Matrix methods for electronic computation of structures

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Matrix Methods for Electronic Computation of Structures

Utilisation de matrices pour formuler les problèmes structuraux en utilisant des calculateurs électroniques

Matrizenmethoden zur Lösung statischer Probleme mittels Elektronenrechner

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Force- and Displacement Transformations. Clebsch's Theorem

In the loaded cantilever, Fig. 1, a small segment near the left end is studied as a "member". The structure is statically determinate, so we can write directly, by statics, the "member force" N as linear expressions in the structure loads P (using the notations of the figure):

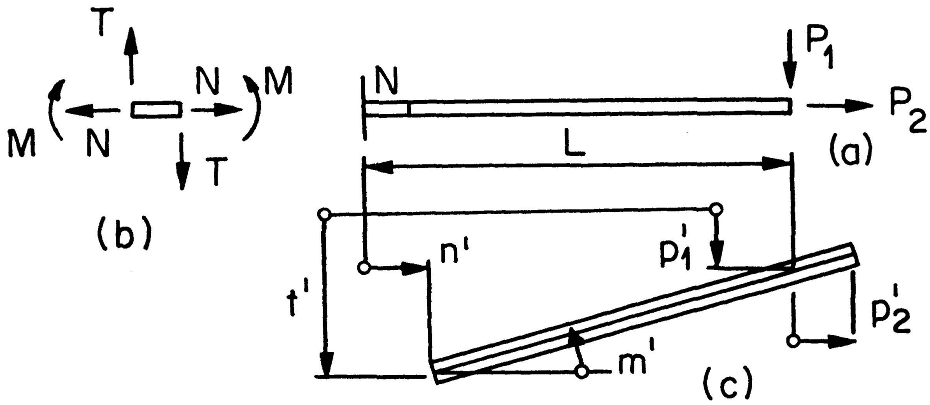

Fig. 1.

$$N = \begin{bmatrix} N \\ T \\ M \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ -L & 0 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = BP. \tag{1}$$

The matrix B is a *force transformation* that transforms the structure loads P into the member forces N. To each N we associate a (covariant) member deformation n' and to each force P a displacement p'. The primes indicate that n', p' are independent of N, P. We write directly by geometry, the displacements p' as a linear expressions in the member deformations n':

$$p' = \begin{bmatrix} p_1' \\ p_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 & -L \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} n' \\ t' \\ m' \end{bmatrix} = B^* n'. \tag{2}$$

Transposition is denoted by an asterisk. We note that

the *displacement transformation* B^*
is the transpose of the force transformation B. (3)

This fact can be proved in general for statically determinate or indeterminate first order theory structures, and it is called *Clebsch's theorem.* In the indeterminate case with non-linear members the elements of B and B^* will be variable, see (8). In ordinary structural analysis Clebsch's theorem can replace completely Maxwell-Mohr's theorem of virtual work and all other work theorems. In the analysis of second-order theory structures, force and displacement transformations can still be established but they are no longer transposes.

Flexibility

No restriction has yet been made on any force-deformation relationship, as f in $n = fN$. In many cases it is sufficient to treat *linear* such relations

$$n = fN + n^t. \tag{4}$$

N and n being associated member force and deformation column matrices, and f a *flexibility matrix* with constant elements. The temperature, initial (misfit) or plastic deformation of the member is expressed by the column matrix n^t.

By a suitable selection of small or larger members and of member forces it is always possible to bring the flexibility matrix f upon diagonal form. We then call the choice of member forces *central.* For instance, in a thin beam lamina with axial force N, shear T, and moment M

$$n = \begin{bmatrix} n \\ t \\ m \end{bmatrix} = \begin{bmatrix} \Delta L/EA & 0 & 0 \\ 0 & \beta\Delta L/GA & 0 \\ 0 & 0 & \Delta L/EI \end{bmatrix} \begin{bmatrix} N \\ T \\ M \end{bmatrix} = fN.$$

We assume a suitable subdivision of the structure enabling a central choice of member forces, and combine our transformations

$$p = B^* n, \quad n = fN + n^t, \quad \text{and} \quad N = BP \quad \text{into}$$

$$p = B^* f B P + B^* n^t = g P + p^t \quad \text{with} \tag{5}$$

$$g = B^* f B \quad \text{and} \quad p^t = B^* n^t. \tag{6}$$

We see that g is in general no longer a diagonal matrix. It is however *symmetric* because

$$g^* = B^* f^* B = g$$

since f is central and $f^* = f$ for all diagonal matrices.

By means of (6) we assemble structures from smaller members. These structures can themselves be used as members in the synthesis of still larger structures.

