Toward a generalized treatment of delayed elasticity in concrete

Toward a Generalized Treatment of Delayed Elasticity in Concrete

Vers un traitement généralisé de l'élasticité differée dans le béton

Zu einer allgemeinen Behandlung der verzögerten Elastizitätstheorie im Beton

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Introduction

It has become increasingly important, as structural design concepts become more advanced and more precise in character, that the creep of concrete must be recognized as an important structural property. The discussion which follows is directed toward the development of simplified means for introducing this property into structural design procedures, particularly when the loading is a function of time. The mathematician will find that the presentation is lacking in rigor; and the materials research scientist will find that the postulated properties of concrete do not conform to his observations. Nevertheless, it is hoped that the structural engineer will find some guide-lines which will be helpful in evaluating certain effects which are too often neglected in design.

Experimental work during recent years by various investigators has apparently confirmed, as a reasonable working basis, a principle for the superposition of creep effects in concrete which has been stated as follows [1]:

The strains produced in concrete at any time t by a stress increment applied at time $t_0 (t > t_0)$ are independent of the effects of any stress applied either earlier or later than t_0. The stress increment may be either positive or negative, but stresses which approach the ultimate strength are excluded.

Among those who have investigated its applicability, mention may be made of Backstrom [2], Seed [3], Ross [4], Davies [5], and Pirtz and Carlson [6]. The experimental work indicates that concrete behaves only in approximate conformance with the principle, but that acceptable accuracy may be expected for stress levels less than about one-half the ultimate strength. It is presumed, of course, that the creep characteristics are known for the concrete used, the exposure conditions, and the dimensions of the member. (Transfer of data from a 15-cm-prism to a full-size member may be highly uncertain when drying is involved.) The change in elastic and creep properties with increased hydration (related to age) must be taken into account in any precise calculation; but for certain approximate or comparative calculations this added complication may sometimes be neglected. It has also been suggested that environmental and other factors which influence creep might be introduced by suitably distorting the time scale, a suggestion which offers significant possibilities.

In the discussion which follows it will be assumed that the principle as stated above is acceptable, and it will be shown that in consequence many problems involving creep of concrete may be solved by the conventional methods of the theory of elasticity, utilizing a simplifying mathematical device which permits transferring the time-dependent effects from the material to the boundary conditions.

In general, simplifying assumptions are necessary in all design problems which involve creep of concrete. Frequently the effect of a difference between the environment of the laboratory and that of the service condition is neglected, either for simplicity or from lack of adequate information. Perhaps the greatest deviation from reality is in the assumption often adopted that the elastic and creep properties are stable, that is, unchanging with time. Such simplification is often necessary in order to bring problems into a range which can be handled by available methods.

If superposition of creep effects is accepted, then the time-dependent effect of load may be treated as a delayed elastic response. This in turn suggests that problems involving the type of creep here postulated may be treated by the methods of elasticity, provided that the time dependence is suitably introduced. Studies toward this type of solution for the "elastische Nachwirkung" of steel wires under torsion were undertaken by BOLTZMAN [7] nearly a century ago. JEFFREYS [8] and others have noted that the stress-strain relationship for delayed elasticity might be expressed by substituting a fictitious value of Young's modulus for the true value. However, for loadings which vary with time and location this fictitious value may range from $+\infty$ to $-\infty$; and it may even vary from point to point within the body, a circumstance which leads immediately to great difficulty in the solution of stress problems. Some others, e. g. ZIENKIEWICZ [9], have approached the matter along lines similar to those which will be considered here.

An incomplete treatment of these concepts has been presented in an earlier paper [1], but it appears desirable at this time to develop a more complete comparison of the three-dimensional equations of conventional elasticity with the corresponding equations for delayed elasticity as applied to concrete. To develop the analogy it will be necessary to include the conventional equations, even though they are available elsewhere. The classical notation of elasticity will be used throughout, rather than the abbreviated tensor notation. A generalized treatment will be presented, but in discussing practical applications use will be made of simplifying assumptions.

General Considerations

Concrete is ordinarily put to work to perform its life function of carrying loads when it is a few days to a few weeks old, long before it has developed stable properties. It is important, therefore, that the general stress-strain relationship should include two time variables: the age of the concrete and the elapsed time after loading. For a constant uniaxial stress, σ_0, the creep deformation in the direction of the load is taken to be.

$$Q(T) = \sigma_0 f(K, T-K) = \epsilon(T) - \frac{\sigma_0}{E(T)}, \tag{1}$$

where σ, ϵ, E are stress, total strain, and modulus of elasticity, respectively, K is the age when loaded, and T is the age at which Q is considered. The function $f(K, T-K)$ is taken as a bounded monotonic function, which is associated with the material and does not depend on the loading[1]). Various forms of this function have been proposed in the literature, but the explicit function is not essential to this generalized development, and its form may therefore be left open.

