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Gridworks of Skew Bridges

Les réseaux de poutres croisées des ponts biais

Schiefwinklige Trägerroste

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1. Introduction

In this paper the gridworks of skew bridges consisting of two simply supported, equal longitudinal girders connected by $m-1$ equal and equidistant transverse cross beams are studied. To simplify the solution of the problem, that otherwise would present $3m-3$ statically indeterminate unknowns, trigonometric polynomials are used, giving the exact solution[1]) of the question for loads applied on the intersections of the cross beams with the girders[2]). In this manner, it is not necessary to solve systems with a great number of simultaneous equations. The method here expounded can be extended to the case of bridges with a greater number of longitudinal girders. However, the reinforced concrete bridges with only two girders are the most economical among the bridges with more frequent breadth. And it is just for bridges with two girders that it is essential to consider the effects of torsion on the girders and cross beams in order to take into account the solidarity of the system as a whole. The hypothesis of simply supported cross beams would not permit in that case the consideration of any contribution of one of the girders in the resistance of loads applied on the other (this would not be the case if the grid had more than two longitudinal girders).

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[1]) The solution with use of trigonometric polynomials is not an approximation of the solution with use of Fourier's series [1] [2]; on the contrary, it gives the exact solution when the forces and moments considered act on equidistant positions of the longitudinal girders.

[2]) The process here studied may be generalized for application to the case of loads acting on any position of the cross beam or on its overhanging extensions.

2. Trigonometric Polynomials

The first author who used trigonometric polynomials in the study of grids was Ferraz [3] in 1951. He amply developped its applications to grids in general ignoring the effect of torsion and, taking in consideration this effect, to orthogonal grids.

As it is known, a function P that assumes the values $P_n = 0$, $P_1, \ldots P_{m-1}$, $P_m = 0$ for the values $n=0$, $n=1, \ldots n=m-1$, $n=m$, of the variable n, can be represented by the trigonometric polynomial:

$$P_n = \sum_{k=1}^{m-1} p_k \sin \frac{k \pi n}{m}, \tag{2.1}$$

where p_k are the coefficients obtained with:

$$p_k = \sum_{n=1}^{m-1} P_n \sin \frac{k \pi n}{m}. \tag{2.2}$$

This polynomial can therefore represent the loads P applied externally at the various intersections of the girders with the cross beams, when the notation of Fig. 1 is used. To distinguish the loads on the girder A of those acting on the girder B the corresponding index ($P_{mA}, P_{mB}, p_{kA}, p_{kB}$) will be used. Quantities represented by polynomials of the cosine function like (2.5) will also be used in the solution of the problem.

It is also known that when a girder of span $l = mc$, simply supported at

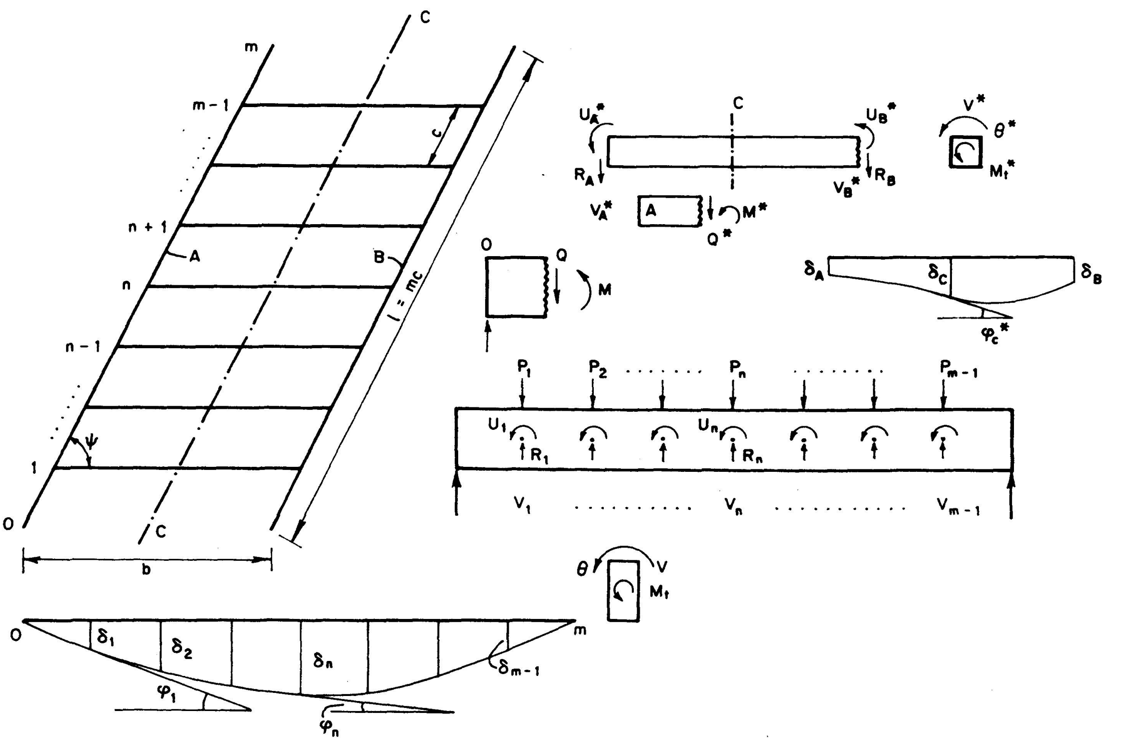

Fig. 1.