Part-inversion

We can split the structure forces P into known forces P and redundant forces R and write (5), skipping p^t, in block matrix form

$$\begin{bmatrix} p \\ r \end{bmatrix} = \begin{bmatrix} c & d \\ d^* & k \end{bmatrix} \begin{bmatrix} P \\ R \end{bmatrix}, \qquad g = \begin{bmatrix} c & d \\ d^* & k \end{bmatrix}. \tag{7}$$

Since g is symmetric, so are c and k. We expand (7) and solve for R only *but not for P.*

$$p = c\,P + d\,R, \quad r = d^*\,P + k\,R$$

$$R = K\,d^*\,P - K\,r, \quad p = c\,P + d\,K\,d^*\,P - d\,K\,r, \quad K = -k^{-1}, \quad \text{or}$$

$$\begin{bmatrix} p \\ -R \end{bmatrix} = \begin{bmatrix} c + d\,K\,d^* & -d\,K \\ -K\,d^* & K \end{bmatrix} \begin{bmatrix} P \\ r \end{bmatrix} = h \begin{bmatrix} P \\ r \end{bmatrix}. \tag{8}$$

From the symmetry of k or K it follows that the matrix h is symmetric, but h would not have become symmetric if we had solved for R instead of $-R$. The orderliness of symmetry being an advantage, we shall always insist upon transfering forces R from the right hand side of (7) into the left hand side after adding a minus sign, while the deformations r moving oppositely shall maintain their sign.

We call the transistion from (7) to (8) *part-inversion;* when the forces R consist of a single force we speak of *row-inversion.* Row-inversion can be performed on any row and is easily programmed for electronic calculations. When a number of rows are to be successively inverted it is suitable always to program to invert the row that has the largest current diagonal element. This prevents stops and decreases errors due to rounding off which propagate more slowly because the restored symmetry for each row inverted, implies that some statical and geometric compatibility conditions are always restored.

Mixed Action

Comparing (8) with (7) we find that the forces P *together* with the displacements r can be considered as (generalized) "forces" while the displacements p and forces $-R$ are thought of as "displacements". In order to avoid confusion we shall say that the *action* $[P^* \; r^*]^*$ will give the *response* $[p^* \; -R^*]^*$ by the *mixed mode* flexibility h applied as in (8). We thus say that the member under the *action* of some forces and some displacements will *respond* by displacements and *negative* forces given by (8). Member flexibilities h defined in a mixed mode often find practical use in structure analysis.

Indeterminate Analysis

Here we confine ourselves to the force method; the deformation and mixed methods are mainly cases of including the concepts of the previous paragraph.

The structure is first subdivided into members with known flexibility under definite member action in the force mode. We cut the structure to form gaps that make a number of member forces R zero, until an *auxiliary structure* is obtained that is statically determinate. The number of cuts necessary is a basic invariant, the *redundancy* of the structure.

We assume the auxiliary to be loaded not only by the given structure forces P but also by arbitrary (pairs of) gap forces R. It is thus possible to calculate its member forces N by statics alone. This means that the force transformation $A = [C \; D]$, applicable to the formula

$$N = [C \; D] \begin{bmatrix} P \\ R \end{bmatrix} = A \begin{bmatrix} P \\ R \end{bmatrix} \tag{9}$$

is deducible by statics alone.

All member flexibilities are known and entered into the "unconnected" flexibility matrix f giving the relation

$$n = f N + n^t,$$

where n^t are the initial member deformations. Clebsch's theorem gives

$$\begin{bmatrix} p \\ r \end{bmatrix} = \begin{bmatrix} C^* \\ D^* \end{bmatrix} n = A^* n. \tag{10}$$

We combine these equations into

$$\begin{gathered} \begin{bmatrix} p \\ r \end{bmatrix} = A^* f A \begin{bmatrix} P \\ R \end{bmatrix} + A^* n^t = g \begin{bmatrix} P \\ R \end{bmatrix} + \begin{bmatrix} p^t \\ r^t \end{bmatrix}, \quad g = A^* f A. \\ \begin{bmatrix} p - p^t \\ r - r^t \end{bmatrix} = g \begin{bmatrix} P \\ R \end{bmatrix} = \begin{bmatrix} c & d \\ d^* & k \end{bmatrix} \begin{bmatrix} P \\ R \end{bmatrix}, \quad c = C^* f C, \quad \text{etc.} \end{gathered} \tag{11}$$

Here the structure loads P and gap displacements $r = 0$ are known while R and p are sought, so we apply part-inversion by (7), (8) and find

$$\begin{bmatrix} p - p^t \\ -R \end{bmatrix} = h \begin{bmatrix} P \\ r - r^t \end{bmatrix}, \quad h \text{ by (8).} \tag{12}$$

This solves p and R. The redundants R can now be entered into (9) to find the structure forces N but it is also simple to deduce from (12), (11), and (8) the expression

$$\begin{bmatrix} p \\ N \end{bmatrix} = \begin{bmatrix} c+dKd^* & C^*+dKD^* \\ C+DKd^* & DKD^* \end{bmatrix} \begin{bmatrix} P \\ n^t \end{bmatrix}, \tag{13}$$

that yields directly the load displacements p and member forces N caused by the structure load P and initial member deformations n^t. The "flexibility" matrix in (13) is symmetric.