For mathematical convenience the creep function $f(K, T-K)$ is taken as the change in strain with respect to the value $1/E(T)$, rather than as a change from the initial value of $1/E$. The elastic strain is thus always equal to $\sigma(T)/E(T)$. In reference [1] this was referred to as the "true creep", in contrast to the "apparent creep" that is ordinarily reported in the literature.

According to the foregoing concepts the total strain for a unilateral continuously varying stress is given by

$$\epsilon(T) = \frac{\sigma(T)}{E(T)} + \int_t^T f(K, T-K) \frac{d\sigma}{dK} dK, \tag{2}$$

where $\sigma(t) = 0$. Any finite jumps are to be accounted for by additions to (2) of terms of type (1). The point during the history of the material at which $K = 0$ is indefinite and must be determined from physical considerations.

In the general 3-dimensional case, Poisson's ratio μ must be introduced. Only limited data are available regarding the value of μ in creep; so it will be assumed, partly to simplify the treatment and partly because presently available data confirm it as reasonably accurate, that the ratio is the same for

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1) It is possible that creep includes a viscous component which increases without limit; but evidence indicates that this component (if present at all) must be a minor one.

creep as for the elastic deformation, and that it is independent of age. Following this assumption we express the total strain under continuously varying triaxial stress by three equations of the form

$$\epsilon_x = \frac{1}{E}[\sigma_x - \mu(\sigma_y + \sigma_z)] + \int_t^T f(K, T-K)\frac{d}{dK}[\sigma_x - \mu(\sigma_y + \sigma_z)]\,dK \qquad (3\text{a, b, c})$$

in which E, ϵ, and σ are functions of T. Corresponding expressions for ϵ_y and ϵ_z may be written by cyclic interchange of the subscripts.

Solving these algebraically insofar as possible for σ_x, σ_y, and σ_z gives three equations of the form

$$\sigma_x + E\int_t^T f(K, T-K)\frac{d\sigma_x}{dK}\,dK = \frac{E}{1+\mu}\epsilon_x + \lambda e \qquad (4\text{a, b, c})$$

in which

$$\lambda = \frac{E\mu}{(1+\mu)(1-2\mu)}, \qquad e = \epsilon_x + \epsilon_y + \epsilon_z.$$

Similarly, for the shear relationship the usual method of analyzing the case of pure shear ($\sigma_x = -\sigma_y, \sigma_z = 0$) is adapted to yield

$$\gamma_{xy} = \frac{\tau_{xy}}{G} + \frac{E}{G}\int_t^T f(K, T-K)\frac{d\tau_{xy}}{dK}\,dK \qquad (5\text{a, b, c})$$

in which γ and τ are shearing strain and stress, respectively, and G is the modulus of elasticity in shear, equal to $\frac{E}{2(1+\mu)}$, all taken as functions of age.

Equations for the stress-strain relation for instantaneous stress changes are, of course, identical to (3), (4), and (5) except that the integrals are replaced by their corresponding constant stress functions.

Having the superposition principle and the stress-strain relationship defined, it is apparent that a solution must exist for the general problem of stress and displacement in the 3-dimensional isotropic body. Some of the mathematical difficulties may be circumvented by a treatment which reduces the problem to one in conventional elasticity, following somewhat the methods developed for studying temperature stresses [10].

Certain of the fundamental equations of the theory of elasticity are given below, each equation representing a group of three in which the remaining two may be obtained by cyclic interchange. The following additional notation is used:

u, v, w = displacements in directions of x, y, z, respectively,
$\bar{X}, \bar{Y}, \bar{Z}$ = surface forces per unit of area,
X, Y, Z = body forces per unit of volume,
l, m, n = direction cosines of the outer normal.

Stress-strain relationship:

$$\epsilon_x = \frac{1}{E}[\sigma_x - \mu(\sigma_y + \sigma_z)], \tag{6a, b, c}$$

$$\sigma_x = \lambda e + 2G\epsilon_x, \tag{7a, b, c}$$

$$\tau_{xy} = G\gamma_{xy}. \tag{8a, b, c}$$

Equilibrium in terms of stress:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X = 0. \tag{9a, b, c}$$

Boundary conditions in terms of stress:

$$\bar{X} = \sigma_x l + \tau_{xy} m + \tau_{xz} n. \tag{10a, b, c}$$

Conditions of compatibility:

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}, \tag{11a, b, c}$$

$$2\frac{\partial^2 \epsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right). \tag{12a, b, c}$$

Equilibrium in terms of displacements:

$$(\lambda + G)\frac{\partial e}{\partial x} + G\nabla^2 u + X = 0. \tag{13a, b, c}$$