the points corresponding to $n=0$ and $n=m$ deforms under the action of the loads P_n, defined by (2.1) and acting downwards, it presents at the points of application of the loads the angular displacements φ_n and the vertical displacements δ_n given by (the positive directions of these displacements are shown in fig. 1; EJ is the flexural rigidity of the girder [3]):

$$\varphi_n = \frac{c^2}{2\,EJ}\sum p_k \beta_k \cos\frac{k\pi n}{m}, \qquad \delta_n = \frac{c^3}{6\,EJ}\sum p_k \alpha_k \sin\frac{k\pi n}{m} \tag{2.3}$$

with
$$\alpha_k = \frac{2+\cos k\pi/m}{2\,(1-\cos k\pi/m)^2}, \qquad \beta_k = \frac{\sin k\pi/m}{2\,(1-\cos k\pi/m)^2}. \tag{2.4}$$

If the simply supported girder is bent by moments U_n (with $U_0=0$ and $U_m=0$) [4]) determined by

$$U_n = C_u + \sum u_k \cos\frac{k\pi n}{m} \tag{2.5}$$

and situated in the plane of the girder, as in Fig. 1, the displacements will be given by:

$$\varphi_n = \frac{c}{EJ}\left(C_\varphi - \sum u_k \gamma_k \cos\frac{k\pi n}{m}\right), \qquad \delta_n = \frac{-c^2}{2\,EJ}\sum u_k \beta_k \sin\frac{k\pi n}{m}, \tag{2.6}$$

where β_k is the same of (2.4) and

$$C_\varphi = -\frac{1}{12}C_u = \frac{1}{12}\sum_e u_k, \qquad \gamma_k = \frac{0{,}5}{1-\cos k\pi/m}. \tag{2.7}$$

If U_n were determined by

$$U_n = \sum u_n \sin\frac{k\pi n}{m} \tag{2.8}$$

the angular displacements would be [5])

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[3]) From now on, for a greater simplicity, the following notations will be used:

$$\sum = \sum_{k=1}^{m}, \qquad \sum_0 = \sum_{k=1,3,\dots}^{m}, \qquad \sum_e = \sum_{k=2,4,\dots}^{m}.$$

The notation $\sum\limits_{0m}$ is equivalent to $\sum\limits_{0}$ if m is even and $\sum\limits_{em}$ to $\sum\limits_{e}$ if m is odd; on the contrary:

$$\sum_{0m} f(k) = \sum_0 f(k) - 0{,}5\,f(m), \qquad \sum_{em} f(k) = \sum_e f(k) - 0{,}5\,f(m).$$

[4]) What leads to:

$$\sum_0 u_k = 0, \qquad \sum_e u_k = -C_u.$$

[5]) λ_k is a function of k such that the sum $\sum u_k \lambda_k \sin(k\pi n/m)$ represents a function of the 2d degree of n. For the use in the formulae of § 4, it is possible to take the approximate value, for k odd:

$$\lambda_k \cong (m-1)\left(\frac{1}{12}+\frac{\gamma_k}{m}\right)$$

and $\lambda_k \cong 0$, for k even.

$$\varphi_n = \frac{c}{EJ} \sum u_k (\lambda_k - \gamma_k) \sin \frac{k\pi n}{m}. \tag{2.9}$$

When the girder is loaded by a system of couples V_n in equilibrium [6]), acting in planes normal to its axis and determined by

$$V_n = C_v + \sum v_k \cos \frac{k\pi n}{m} \tag{2.10}$$

there are angular displacements of the various cross sections of the girder given by (γ_k is given by (2.7) and GJ_t is the torsional rigidity of the girder) [7]):

$$\theta_n = \frac{c}{GJ_t} \left[C_\theta + \sum (v_k - K_k) \gamma_k \cos \frac{k\pi n}{m} \right]. \tag{2.11}$$

The constants C_θ and K_k (K_a and K_b) depend on the conditions of the supports. When the transversal rotations of the ends of the girder are prevented ($\theta_0 = \theta_m = 0$) their values are:

$$K_a = \frac{8}{m^2} \sum_0 v_k \gamma_k, \quad K_b = 2C_v = \frac{-2}{m+1} \sum_e v_k, \quad C_\theta = \frac{m^2-1}{24} K_b - \sum_e v_k \gamma_k. \tag{2.12}$$

If V_n is given by

$$V_n = \sum v_k \sin \frac{k\pi n}{m} \tag{2.13}$$

the more simple expression

$$\theta_n = \frac{c}{CJ_t} \sum v_k \gamma_k \sin \frac{k\pi n}{m} \tag{2.14}$$

is obtained for the same girder.

The values of the coefficients α_k, β_k and γ_k are given in Tables I, II and III.