Design of an Interpretive Matrix Program for Structure Analysis

Structural matrix solutions can best be numerically treated by an interpretative matrix program. Such a program can be constructed in many ways. The following principles were used for a program referred to later on in this paper.

Elements of rectangular matrices are stored consecutively by rows, elements of symmetric matrices by diagonals. Incidence matrices containing in each column at most one unit element, are defined by the sequence of row numbers of these units. We pack this sequence by storing ten, five or three row numbers in a ten-digit decimal word. In each case a short indirect "matrix address" reaches an ingress field of two words. The first ingress word stores the real address of the first element and of the first (non-indexed) matrix. The second word is a format word for the matrix, giving matrix type (rectangular, symmetric, or incidence), number of rows or diagonals, and number of columns.

A structural solution in matrix form is programmed by a sequence of pseudo-instructions. Each such consists of a pseudo-operation and up to three matrix addresses (two for operands and one for results), each with index tags. When the pseudo-program is run each pseudo-instruction is interpreted and executed in order. Branch pseudo-instructions and the matrix index admit loops to be formed. The pseudo-instruction "branch and load location" simplifies the access to subroutines. The complete pseudo-program is printed in the output.

A main principle in computer use is to let the computer make all possible work, such as preparation of input material, even if it is thought of as being exceedingly simple. Incidence matrix multiplications for instance are useful in selecting or adding data from other matrices. Fragments of the output from the analysis of a pile-group in space are shown in Fig. 2.

Beams

Structural analysis of beams, frames, arches, and suspension bridges motivates the establishment of a special beam theory. In it we subdivide the beam by cross-section joints into members of finite length. For loads at the joints,

ÄLVSBGBRO SOAM PILE SOA 14 FEB 1964 P 04

3131	TRPO	40	00	08	00		100	3	3
3132	VEMA	22	00	20	00		100	3	3
3133	MURR	20	08	10	00		100	3	3
3134	TRPO	38	00	26	00		100	3	6
3135	TRPO	36	00	28	00		100	3	6
3136	MPSC	44	02	60	10		100	1	6
3137	MTRR	44	60	32	20		100	6	6
3138	CORS	32	00	26	00		10	6	6
3139	ADD	04	26	04	00		10	6	6
3140	BIXX	00	25	00	00	1			
3141	ROWI	04	34	04	00	38	10	6	6
3142	COPN	04	00	04	00	38	10	6	6

PILE GROUP FLEXIBILITY

3149	MAOU	05	00	04	00	38	10	6	6

Moments loads

	Translations			Rotations			
E 8–	1	2	3	4	5	6	
1	22339	20994	432	184	284	3740	Main diag.
2	416	–		2–	–		1st side diag.
3		589–		–			
4	78	200	97–				
5	971	–					
6	–						

3150	COSR	04	00	32	00	38	100	6	6
3169	MURR	80	60	70	00	38	100	21	38

AXIAL PILE FORCE FOR PIER LOAD

3178	MAOU	03	00	70	00	38	100	21	38

. . . Loading cases . . . / Load combinations . . .

	Pile numbers									
E 1–	1	2	3	4	5	6	7	8	9	10 etc.
	Axial forces									
1	283	283	283	283	283	236	250	293	256	256
2	317	317	317	317	317	264	280	328	286	286
3	77	77	77	77	77	64	68	80	70	70
4	11–	11–	11–	11–	11–	9–	10–	11–	10–	10–
5	54	36	10	15–	33–	56	38	41	6–	23–
6	54–	36–	10–	15	33	56–	38–	41–	6	23
7	27	28	30	32	33	8	9	17	15	16
8	32–	31–	30–	28–	27–	60	61	19–	15–	14–
9	10	4	5–	14–	21–	16	10	62–	62	56
10	43–	35–	24–	13–	5–	67	75	66	100–	92–
11	26	31	39	46	52	101–	96–	106	67–	63–
12	2	2		1–	2–	3	2	2	1–	1–
13	2	1		1–	1–	2	1	1	1–	1–
14	22	28	37	45	51	11–	4–	23	12	18
15	360	360	360	360	360	300	318	373	326	326
16	362	363	364	365	366	388	408	389	341	342
17	404	397	388	379	373	345	357	346	418	412
18	337	369	414	460	491	327	378	523	213	242
19	237	261	296	330	355	258	296	243	282	304
20	468	457	441	426	414	445	453	473	361	351
21	265	293	332	372	400	251	294	294	296	322

Fig. 2.

the solutions involve no approximations. Complementary solutions hold for loads between the member joints. As an introduction to beam theory we consider a simple geometric problem.

