a \\ y=y_i}} = \frac{a}{D\pi^2}\sum_{i=1}^{5} M_i(\phi'_{\eta_{i-1}} - \phi'_{\eta_{i+1}}) + M'_i(\phi_{\eta_{i-1}} - \phi_{\eta_{i+1}})\,, \tag{13}$$

while the slopes of the top plate due to the joint moments M at the compatible points ($x=a$, $y=y_i$) are given as follows

$$(\theta'_M)_{\substack{x=a \\ y=y_i}} = \frac{a}{D\pi^2}\sum_{i=1}^{5}(\phi_{\eta_{i-1}} - \phi_{\eta_{i+1}} + \phi'_{\eta_{i-1}} - \phi'_{\eta_{i+1}})M_i\,. \tag{14}$$

The continuity condition at a compatible point P_i upon a joint ($z=0$, $y=y_i$) of each plate is expressed in the form

$$\sum M_{p_i} = 0\,, \tag{15}$$

$$(\theta_M + \theta_L)_{\substack{z=0 \\ y=y_i}} = (\theta'_M + \theta'_L)_{\substack{x=a \\ y=y_i}}\,, \tag{16}$$

in which θ_L, θ'_L are the slopes of each plate due to the given load.

The boundary condition of the edge ($z=a$, $y=y_i$) of the vertical walls is given by

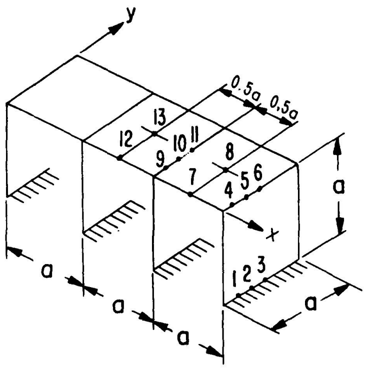
$$(\theta_M + \theta_L)_{\substack{z=a \\ y=y_i}} = 0\,. \tag{17}$$

From the results of Eqs. (15), (16) and (17), 6 equations containing the unknown moments M_1, M_2, M_3, M'_1, M'_2 and M'_3 are finally obtained as shown in Table 1.

Table 1

M Location	M_1	M_2	M_3	M_1'	M_2'	M_3'	$\theta_L\left(\times\frac{D\pi^2}{a}\right)$
$P(z=0, y=0.1a)$	$4(\phi_0-\phi_{0.2a})$ $-2(\phi_0'-\phi_{0.2a}')$	$4(\phi_{0.2a}-\phi_{0.4a})$ $-2(\phi_{0.2a}'-\phi_{0.1a}')$	$2(\phi_{0.4a}-\phi_{0.6a})$ $-(\phi_{0.4a}'-\phi_{0.6a}')$	$2(\phi_0'-\phi_{0.2a}')$	$2(\phi_{0.4a}'-\phi_{0.2a}')$	$\phi_{0.6a}'-\phi_{0.4a}'$	$\theta_L'-\theta_L$
$P(z=0, y=0.3a)$	$4(\phi_0-\phi_{0.2a})$ $-2(\phi_0'-\phi_{0.2a}')$	$4(\phi_{0.2a}-\phi_{0.4a})$ $-2(\phi_{0.2a}'-\phi_{0.4a}')$	$2(\phi_{0.4a}-\phi_{0.6a})$ $-(\phi_{0.4a}'-\phi_{0.6a}')$	$2(\phi_0'-\phi_{0.2a}')$	$2(\phi_{0.4a}'-\phi_{0.2a}')$	$\phi_{0.6a}'-\phi_{0.4a}'$	$\theta_L'-\theta_L$
$P(z=0, y=0.5a)$	$4(\phi_0-\phi_{0.2a})$ $-2(\phi_0'-\phi_{0.2a}')$	$4(\phi_{0.2a}-\phi_{0.4a})$ $-2(\phi_{0.2a}'-\phi_{0.4a}')$	$2(\phi_{0.4a}-\phi_{0.6a})$ $-(\phi_{0.4a}'-\phi_{0.6a}')$	$2(\phi_0'-\phi_{0.2a}')$	$2(\phi_{0.4a}'-\phi_{0.2a}')$	$\phi_{0.6a}'-\phi_{0.4a}'$	$\theta_L'-\theta_L$
$P(z=a, y=0.1a)$	$2(\phi_0'-\phi_{0.2a}')$	$2(\phi_{0.4a}'-\phi_{0.2a}')$	$\phi_{0.6a}'-\phi_{0.4a}'$	$2(\phi_0-\phi_{0.2a})$	$2(\phi_{0.4a}-\phi_{0.2a})$	$\phi_{0.6a}-\phi_{0.4a}$	$-\theta_L$
$P(z=a, y=0.3a)$	$2(\phi_0'-\phi_{0.2a}')$	$2(\phi_{0.4a}'-\phi_{0.2a}')$	$\phi_{0.6a}'-\phi_{0.4a}'$	$2(\phi_0-\phi_{0.2a})$	$2(\phi_{0.4a}-\phi_{0.2a})$	$\phi_{0.6a}-\phi_{0.4a}$	$-\theta_L$
$P(z=a, y=0.5a)$	$2(\phi_0'-\phi_{0.2a}')$	$2(\phi_{0.4a}'-\phi_{0.2a}')$	$\phi_{0.6a}'-\phi_{0.4a}'$	$2(\phi_0-\phi_{0.2a})$	$2(\phi_{0.4a}-\phi_{0.2a})$	$\phi_{0.6a}-\phi_{0.4a}$	$-\theta_L$