3. Internal Forces and Moments

The cross beams are subjected at their ends (Fig. 1) to vertical forces R_A and R_B, to moments U_A^* and U_B^* acting in the vertical plane that contains its axis and torques V_A^* and V_B^*. By a simple consideration of equilibrium the

[6]) That is, which sum is null, what leads to:

$$\sum_e v_k = -(m+1) C_v.$$

[7]) On what follows K_k and C_k are constants meaning respectively K_b and C_b for k even and different from m, K_a and C_a for k odd and different from m, 0,5 K_b and 0,5 C_b for $k = m$ even, and 0,5 K_a and 0,5 C_a for $k = m$ odd. These conditions lead to:

$$\frac{1}{K_b} \sum_e K_k = \frac{1}{K_a} \sum_0 K_k = \frac{1}{C_b} \sum_e C_k = \frac{1}{C_a} \sum_0 C_k = \frac{m}{2},$$

$$\frac{1}{K_b} \sum_e \gamma_k K_k = \frac{1}{C_b} \sum_e \gamma_k C_k = \frac{m^2-1}{24}, \qquad \frac{1}{K_a} \sum_0 \gamma_k K_k = \frac{1}{C_a} \sum_0 \gamma_k C_k = \frac{m^2}{8}.$$

Table I. Values of α_k

m	$k =$															
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	1,000	0,125														
3	5,000	0,333	0,125													
4	15,778	1,000	0,222	0,125												
5	38,507	2,418	0,493	0,182	0,125											
6	79,837	5,000	1,000	0,333	0,163	0,125										
7	147,90	9,253	1,838	0,595	0,261	0,152	0,125									
8	252,31	15,778	3,126	1,000	0,423	0,222	0,145	0,125								
9	404,14	25,267	5,000	1,592	0,663	0,333	0,198	0,141	0,125							
10	615,98	38,507	7,615	2,418	1,000	0,493	0,280	0,182	0,138	0,125						
11	901,84	56,373	11,144	3,534	1,456	0,712	0,395	0,246	0,171	0,135	0,125					
12	1277,3	79,837	15,778	5,000	2,056	1,000	0,549	0,333	0,222	0,163	0,134	0,125				
13	1759,3	109,96	21,728	6,882	2,826	1,371	0,748	0,448	0,291	0,205	0,157	0,132	0,125			
14	2366,3	147,90	29,222	9,253	3,797	1,838	1,000	0,595	0,381	0,261	0,192	0,152	0,131	0,125		
15	3118,4	194,90	38,507	12,191	5,000	2,418	1,312	0,777	0,493	0,333	0,239	0,182	0,148	0,131	0,125	
16	4036,6	252,31	49,846	15,778	6,469	3,126	1,694	1,000	0,632	0,423	0,298	0,222	0,174	0,145	0,130	0,125

Table II. Values of β_k

m	$k =$															
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	0,500	0														
3	1,732	0,192	0													
4	4,121	0,500	0,121	0												
5	8,057	0,996	0,278	0,0898	0											
6	13,928	1,732	0,500	0,192	0,0718	0										
7	22,121	2,758	0,806	0,326	0,148	0,0600	0									
8	33,023	4,121	1,212	0,500	0,242	0,121	0,0517	0								
9	47,020	5,872	1,732	0,721	0,357	0,192	0,103	0,0455	0							
10	64,502	8,057	2,381	0,996	0,500	0,278	0,160	0,0898	0,0406	0						
11	85,851	10,727	3,172	1,331	0,673	0,379	0,227	0,138	0,0797	0,0367	0					
12	111,46	13,928	4,121	1,732	0,879	0,500	0,305	0,192	0,121	0,0718	0,0335	0				
13	141,71	17,710	5,242	2,206	1,122	0,642	0,395	0,255	0,167	0,108	0,0654	0,0308	0			
14	177,00	22,121	6,550	2,758	1,406	0,806	0,500	0,326	0,219	0,148	0,0982	0,0600	0,0285	0		
15	217,70	27,208	8,057	3,394	1,732	0,996	0,620	0,408	0,278	0,192	0,133	0,0898	0,0555	0,0266	0	
16	264,19	33,023	9,780	4,121	2,105	1,212	0,757	0,500	0,343	0,242	0,172	0,121	0,0828	0,0517	0,0249	0

Table III. Values of γ_k

m	k = 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	0,500	0,250														
3	1,000	0,333	0,250													
4	1,707	0,500	0,293	0,250												
5	2,618	0,724	0,382	0,276	0,250											
6	3,732	1,000	0,500	0,333	0,268	0,250										
7	5,049	1,328	0,643	0,409	0,308	0,263	0,250									
8	6,569	1,707	0,810	0,500	0,362	0,293	0,260	0,250								
9	8,291	2,137	1,000	0,605	0,426	0,333	0,283	0,258	0,250							
10	10,216	2,618	1,213	0,724	0,500	0,382	0,315	0,276	0,256	0,250						
11	12,344	3,150	1,449	0,855	0,583	0,438	0,353	0,302	0,272	0,255	0,250					
12	14,674	3,732	1,707	1,000	0,675	0,500	0,397	0,333	0,293	0,268	0,254	0,250				
13	17,207	4,365	1,988	1,158	0,775	0,569	0,446	0,369	0,319	0,286	0,265	0,254	0,250			
14	19,943	5,049	2,292	1,328	0,883	0,643	0,500	0,409	0,349	0,308	0,281	0,263	0,253	0,250		
15	22,881	5,783	2,618	1,511	1,000	0,724	0,558	0,453	0,382	0,333	0,300	0,276	0,261	0,253	0,250	
16	26,021	6,569	2,967	1,707	1,125	0,810	0,621	0,500	0,418	0,362	0,321	0,293	0,273	0,260	0,252	0,250

following relations are obtained:

$$R_B = -R_A, \qquad U_B^* = -U_A^* - b\,R_A, \qquad V_B^* = -V_A^*. \tag{3.1}$$

The same forces R_A and R_B act on the girders A and B and the moments U^* and V^* are related to the moments U and V that act on the girders in the section of contact by the equalities (also valid when A is replaced by B):

$$U_A^* = V_A \sin\psi - U_A \cos\psi, \qquad V_A^* = -U_A \sin\psi - V_A \cos\psi. \tag{3.2}$$