Polygons

A polygon is given by the lengths z_k of its perpendicles to a base line and by the intervals, x_{jk}, x_{kl}, of their foot-points on the base line, Fig. 3. We assume further that $z_0 = z_n = 0$ at the ends of the polygon. We define the affine angle A_{jk} of the polygon side jk as

$$A_{jk} = (z_k - z_j)/x_{jk} = z_{jk}/x_{jk}, \quad \text{or}$$

$$A = \begin{bmatrix} A_{01} \\ A_{12} \\ \cdot \\ A_{mn} \end{bmatrix} = \begin{bmatrix} x_{01}^{-1} & & & \\ & x_{12}^{-1} & & \\ & & \cdot & \\ & & & x_{mn}^{-1} \end{bmatrix} \begin{bmatrix} z_1 - 0 \\ z_2 - z_1 \\ \cdot \\ 0 - z_m \end{bmatrix} =$$
$$\begin{bmatrix} x_{01}^{-1} & & & \\ & x_{12}^{-1} & & \\ & & \cdot & \\ & & & x_{mn}^{-1} \end{bmatrix} \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & \cdot & \cdot & \\ & & & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \cdot \\ z_m \end{bmatrix} = \Delta\, x^{-1} dz \tag{14}$$

or, superfix D indicating a diagonal matrix of the elements $1/x_{jk}$:

$$A = X'z \quad \text{with} \quad X' = (1/x_{jk})^D d. \tag{15}$$

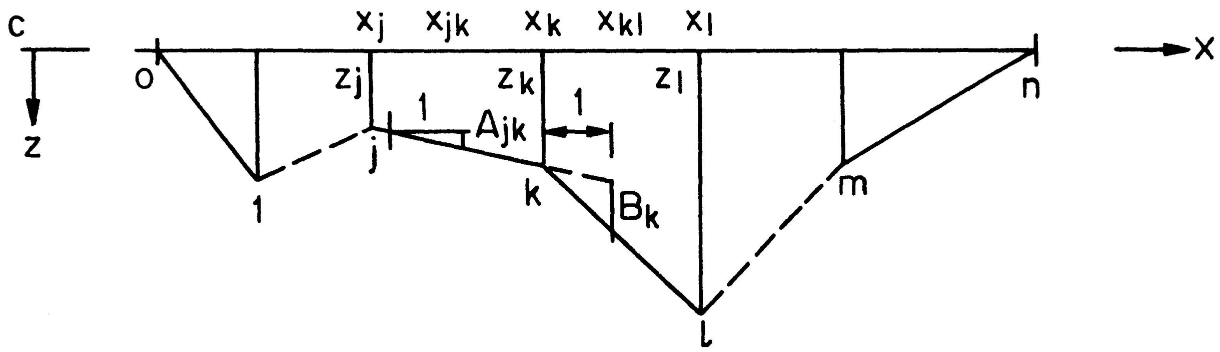

Fig. 3.

We define the affine angle break at the polygon corner k as

$$B_k = A_{kl} - A_{jk}, \quad \text{or}$$

$$B = \begin{bmatrix} B_1 \\ B_2 \\ \cdot \\ B_m \end{bmatrix} = - \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \cdot & \cdot & \\ & & & 1 & -1 \end{bmatrix} \begin{bmatrix} A_{01} \\ A_{12} \\ \cdot \\ A_{mn} \end{bmatrix} = -d^* A = -d^* (1/x_{jk})^D dz, \quad \text{or} \tag{16}$$

$$B = Xz \quad \text{with} \quad X = -d^* X' = -d^* (1/x_{jk})^D d. \tag{17}$$

The second-order difference operator X is a symmetric three-diagonal (band) matrix. Logically it could have been denoted by X'' but X is used for shortness.

Beam Moments

Fig. 4 shows a loaded simple beam and its moment polygon. It is also a string polygon on the loads P drawn for a unit horizontal force; its force

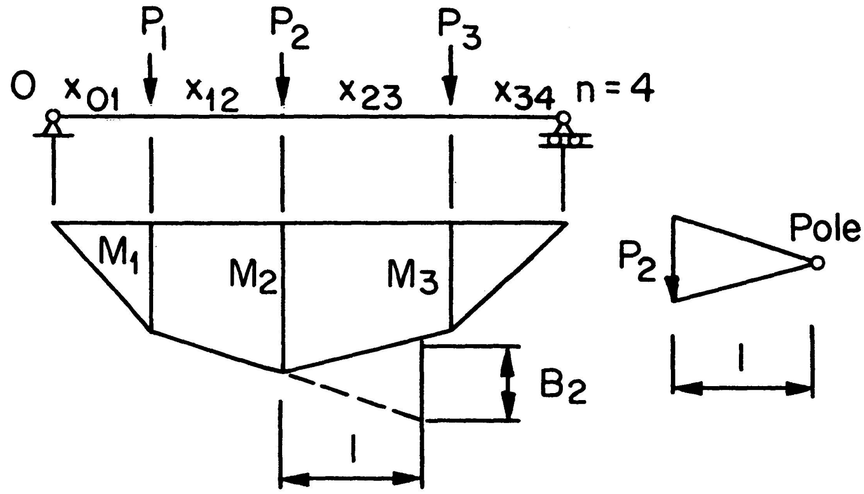

Fig. 4.