Table 2



Remarks:

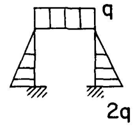
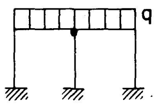
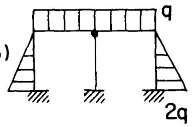
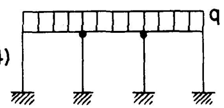
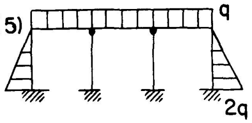
Values 1) are the results by present analysis.

Values 2) are the results by Rahmen-theory.

Values 3) are the differences (%).

In this examples, the top joints of intermediate walls are assumed hinged ones.

All elementary plates have the same stiffness.

Type	Joint Moment ($\times qa^2$)								
	M_1 $y_i=0.1a$	M_2 $y_i=0.3a$	M_3 $y_i=0.5a$	M_4 $y_i=0.1a$	M_5 $y_i=0.3a$	M_6 $y_i=0.5a$	M_9 $y_i=0.1a$	M_{10} $y_i=0.3a$	M_{11} $y_i=0.5a$
(1) 	-0.0909 1) -0.0944 2) -4 3)	-0.102 -0.0944 7	-0.102 -0.0944 7	-0.0835 -0.0778 7	-0.0682 -0.0778 -14	-0.0670 -0.0778 -16			
(2) 	0.0245 1) 0.0208 2) 15 3)	0.0234 0.0208 11	0.0232 0.0208 10	-0.0533 -0.0417 22	-0.0443 -0.0417 6	-0.0430 -0.0417 3	-0.1251 -0.1042 17	-0.1122 -0.1042 7	-0.1093 -0.1042 5
(3) 	-0.0869 1) -0.0958 2) -10 3)	-0.0980 -0.0958 2	-0.0980 -0.0958 2	-0.0910 -0.0750 18	-0.0766 -0.0750 2	-0.0749 -0.0750 0	-0.1134 -0.0875 23	-0.0952 -0.0875 8	-0.0925 -0.0875 6
(4) 	0.0288 1) 0.0227 2) 21 3)	0.0278 0.0227 18	0.0273 0.0227 17	-0.0618 -0.0455 26	-0.0527 -0.0455 14	-0.0513 -0.0455 11	-0.0970 -0.0909 6	-0.0824 -0.0909 -10	-0.0803 -0.0909 -13
(5) 	-0.0843 1) -0.0955 2) -13 3)	-0.0955 -0.0955 0	-0.0955 -0.0955 0	-0.0962 -0.0758 21	-0.0816 -0.0758 7	-0.0799 -0.0758 5	-0.0959 -0.0848 12	-0.0771 -0.0848 -10	-0.0751 -0.0848 -13