On the cross section of the middle of the transverse beam act a shear force Q_c^*, a bending moment M_c^* and a twisting moment M_{tc}^* (Fig. 2):

$$Q_c^* = -R_A, \qquad M_c^* = -U_A^* - \tfrac{1}{2} b\,R_A, \qquad M_{tc}^* = -V_A^*. \tag{3.3}$$

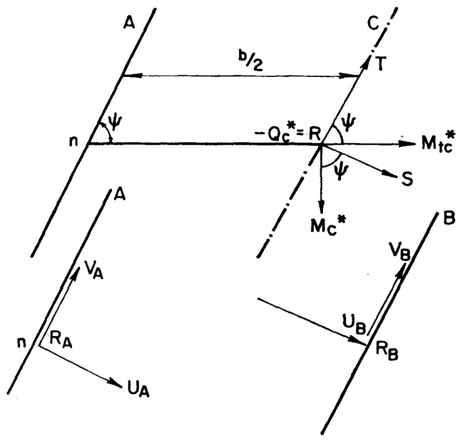

Fig. 2.

The following internal forces and moments (Fig. 2) act on the section of the middle of the cross beam parallel to the girders (they will be taken as unknowns of the statically indeterminate problem):

$$R = R_A, \qquad S = U_A - R\frac{b}{2}\cos\psi, \qquad T = V_A + R\frac{b}{2}\sin\psi. \tag{3.4}$$

There will be one of each kind of these unknowns for each cross beam, that is, for each n; so they can be represented by trigonometric polynomials:

$$R_n = \sum r_k \sin\frac{k\pi n}{m}, \qquad S_n = C_s + \sum s_k \cos\frac{k\pi n}{m}, \qquad T_n = C_t + \sum t_k \cos\frac{k\pi n}{m}. \tag{3.5}$$

If R, S and T are determined, all the stress resultants on the longitudinal girders will be known

$$R_{nA} = R_n, \qquad U_{nA} = S_n + R_n\frac{b}{2}\cos\psi, \qquad V_{nA} = T_n - R_n\frac{b}{2}\sin\psi, \tag{3.6}$$

$$R_{nB} = -R_n, \qquad U_{nB} = U_{nA} - 2\,S_n, \qquad V_{nB} = V_{nA} - 2\,T_n \tag{3.7}$$

and consequently through (3.2) and (3.3) those of the cross beams.

Using (3.6) and (3.7) and comparing (3.5) with the developments of U_A and U_B (2.5), V_A and V_B (2.10), R_A and R_B (3.5), the following relations are obtained:

$$r_{kB} = -r_{kA} = -r_k, \qquad u_{kB} = u_{kA} - 2\,s_k, \qquad v_{kB} = v_{kA} - 2\,t_k. \tag{3.8}$$

The relations:

$$U_{nA} + U_{nB} = R_n\, b \cos\psi, \qquad V_{nA} + V_{nB} = -R_n\, b \sin\psi \tag{3.9}$$

are also valid.

4. Determination of the Unknowns

The compatibility of the displacements is used to find the unknowns. The differences of the linear and angular displacements between the two extremities of each cross beam are (Fig. 1 and formulae (3.2) and (3.4)):

$$\theta_B^* - \theta_A^* = -V_A^* \frac{b}{G J_t^*} = (U_A \sin\psi + V_A \cos\psi)\frac{b}{G J_t^*} = (S \sin\psi + T\cos\psi)\frac{b}{G J_t^*}, \tag{4.1}$$

$$\varphi_B^* - \varphi_A^* = R_A \frac{b^2}{2\,E J^*} + U_A^* \frac{b}{E J^*} = (T \sin\psi - S\cos\psi)\frac{b}{E J^*}, \tag{4.2}$$

$$\delta_B - \delta_A = \varphi_A^* b + R_A \frac{b^3}{6\,E J^*} + U_A^* \frac{b^2}{2\,E J^*} = (\varphi_B^* + \varphi_A^*)\frac{b}{2} - R\frac{b^3}{12\,E J^*}, \tag{4.3}$$

where b is the length of the cross beam, EJ^* its flexural rigidity and $G J_t^*$ its torsional rigidity.

The vertical displacements of the girders A and B at the intersections with the cross beams are equal to those of these beams (δ_A and δ_B); the angular displacements are (these formulae are also valid when A is replaced by B):

$$\varphi_A = \varphi_A^* \cos\psi - \theta_A^* \sin\psi, \qquad \theta_A = \varphi_A^* \sin\psi + \theta_A^* \cos\psi \tag{4.4}$$

expressions that compared with (4.1) and (4.2) lead to [8]):

$$\theta_B - \theta_A = \frac{c}{G J_t}\left(\omega_1 S + \omega_3 T + 0{,}5\, C_b' + \sum C_k' \cos\frac{k\pi n}{m}\right), \tag{4.5}$$

$$\varphi_B - \varphi_A = \frac{c}{EJ}\left(-\omega_4 S - \omega_2 T + 0{,}5\, C_b + \sum C_k \cos\frac{k\pi n}{m}\right), \tag{4.6}$$

where the ω are the following constants that only depend of the dimensions of the structure