Boundary conditions in terms of displacements:

$$\bar{X} = \lambda e l + G\left(\frac{\partial u}{\partial x}l + \frac{\partial u}{\partial y}m + \frac{\partial u}{\partial z}n\right) + G\left(\frac{\partial u}{\partial x}l + \frac{\partial v}{\partial x}m + \frac{\partial w}{\partial x}n\right). \tag{14a, b, c}$$

Eqs. (9), (10), (11), and (12) or eqs. (13) and (14) are independently sufficient for the solution of most problems in conventional elasticity provided the boundary conditions are suitably specified. These two systems of equations are related through the stress-strain relationships (7) and (8).

Corresponding equations may be written out for the solution of problems involving a material which exibits creep in the manner here postulated. The derivation is straightforward, following the treatment in conventional elasticity; so it need not be given in detail. The stress-strain relationship has been given by eqs. (3), (4), and (5) which are analogous to (6), (7), and (8). The equations of static equilibrium in terms of stress and boundary conditions in terms of stress evidently remain unchanged, the inertia due to creep being negligible. It is evident, also, that if the pure elastic strains at any time satisfy the equations of compatibility, then so do the total strains. Insofar as the principle of superposition is applicable, then, solutions reached by the use of eqs. (9) through (12) are valid either for the so-called perfectly elastic materials or for materials which exhibit combined elasticity and creep.

The equations of equilibrium in terms of displacements, analogous to eqs. (13), are reached by substitution of (4) and (5) into (9), yielding

$$(\lambda + G)\frac{\partial e}{\partial x} + G\nabla^2 u + X + E\int_t^T f(K, T-T)\frac{dX}{dK}\,dK = 0. \qquad (15\text{a, b, c})$$

Similarly, by substitution of eqs. (4) and (5) into (10), the boundary conditions in displacements are given as

$$\begin{aligned}\lambda l e + G\left(\frac{\partial u}{\partial x}l + \frac{\partial u}{\partial y}m + \frac{\partial u}{\partial z}n\right) + G\left(\frac{\partial u}{\partial x}l + \frac{\partial v}{\partial x}m + \frac{\partial w}{\partial x}n\right)& \\ = \bar{X} + E\int_t^T f(K, T-K)\frac{d\bar{X}}{dK}\,dK.&\end{aligned} \qquad (16\text{a, b, c})$$

Eqs. (15) and (16) are the same as eqs. (13) and (14) of conventional elasticity except for the time-dependent terms. These terms enter into the equations in the same way as do the terms X and $\bar{X}$, indicating that the displacements are the same as those produced in a perfectly elastic material by body forces X' and surface forces $\bar{X}'$, where

$$X' = X + E\int_t^T f(K, T-K)\frac{dX}{dK}\,dK, \qquad (17\text{a, b, c})$$

$$\bar{X}' = \bar{X} + E\int_t^T f(K, T-K)\frac{d\bar{X}}{dK}\,dK. \qquad (18\text{a, b, c})$$

The above development has not included instantaneous stress changes; but it is apparent from the generality of the method that the integrals may be replaced throughout by the corresponding expressions for creep due to constant load, and that such terms may be added to the terms involving the integrals.

In the case of a body with no previous stress-history, subjected to known surface forces and body forces which are functions of the age of the material, the problem of stress and displacement is then solved as follows: at time T the material is replaced by a perfectly elastic material of Young's modulus $E(T)$ and Poisson's Ratio μ, and the stress distribution and displacements are determined by the methods of conventional elasticity, using for the stress computations the true values of the known forces and for the displacement computations the fictitious values of the forces as given by eqs. (17) and (18).

When the boundary conditions are given in terms of displacements, or when strains are specified, the use of a creep function leads to integral equations of some difficulty. It is expedient for this case to introduce a "relaxation function", so-called because it defines the way in which a uniaxial stress relaxes under a constant strain. This function, like the creep function, may be determined experimentally. Denoting by $\phi(K, T-K)$ the reduction in stress from the initial value, when strain is constant at ϵ_0, we have

$$\epsilon_0 \phi(K, T-K) = E(T)\epsilon_0 - \sigma(T). \qquad (19)$$

It will be noted that the dimensions of ϕ are the same as those of modulus of elasticity.