polygon is partly shown. Congruency yields $-B_1=P_1$, or generally $-B=P$. But $B=X\,M$ by (17), so

$$P = -X\,M. \tag{18}$$

Transverse forces are calculated by $V_{jk}=(M_k-M_j)/x_{jk}$ or

$$V = X'\,M, \tag{19}$$

and we can close the argument by forming $P=d^*\,V=d^*\,X'\,M=-X\,M$. The boundary conditions $M_0=M_4=0$ are included in (18), so its complete solution is

$$M = -X^{-1}\,P = C\,P, \qquad C = -X^{-1}.$$

C is obviously the force transformation that leads from the structure loads P to the member end moments M. In the limit $\Delta x \to 0$ we find that (18) corresponds exactly to the well known differential equation $p = -M''$ with its boundary conditions.

Inversion of Band Matrices

The inversion of the symmetric band matrix X is substantially shortened by the following algorithm.

If T is invertible we can write the following block matrix identities

$$Y = \begin{bmatrix} S & T \\ U & V \end{bmatrix}, \quad Y\begin{bmatrix} I \\ -T^{-1}S \end{bmatrix} = \begin{bmatrix} 0 \\ W \end{bmatrix}, \quad W = U - VT^{-1}S, \tag{20}$$

and

$$[VT^{-1} \;\; I]\,Y = [W \;\; 0]. \tag{21}$$

By forming $YY^{-1} = I$ we verify that

$$Y^{-1} = \begin{bmatrix} 0 & 0 \\ T^{-1} & 0 \end{bmatrix} + \begin{bmatrix} I \\ -T^{-1}S \end{bmatrix} W^{-1}[-VT^{-1} \;\; I]. \tag{22}$$

This formula is used for the case that T is a lower triangle matrix which occurs when Y is a band matrix. The inverse T^{-1} also will be a lower triangle matrix that, by (22), does not affect the upper triangle of Y^{-1}. When Y, and Y^{-1}, is symmetric, Y^{-1} is equal to the upper triangle of the second term of right hand side of (22), complemented to symmetry. — The whole procedure is illustrated by the following example

$$Y = \begin{bmatrix} S & T \\ U & V \end{bmatrix} = \begin{bmatrix} -2 & 1 & & \\ 1 & -3 & 5 & \\ & 5 & -4 & 1 \\ & & 1 & -5 \end{bmatrix},$$

$$Y = \begin{bmatrix} I \\ -T^{-1}S \end{bmatrix} = \begin{bmatrix} a = 1 \\ b = 2 \\ c = 1 \\ d = -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ e = 31 \end{bmatrix} = \begin{bmatrix} 0 \\ \\ \\ W \end{bmatrix},$$

$$[-VT^{-1} \;\; I]\,Y = [i = -68/5, \;\; h = 19/5, \;\; g = 5, \;\; f = 1]\,Y = [W \;\; 0] = [j = 31 \;\; 0 \;\; 0 \;\; 0].$$

Explanations: a = unit matrix, see (20); $-2\times 1 + 1b = 0$, b = 2; $1\times 1 - 3\times 2 + 5c = 0$, $c = 1$; $5\times 2 - 4\times 1 + 1d = 0$, $d = -6$; $1\times 1 + 5\times 6 = e = 31$; f = unit matrix, see (21); $-5\times 1 + 1g = 0$, $g = 5$; $1\times 1 - 4\times 5 + 5h = 0$; $h = 19/5$; $5\times 5 - 3\times 19/5 + 1i = 0$, $i = -68/5$; $1\times 19/5 + 2\times 68/5 = j = 31 = e$ (check!). We then compose by (22)

$$Y^{-1}_{up.tri.} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ -6 \end{bmatrix} (1/31)[-68 \;\; 19 \;\; 25 \;\; 5]/5, \qquad Y^{-1} = \begin{bmatrix} -68 & 19 & 25 & 5 \\ & 38 & 50 & 10 \\ & & 25 & 5 \\ \text{sym.} & & & -30 \end{bmatrix} /155$$

Computer subroutines are easily programmed for inversion by this method of 3-diagonal, 5-diagonal, etc. symmetric matrices (collectively called band-matrices). Complete inversion of a 3-diagonal matrix requires $\frac{1}{2}n^2 + 6\frac{1}{2}n - 3$ multiplications or divisions whereas row inversion of all rows requires $\frac{1}{2}n^2(n+1)$ multiplications.