[8]) The terms added in (4.5) and (4.6)

$$0{,}5\, C_b + \sum C_k \cos\frac{k\pi n}{m}$$

correspond to values of $\theta_B - \theta_A$ and $\varphi_B - \varphi_A$ null in all the intersections of the longitudinal girders with the cross beams, but not in the free extremities of the girder; they are added because the verification of the equalities (4.5) and (4.6) in those extremities is not necessary.

$$\omega_1 = \left(\frac{G J_t}{G J_t^*} - \frac{G J_t}{E J^*}\right) \frac{b}{c} \sin\psi \cos\psi, \qquad \omega_2 = \frac{E J}{G J_t} \omega_1,$$
$$\omega_3 = \left(\frac{G J_t}{G J_t^*} \cos^2\psi + \frac{G J_t}{E J^*} \sin^2\psi\right) \frac{b}{c}, \qquad \omega_4 = \left(\frac{E J}{E J^*} \cos^2\psi + \frac{E J}{G J_t^*} \sin^2\psi\right) \frac{b}{c}. \tag{4.7}$$

From (2.11) and (3.8) the expression

$$\theta_B - \theta_A = K_\theta - \frac{c}{G J_t} \sum (2 t_k + K_k) \gamma_k \cos\frac{k \pi n}{m} \tag{4.8}$$

is obtained; this one, compared with (4.5) and (3.5) gives

$$\omega_1 s_k + (\omega_3 + 2\gamma_k) t_k = -\gamma_k K_k - C_k' = -K_k' \tag{4.9}$$

whence

$$s_k = \eta_k' t_k - \frac{1}{\omega_1} K_k' \tag{4.10}$$

with

$$\eta_k' = -\frac{\omega_3 + 2\gamma_k}{\omega_1}, \tag{4.11}$$

Similarly, using (2.3) to (2.7) and (3.8), the expression

$$\varphi_B - \varphi_A = K_\varphi + \frac{c}{E J} \sum (c r_k \beta_k - 0{,}5\, c p_k \beta_k + 2 s_k \gamma_k) \cos\frac{k \pi n}{m} \tag{4.12}$$

is obtained; this one, compared with (4.6) and (3.5) gives

$$(\omega_4 + 2\gamma_k) s_k + \omega_2 t_k + c \beta_k r_k = 0{,}5\, c \beta_k p_k + C_k \tag{4.13}$$

whence, using 4.10 ($k \neq m$):

$$r_k = \frac{p_k}{2} + \frac{C_k}{c \beta_k} - \eta_k t_k + \bar{\eta}_k K_k' \tag{4.14}$$

with

$$\eta_k = \frac{\omega_2 + (\omega_4 + 2\gamma_k) \eta_k'}{c \beta_k}, \qquad \bar{\eta}_k = \frac{\omega_4 + 2\gamma_k}{c \beta_k \omega_1}. \tag{4.15}$$

Finally from (2.3) to (2.7) and (3.8) it is also obtained:

$$\delta_B - \delta_A = \frac{c^2}{6 E J} \sum (2 c r_k \alpha_k - c p_k \alpha_k + 6 s_k \beta_k) \sin\frac{k \pi n}{m}. \tag{4.16}$$

Also, knowing that

$$\varphi_A^* + \varphi_B^* = (\varphi_A + \varphi_B) \cos\psi + (\theta_A + \theta_B) \sin\psi \tag{4.17}$$

and using (3.9), (2.9) and (2.14), it is verified that (4.3) can be written

$$\delta_B - \delta_A = \frac{-c^3}{6 E J} \sum (3 \omega_5 \gamma_k + \omega_6 - \omega_7 \lambda_k) r_k \sin\frac{k \pi n}{m}, \tag{4.18}$$

where

$$\omega_5 = \left(\cos^2\psi + \frac{E J}{G J_t} \sin^2\psi\right) \frac{b^2}{c^2}, \qquad \omega_6 = \frac{b^3}{2 c^3} \frac{E J}{E J^*}, \qquad \omega_7 = \frac{3 b^2}{c^2} \cos^2\psi. \tag{4.19}$$

Putting
$$\eta_k'' = 3\,\omega_5\gamma_k + \omega_6 - \omega_7\lambda_k + 2\,\alpha_k \tag{4.20}$$
and comparing (4.16) and (4.18), $(k \neq m)$:
$$c\,\eta_k'' r_k + 6\,\beta_k s_k = c\,\alpha_k p_k \tag{4.21}$$
is obtained, whence, with (4.10) and (4.14):
$$\begin{aligned} t_k &= \frac{c\,p_k(\alpha_k - 0{,}5\,\eta_k'') - (\eta_k''/\beta_k)\,C_k + (6\,\beta_k/\omega_1 - c\,\bar\eta_k\,\eta_k'')\,K_k'}{6\,\beta_k\,\eta_k' - c\,\eta_k\,\eta_k''} \\ &= c\,\eta_k^*\,p_k + \eta_k^{**}\,C_k + \bar\eta_k^*\,K_k'. \end{aligned} \tag{4.22}$$