A solution of the symmetrical bending in certain multispan-type structures may be simply obtained by the combination of Eqs. (15), (16) and (17). Fig. 6 shows the continuity of three elementary plates at a certain point P_i in the ith section of the joint. The continuity equations at the point are expressed in the form

$$\theta_i = \theta'_i = \theta''_i , \tag{18}$$

$$\sum M_{p_i} = 0 . \tag{19}$$

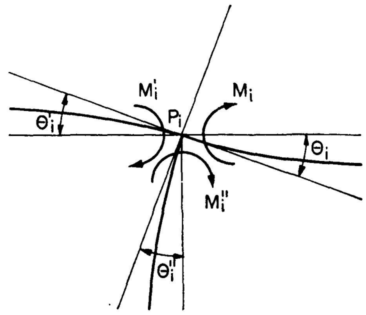


Fig. 6.

Table 3

Type	Span Moment ($\times qa^2$)			
	M_7 $y_i=0$	M_8 $y_i=0.5a$	M_{12} $y_i=0$	M_{13} $y_i=0.5a$
(1)	0.0589 1) 0.0472 2) 25 3)	0.0509 0.0472 8		
(2)	0.0513 1) 0.0520 2) -1 3)	0.0432 0.0520 -17		
(3)	0.0417 1) 0.0437 2) -5 3)	0.0336 0.0437 -23		
(4)	0.0610 1) 0.0568 2) 7 3)	0.0530 0.0568 -7	0.0457 0.0341 34	0.0375 0.0341 10
(5)	0.0477 1) 0.0447 2) 7 3)	0.0396 0.0447 -11	0.0491 0.0402 22	0.0409 0.0402 2

Remarks: the same as in Table 2.

In Tables 2 and 3 are found some of the numerical values of moments for several types of loading, worked out by the proposed method of analysis are given, and each value is compared with that obtained by the ordinary Rahmentheory.

5. Conclusion

An approximation method of analysis is proposed for the solution of solid structures, with culvert-type structures as examples.

As is evident from some numerical examples relating to the symmetrical loading of the structures, which are tabulated in Tables 2 and 3, the results clearly show the characteristics of solid structures. Special interest attaches to the results of the analysis when compared with those resulting from the Rahmentheory, e.g., the values obtained in the vicinity of the free edges are appreciably different from those of the Rahmentheory.

The numerical examples worked out in the paper relate solely to symmetrical bending; examples relating to asymmetrical bending are to be published shortly.

References

1. M. Kurata and H. Okamura: A Method of Approximation for the Solution of Plate Problems. Transactions of the Faculty of Engineering, Osaka City University. Vol. 3, Dec., 1961.

Summary

The authors propose a method of analysis for a solid structure composed of plate elements. Some examples of the application of the method for purposes of practical design are worked out with regard to the bending problem of a culvert structure. The results are compared with those obtained by means of the Rahmentheory, and some interesting differences between them are observed.

Résumé

Les auteurs présentent une méthode de calcul des ouvrages continus formés de plaques. L'application pratique en est illustrée sur quelques exemples relatifs aux problèmes de flexion dans un ouvrage de type ponceau. On compare les résultats avec ceux obtenus selon la théorie des cadres, d'où il ressort certaines différences intéressantes qui doivent être signalées.

Zusammenfassung

Es wird ein Verfahren zur Berechnung von Plattensystemen dargestellt. Die Methode wird an einigen Biegeproblemen eines Durchlasses praktisch angewendet und die Ergebnisse mit den Berechnungen nach der Rahmentheorie verglichen, woraus einige interessante Unterschiede zutage treten.