Bending Deformations of a Beam

A beam, Fig. 5, is transversely loaded at its member (segment) joints, so its moment curve is a polygon with ordinates M_1, M_2, M_3.

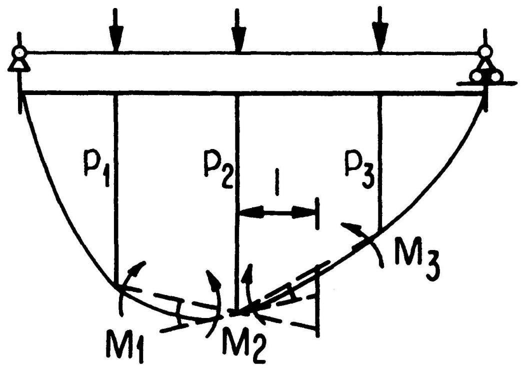

Fig. 5.

The angle between the chord in the segment jk of the deflection curve and the tangent to the beam axis at k is $f_{kj} M_k + g_{jk} M_j$. Here f_{kj} is the end-flexibility at k of the member kj acting as a simple beam (f_{kj} = angle at k caused by a unit end-moment at k) and g_{jk} the carry-over flexibility ($g_{jk} = g_{kj}$ = angle at k for a unit moment at j). The angle-break between the member chords at 2 is the sum of two such angles:

$$m_2 = g_{12} M_1 + (f_{21} + f_{23}) M_2 + g_{23} M_3 .$$

For all member boundaries we write

$$m = f^b M, \qquad f^b = \begin{bmatrix} f_{10}+f_{12} & g_{12} & 0 & \cdot \\ g_{12} & f_{21}+f_{22} & g_{23} & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \qquad M_0 = M_n = 0 .$$

The effect of shear deformations is mostly small. It can be accurately accounted for simply by adding to f^b another three-diagonal matrix $f^{sh} = -d^* f^{sh\,D} d$ with $f^{sh\,D} = (\beta/G A s)^D_{jk}$. We write $f^b + f^{sh} = f$.

The deflection curve ordinates p_1, p_2, p_3 at the member joints form a polygon with negative angle breaks $-X p$, see (16), that are the same as m, so $m = -X p$ or $p = C m$, $C = -X^{-1}$.

The symmetric displacement transformation is C^* also in force of Clebsch's theorem. To sum up: for the simple beam we have

$$M = C P, \qquad m = f M, \qquad p = C^* m, \qquad C = -X^{-1} \tag{23}$$

and f is a three-diagonal matrix. The continuous equivalent of $-X p = f M$ is $-p'' = M/E I$.

In a beam that is not straight an axial force will carry part of the transverse load because of *bar-chain action*. The bar-chain action of beam subjected to a (variable) axial force and to the action of *side-support springs* can be easily incorporated into a general beam theory.

Suspension Bridge Analysis

The notions so far defined suffice for formulating effectively in matrices, for subsequent computer application, second-order theory analyses of column-beams, arches, suspension bridges, etc. A suspension bridge analysis will be outlined here as an example. The characteristic design features of a suspension bridge are assumed to be known to the reader.

For members are chosen the truss segments (between hangers), the hangers, and the cable segments (between hanger connections, saddle entrances and anchors). Fig. 6 shows a piece of the cable, one hanger, and the stiffening truss.

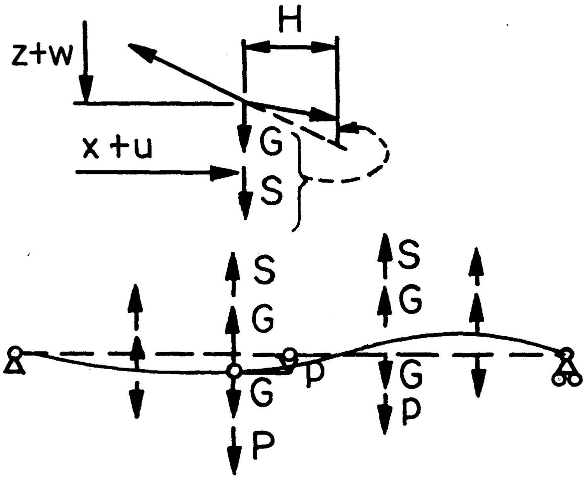

Fig. 6.

Structure dead and live loads G and P are acting on the truss at the hangers. When the cable is cut a determinate auxiliary structure is formed. The cable horizontal force $H = H^G + H^P$ is applied as a redundant structure load. Member forces are: the moments M in the truss, member joints, the forces $G + S$ in the hangers, and the tensions $N^G + N$ in the cable members.