For $k=m$, (4.13) is used (remark that $\beta_m = 0$ and $\gamma_m = 0{,}25$):
$$s_m = \frac{C_m - \omega_2\,t_m}{\omega_4 + 0{,}5} \tag{4.23}$$
that compared with (4.10) permits to write
$$t_m = \frac{C_m - (\omega_4 + 0{,}5)\,K_m'/\omega_1}{\omega_2 + (\omega_4 + 0{,}5)\,\eta_m'} \tag{4.24}$$
or, with the notation of the last member of (4.22):
$$\eta_m^* = 0, \qquad \eta_m^{**} = \frac{1}{\omega_2 + (\omega_4 + 0{,}5)\,\eta_m'}, \qquad \bar\eta_m^* = \frac{\omega_4 + 0{,}5}{\omega_1}\,\eta_m^{**}. \tag{4.25}$$

The constants K_a, K_b, C_a, C_b, C_s and C_t are determined in view of the conditions at the extremities: $S_0 = S_m = 0$ (from 3.6 and 3.7 with $U_0 = U_m = 0$ and $R_0 = R_m = 0$) and those defined by (2.7) and $(2.12)_j$ that are[9])
$$\sum_0 s_k = 0, \qquad \sum_e s_k = -C_s, \qquad C_b - 2\,\omega_2 C_t = C_s(2\,\omega_4 + 1/3), \tag{4.26}$$
$$\begin{aligned} &\sum_0 t_k\gamma_k = -\frac{m^2}{16}K_a, \qquad \sum_e t_k = \frac{m+1}{4}K_b, \qquad C_t = -\frac{1}{4}K_b, \\ &\omega_1 C_s + \omega_3 C_t + 0{,}5\,C_b' = \frac{m^2-1}{4}K_b + 2\sum_e \gamma_k t_k. \end{aligned} \tag{4.27}$$

From the first equalities (4.26) and (4.27), using (4.10) and (4.22) and putting $C_a' = 0$, the equations
$$\begin{aligned} C_a \sum_{0m} \eta_k'\,\eta_k^{**} + K_a\left(\sum_{0m} \eta_k'\,\bar\eta_k^* - \frac{m^2}{8\,\omega_1}\right) &= -c\sum_0 \eta_k'\,\eta_k^*\,p_k, \\ C_a \sum_{0m} \gamma_k\,\eta_k^{**} + K_a\left(\sum_{0m} \gamma_k\,\bar\eta_k^* + \frac{m^2}{16}\right) &= -c\sum_0 \gamma_k\,\eta_k^*\,p_k \end{aligned} \tag{4.28}$$
are obtained, from which C_a and $K_a = K_k'/\gamma_k$ are determined.

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[9]) The 3d condition proceeds from (2.7) combined with (3.8) and (4.6):
$$C_{\varphi B} - C_{\varphi A} = \frac{1}{12}\sum_e (u_{kB} - u_{kA}) = -\frac{1}{6}\sum_e s_k = -\omega_4 C_s - \omega_2 C_t + 0{,}5\,C_b.$$

The equations that determine C_b, C_b' and K_b are:

$$C_b \sum_{em} \eta_k^{**} + C_b' \sum_{em} \bar{\eta}_k^{*} + K_b \left(\sum_{em} \gamma_k \bar{\eta}_k^{*} - \frac{m+1}{4}\right) = -c \sum_{e} \eta_k^{*} p_k, \tag{4.29}$$

$$C_b \left(\sum_{em} \eta_k' \eta_k^{**} + \omega_8\right) + C_b' \left(\sum_{em} \eta_k' \bar{\eta}_k^{*} - \frac{m}{2\,\omega_1}\right) + K_b \left(\sum_{em} \gamma_k \eta_k' \bar{\eta}_k^{*} + \frac{2\,\omega_9}{\omega_1}\right) = -c \sum_{e} \eta_k' \eta_k^{*} p_k,$$

$$C_b \left(\sum_{em} \gamma_k \eta_k^{**} - 0{,}5\,\omega_1 \omega_8\right) + C_b' \left(\sum_{em} \gamma_k \bar{\eta}_k^{*} - 0{,}25\right) + K_b \left(\sum_{em} \gamma_k^2 \bar{\eta}_k^{*} - \omega_9 - \frac{\omega_3}{8}\right)$$
$$= -c \sum_{e} \gamma_k \eta_k^{*} p_k,$$

where
$$\omega_8 = \frac{3}{6\,\omega_4 + 1}, \qquad \omega_9 = \frac{1}{48}\left(12\,\omega_1 \omega_2 \omega_8 + 1 - m^2\right). \tag{4.30}$$

Finally, C_s and C_t are obtained from:

$$C_s = -\sum_{e} s_k, \qquad C_t = -0{,}25\,K_b. \tag{4.31}$$

So, all the necessary elements to solve the problem are known. The solution can be obtained with the following sequence:

a) Determine the quantities ω that only depend of E/G and of the dimension of the structure.
b) Determine the constants η that depend of the ω and of the coefficients of the Tables I, II, III.
c) Calculate the following constants, that depend of the load (p_k): K_a and C_a (from 4.28), K_b, C_b and C_b' (from 4.29).
d) Calculate t_k (4.22), s_k (4.10), r_k (4.14), C_s and C_t (4.31) and, after, R_n, S_n and T_n (3.5).
e) Using (3.6) and (3.7), all the forces and moments (R, U and V) that act on the longitudinal girder will be known and, through (3.1), (3.2) and (3.3), the forces and moments that act on the cross beams.