The equilibrium conditions for the upper hanger connections are for total load and for dead load only, see (17),

$$-H X^h (z+w) = G + S, \qquad -H^G X z = G. \tag{24}$$

The second difference operator X is based upon the *dead* load horizontal intervals x_{jk} of the *cable* members, and X^h upon the *total* load cable member intervals $x_{jk} + u_{jk}$. The two Eqs. (24) are subtracted and the secondary movement effect is reduced by differentiation, see (Sc 9) in [1].

$$S = -H^P X z - H X^U w, \qquad X^U = X - d^* (z_{jk}^2/x_{jk}^3)^D d, \qquad \text{see (17)}. \tag{25}$$

Under the dead load G or *form load* G the cable hangs with ordinates z while the stiffening truss carries no load and takes no moments. Of the live load P,

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[1]) S. O. Asplund, Structural Mechanics-Classical and Matrix Methods, Prentice-Hall Inc., Englewood Cliffs, N. J., 1966.

the hangers take S and the truss $P-S=-X^g M$. Here the second difference operator X^g is based upon the horizontal intervals of the *truss* members. We also write $-(X^g)^{-1}=C$, $M=C(P-S)$. For the cable tensions we simply write $N=\sec c\, H^P$ where $\sec c$ is the column of secants of the cable member angles c.

Force Transformation

By what has just been said we array in matrix form, first S from (25), then $M=CP-CS$, and finally N:

$$\begin{bmatrix} M \\ S \\ N \end{bmatrix} = \begin{bmatrix} C & C X z \\ 0 & -X z \\ 0 & \sec c \end{bmatrix} \begin{bmatrix} P \\ H^P \end{bmatrix} + H \begin{bmatrix} C \\ -I \\ 0 \end{bmatrix} X^U w = A \begin{bmatrix} P \\ H^P \end{bmatrix} + H\, C'\, X^U w.$$

Flexibility

The connected member flexibility is f' in

$$\begin{bmatrix} m \\ s \\ n \end{bmatrix} = f' \begin{bmatrix} M \\ S \\ N \end{bmatrix} + \begin{bmatrix} m^t \\ s^t \\ n^t \end{bmatrix}, \qquad f' = \begin{bmatrix} f & & \\ & f^{su} & \\ & & f^n \end{bmatrix} \tag{26}$$

with $\qquad f = f^b + f^s, \qquad f^{su} = (L/EA)^D_{hanger}, \qquad f^n = (s/EA)^D_{cable}.$

The bending and shear flexibility of the stiffening truss is expressed by f, the hanger and cable flexibilities by f^{su} and f^n.

Revision of the Force Transformation

The cable deflections w equal the truss deflections p minus the hanger stretch s:

$$w = p - s = C^* f M - f^{su} S = [C^* \quad -I \quad 0]\, f' \begin{bmatrix} M \\ S \\ N \end{bmatrix} = C'^* f' \begin{bmatrix} M \\ S \\ N \end{bmatrix}.$$

By this formula we move the w-terms in the previous force transformation formula to the left hand side and obtain

$$(I - H\, C'\, X^U C'^* f') \begin{bmatrix} M \\ S \\ N \end{bmatrix} = B \begin{bmatrix} M \\ S \\ N \end{bmatrix} = A \begin{bmatrix} P \\ H^P \end{bmatrix}, \qquad \begin{bmatrix} M \\ S \\ N \end{bmatrix} = A^s \begin{bmatrix} P \\ H^P \end{bmatrix}. \tag{27}$$

The second-order theory force transformation is $A^s = B^{-1} A$.

Displacement Transformation

It can be directly verified in this case, see (Sc 16) in [1]), that the displacement transformation A^* equals the transposed force transformation when the contributions by displacements are neglected:

$$\begin{bmatrix} p \\ h^P \end{bmatrix} = A^* \begin{bmatrix} m \\ s \\ n \end{bmatrix}, \qquad A = \begin{bmatrix} C & C X z \\ 0 & -X z \\ 0 & \sec c \end{bmatrix}. \tag{28}$$

We obtain by (28), (26) and (27)

$$\begin{bmatrix} p \\ h^P \end{bmatrix} = A^* \left(f' A^s \begin{bmatrix} P \\ H^P \end{bmatrix} + \begin{bmatrix} m^t \\ s^t \\ n^t \end{bmatrix} \right),$$

or $$\begin{bmatrix} p - p^t \\ h^P - h^t \end{bmatrix} = g \begin{bmatrix} P \\ H^P \end{bmatrix}, \qquad g = A^* f' A^s = A^* f'^s A, \qquad \begin{bmatrix} p^t \\ h^t \end{bmatrix} = A^* \begin{bmatrix} m^t \\ s^t \\ n^t \end{bmatrix}. \tag{29}$$

Second-order Flexibility

Further, we note that $f' B^{-1} = f'^s$ is symmetric because its inverse is symmetric:

$$F'^s = B (f')^{-1} = B F' = F' - H C' X^U C'^*,$$
$$B = (f'^s)^{-1} f' = F'^s f'.$$

The second-order theory structure thus behaves like a first-order theory suspension bridge that has a flexibility of $f'^s = f' B^{-1}$ instead of f'.