5. Example

Consider a gridwork with girders long $l = 24$ meters, divided in $m = 6$ parts $c = 4$ meters long, by five cross beams $b = 6$ meters long; it is known that $\tan\psi = 2$, $J = J^* = 1{,}6\,J_t = 1{,}6\,J_t^*$ and $E/G = 2{,}5$. It is asked the effect of one load $P = 1000$ kg applied in the intersection of the girder A with the cross beam $n = 2$.

With the previous formulae, the coefficients ω, which depend only of the dimensions of the grid, are calculated:

$\omega_1 = 0{,}45$, $\omega_2 = 1{,}8$, $\omega_3 = 0{,}6$, $\omega_4 = 5{,}1$, $\omega_5 = 7{,}65$,
$\omega_6 = 1{,}6875$, $\omega_7 = 1{,}35$, $\omega_8 = 0{,}09494$, $\omega_9 = -0{,}70994$.

Then, the coefficients η, that depend also from the functions tabulated in the Tables I, II and III, are determined; the results are given in the Table IV. The three last lines of this Table, where are already given the unknowns t_k, s_k and r_k, were obtained after the determination of the constants C_a, K_a, C_b, C_b' and K_b with (4.28) and (4.29):

$$\begin{aligned} 0{,}51613\,C_a + 0{,}29746\,K_a &= -87{,}463\,c, \\ -0{,}07440\,C_a + 0{,}25716\,K_a &= 18{,}420\,c. \end{aligned} \qquad (5.1)$$

$$\begin{aligned} -0{,}13926\,C_b - 1{,}8154\,C_b' - 2{,}5807\,K_b &= 3{,}1054\,c, \\ 0{,}55324\,C_b - 0{,}5542\,C_b' + 0{,}3307\,K_b &= -23{,}2882\,c, \\ -0{,}08872\,C_b - 1{,}0807\,C_b' + 0{,}0998\,K_b &= 4{,}3082\,c, \end{aligned} \qquad (5.2)$$

Table IV

$k=$	1	2	3	4	5	6
p_k (kg)	288,68	288,68	0	−288,68	− 288,68	0
η_k'	−17,9202	− 5,7778	− 3,5556	− 2,8148	− 2,5242	− 2,4444
$c\,\eta_k$	−16,036	−22,645	−39,778	−74,991	−173,07	—
$c\,\overline{\eta}_k$	2,0046	9,1093	27,111	66,588	174,43	—
η_k''	242,250	34,6375	14,0375	10,0042	7,2987	—
$1000\,\eta_k^*$	−17,296	−17,007	−10,989	− 6,250	− 2,762	0
$1000\,\eta_k^{**}$	− 7,286	−27,609	−51,258	−69,592	− 80,546	−84,112
$\overline{\eta}_k^*$	− 0,12563	− 0,40372	− 0,68266	− 0,88837	− 1,00797	− 1,04673
$\gamma_k\,\overline{\eta}_k^*$	− 0,46886	− 0,40372	− 0,34133	− 0,29612	− 0,27009	− 0,26168
K_k'	72,368	1,0355	9,6955	0,1358	5,1958	− 0,1454
$\eta_k^*\,p_k$ (kg)	− 4,9930	− 4,9096	0	1,8042	0,7973	0
t_k/c (kg)	−12,769	− 4,134	2,640	4,691	10,109	1,970
s_k/c (kg)	68,007	21,584	−30,932	−13,506	− 37,063	− 4,492
r_k (kg)	71,676	35,208	6,609	− 8,071	− 4,308	—

Table V

$n=$	1	2	3	4	5
T_n (kgm)	−106,3	− 10,6	25,9	23,1	52,2
S_n (kgm)	437,8	137,1	−136,7	−234,1	−290,2
$R_n = R_{nA} = -R_{nB}$ (kg)	63,8	103,3	60,8	28,3	16,8
U_{nA} (kgm)	523,4	273,7	− 55,1	−196,1	−267,7
U_{nB} (kgm)	−352,2	− 0,5	218,3	272,1	312,7
V_{nA} (kgm)	−277,5	−266,6	−137,2	− 52,8	− 7,1
V_{nB} (kgm)	− 64,9	−245,4	−189,0	− 99,0	−111,5

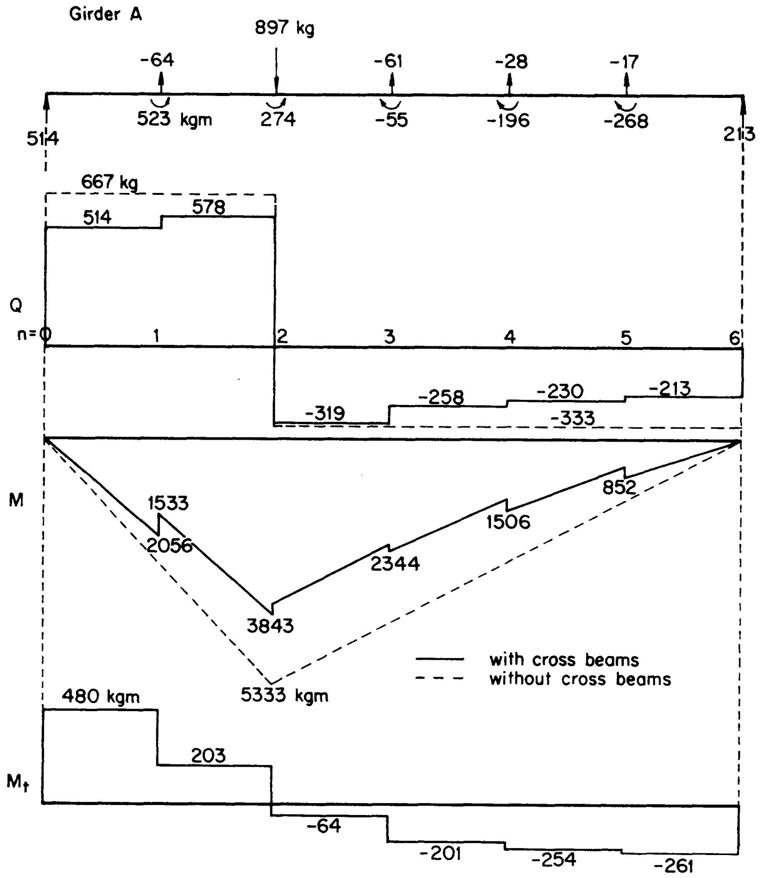

Fig. 3.

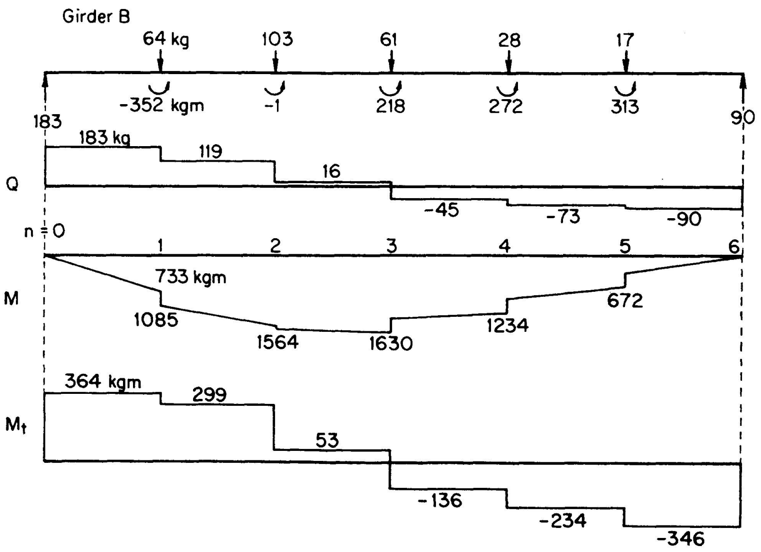

Fig. 4.

with the results:

$$C_a = -180{,}63\,c, \quad K_a = 19{,}391\,c,$$
$$C_b = -43{,}216\,c, \quad C_b' = -0{,}3141\,c, \quad K_b = 1{,}3496\,c. \tag{5.3}$$

From (4.31):

$$C_t = -0{,}3774\,c, \qquad C_s = -3{,}586\,c. \tag{5.4}$$

The statically indeterminate unknowns R, S and T and the forces and moments acting in the girders and in the cross beams are obtained with (3.5), (3.6), (3.7) and (3.2). The results are given in the Table V and in the diagrams of Figs. 3 and 4.

Observe that, for the determination of the effects of other applied loads it is not necessary to recalculate the coefficients ω and η; the same values already obtained can be used.

The time spended to solve this example, without electronic computers, was much more brief than the necessary one to solve it by the usual methods.

References

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Summary

The gridworks of skew bridges consisting of two simply supported, equal longitudinal girders connected by $m-1$ equal and equidistant transverse cross beams are studied. To simplify the solution of the problem, that otherwise would present $3m-3$ statically indeterminate unknowns, trigonometric polynomials are used, giving the exact solution of the question for loads applied on the intersections of the cross beams with the girders. In this manner, it is not necessary to solve systems with a great number of simultaneous equations. All effects of torsion and bending of the longitudinal girders and also of the cross beams are considered.

Résumé

On étudie les réseaux de poutres croisées de ponts biais composés de deux longerons égaux simplement appuyés solidarisés par $m-1$ poutres transversales égales et équidistantes. Pour simplifier la résolution du problème, qui autrement aurait des inconnues hyperstatiques au nombre de $3m-3$, on fait usage de polynômes trigonométriques qui permettent d'obtenir la solution exacte dans le cas de charges appliquées aux intersections des traverses et des longerons. Ce qui évite la résolution de systèmes comportant un grand nombre d'équations simultanées. On considère tous les effets de torsion et de flexion des longerons ainsi que des poutres transversales.

Zusammenfassung

Der Verfasser untersucht schiefwinklige Trägerroste, bestehend aus zwei gleichen, einfach gelagerten Längsträgern, die durch $m-1$ gleiche und äquidistante Querträger miteinander verbunden sind. Um die Lösung des Problems zu vereinfachen, welches sonst $3m-3$ statische Unbekannte aufweisen würde, werden trigonometrische Polynome eingeführt, die für in den Schnittpunkten Querträger-Längsträger aufgebrachte Lasten eine exakte Lösung des Problems ergeben. Auf diese Weise ist es nicht mehr notwendig, Gleichungssysteme mit einer großen Anzahl von Simultangleichungen aufzulösen. Alle Verdrehungs- und Biege-Einflüsse der Längsträger und der Querträger werden berücksichtigt.