Solution of Force-method Equations

We row-invert (29) by the last row

$$\begin{bmatrix} p - p^t \\ -H^P \end{bmatrix} = h \begin{bmatrix} P \\ -h^t \end{bmatrix}. \tag{30}$$

Using the last row we assemble, see (8),

$$\begin{bmatrix} P \\ H^P \end{bmatrix} = \begin{bmatrix} I & 0 \\ K d^* & K \end{bmatrix} \begin{bmatrix} P \\ h^t \end{bmatrix} = E \begin{bmatrix} P \\ h^t \end{bmatrix} \tag{31}$$

that we enter into (27) to find the member forces

$$\begin{bmatrix} M \\ S \\ N \end{bmatrix} = A^s E \begin{bmatrix} P \\ h^t \end{bmatrix}. \tag{32}$$

Since we have found H^P and know its simple relation to N we can omit all rows N in this solution.

The load deflections p are found in (30), the cable deflections by $w = p - f^{su} S$ and, finally, the horizontal cable displacements, see (Sc 7) in [1],

$$u = X^{-1} d^* (z_{jk}/x_{jk}^2)^D dw. \tag{33}$$

The transverse forces are calculated by (19) and the changes in the cable slopes by $w' = X' w$. Each force or displacement G is found in familiar influence coefficient form with correction for inelastic changes h^t

$$G = G^i P + G^t h^t.$$

In such an analysis frequent use is found for band-matrix and row inversions that considerably reduce computer and programming time.

It is often advantageous to carry other minor effects, such as cable strains and side-span action, into the correction terms. Thereby a computer can prepare dimensionless influence and correction tables by which large classes of suspension bridges can be designed, see Si in [1].

Torsion Box Action

The stiffening girders, together with the systems of upper and lower wind laterals, act also as a torsion box transmitting local antisymmetric loads upon the bridge deck to a longer portion of the stiffening truss, thus reducing the bending moments in the trusses but considerably increasing the transverse forces in the wind bracing.

This structural problem also is readily formulated in matrix language and solved by computer. The 418 m suspension span now completed in Göteborg was analyzed accordingly.

Summary

The analysis of a structure can always suitable be subdivided into three stages: the force, the flexibility, and the displacement transformations. In first-order theory structures the force and displacement transformations are transposes, in second-order theory structures they are not. When *action* is composed of loads *and* displacements the *response* refers suitably to associated displacements *and negative* forces.

The force and the mixed method equations are formulated and solved by successive row-inversions. In beam-theory the inversion of band-matrices is described. To demonstrate the versatility of matrix methods the suspension bridge theory is outlined and its numerical treatment indicated. Properties of an interpretive matrix program are sketched. Computer applications are described.

Résumé

Il est toujours possible de diviser l'étude d'une structure en trois parties: transformations des efforts, des flexibilités et des déplacements. Pour les structures qui obéissent à une théorie du premier ordre, les transformations des efforts et des déplacements sont transposés; elles ne le sont plus dans une théorie du second ordre. Lorsque les *sollicitations* comprennent des charges *et* des déplacements, la *réponse* correspondra à des déplacements associés et à des efforts *négatifs*.

L'auteur établit les équations de la méthode des forces et de la méthode mixte; il les résout par des inversions de lignes successives. Dans la théorie des poutres, on décrit l'inversion de matrices-bandes. Pour montrer la souplesse des méthodes matricielles, on esquisse la théorie des ponts suspendus et on en présente l'étude numérique. On esquisse les propriétés du programme matriciel correspondant et on décrit des applications pour les calculateurs électroniques.

Zusammenfassung

Die Berechnung eines Tragwerkes kann immer in drei Teile unterteilt werden: Kraft-, Verformungs- und Verschiebungs-Transformation. In Tragwerken, die sich nach der Theorie erster Ordnung verhalten, sind die Kraft- und Verformungs-Transformationen transponiert, jedoch ist dies nicht mehr gültig bei Tragwerken, die sich nach der Theorie zweiter Ordnung verhalten. Falls sich die Beanspruchung aus Lasten *und* Verschiebungen zusammensetzt, reagiert das Tragwerk hierauf durch geeignete zusammenhängende Verschiebungen *und negative* Kräfte.

Die Gleichungen der Kraftmethode und der gemischten Methode werden angegeben und durch sukzessive Linien-Umkehrungen gelöst. In der Balkentheorie wird die Umkehrung von Band-Matrizen beschrieben. Um die Vielseitigkeit der Matrizen-Methoden zu zeigen, wird die Hängebrückentheorie benutzt und die numerische Behandlung aufgezeigt. Eigenschaften eines wiedergegebenen Matrizen-Programmes werden skizziert sowie Anwendungen von Rechenautomaten beschrieben.