For a continuously varying strain, with $\epsilon(t)=0$, the principle of superposition leads to

$$\sigma(T) = E(T)\,\epsilon(T) - \int_t^T \phi(K, T-K)\frac{d\epsilon}{dK}\,dK. \tag{20}$$

The treatment which was applied to the creep function now yields the equations of equilibrium in displacements as

$$(\lambda+G)\frac{\partial e}{\partial x} + G\nabla^2 u - \frac{1}{E}\int_t^T \phi(K, T-K)\frac{d}{dK}\left[(\lambda+G)\frac{\partial e}{\partial x} + G\nabla^2 u\right] dK + X = 0 \tag{21a, b, c}$$

and the boundary conditions as

$$\bar{X} = \lambda e l + G\left(\frac{\partial u}{\partial x}l + \frac{\partial u}{\partial y}m + \frac{\partial u}{\partial z}n\right) + G\left(\frac{\partial u}{\partial x}l + \frac{\partial v}{\partial x}m + \frac{\partial w}{\partial x}n\right) - \frac{1}{E}\int_t^T \phi(K, T-K)\cdot$$
$$\cdot\frac{d}{dK}\left[\lambda e l + G\left(\frac{\partial u}{\partial x}l + \frac{\partial u}{\partial y}m + \frac{\partial u}{\partial z}n\right) + G\left(\frac{\partial u}{\partial x}l + \frac{\partial v}{\partial x}m + \frac{\partial w}{\partial x}n\right)\right] dK. \tag{22a, b, c}$$

If we let

$$u' = u - \frac{1}{E}\int_t^T \phi(K, T-K)\frac{du}{dK}\,dK \tag{23a, b, c}$$

and similarly for v', w', and e', then (21) and (22) become, respectively,

$$(\lambda+G)\frac{\partial e'}{\partial x} + G\nabla^2 u' + X = 0, \tag{24a, b, c}$$

$$\bar{X} = \lambda e' l + G\left(\frac{\partial u'}{\partial x}l + \frac{\partial u'}{\partial y}m + \frac{\partial u'}{\partial z}n\right) + G\left(\frac{\partial u'}{\partial x}l + \frac{\partial v'}{\partial x}m + \frac{\partial w'}{\partial x}n\right), \tag{25a, b, c}$$

which are equivalent to eqs. (13) and (14).

Therefore, the quantities X, Y, Z and $\bar{X}$, $\bar{Y}$, $\bar{Z}$, which at age T produce the displacements u, v, w in the actual material, are the same as those which would produce displacements u', v', w' in a perfectly elastic material whose properties at age T are specified by E and μ. If $\epsilon(0) \neq 0$, the solution may be reached, as before, by superposing the solution for variable strain on that for constant strain. In determining stresses from displacements, then, the problem is reduced to one in conventional elasticity by substituting for the true displacements their fictitious values as given by equations (23).

Considerations given above lead to the conclusion that any problem which can be solved by the methods of conventional elasticity can also be solved for delayed elasticity. It is obvious, however, that solutions may become rather involved when age-dependent properties of the material are introduced; and still further difficulties arise when the boundary conditions are mixed in terms of forces and displacements. The extent of these mathematical difficulties may be judged from a review of the papers of Theme Va of the Seventh Congress of this International Association. Refuge must be taken in simplifying assumptions, and the question may sometimes arise as to whether a frankly inaccurate solution is better than no solution at all. The answer will usually be in the affirmative, but frequently a qualitative or intuitive evaluation must be made of the effect of deviation from reality.

Simplified Applications

In introducing the simplifications which follow, there is no intent to detract from the potential usefulness of the general treatment given above. However, the complete analysis of a 3-dimensional problem including age effects will ordinarily require an extensive laboratory investigation of the concrete, plus a computerized solution of the equations, and that is beyond the scope of this paper.

In the material which follows, and in the examples, we shall omit the age factors K and T, and work only with a simple exponential creep function, $Q = \alpha(1-e^{-rt})$ where α and r are constants of the material, t is time, and e is the base of natural logarithms. The total strain for constant unit stress is then

$$\epsilon(t) = \frac{1}{E} + Q = \frac{1}{E} + \alpha(1-e^{-rt}). \tag{26}$$

The strain due to a varying stress, $\sigma(\tau)$, with the boundary condition $\sigma(0)=0$, is then given by

$$\epsilon(t) = \frac{\sigma(\tau)}{E} + \alpha\int_0^t (1-e^{-r(t-\tau)})\frac{d\sigma(\tau)}{d\tau}d\tau. \tag{27}$$

For a constant sustained strain $\epsilon = \sigma_0/E$, the corresponding function which gives the stress at any time t is[2])

$$\sigma(t) = \frac{\sigma_0}{E\alpha+1}(1+E\alpha\, e^{-rt(E\alpha+1)}), \tag{28}$$

where α and r have the same values as in (26). The right-hand member with σ_0 = unity, may be called the relaxation function, R.

For the generalized treatment given previously it was found more convenient to define the reduction in stress from the initial value, that is $\sigma(0)-\sigma(t)$. Expressed in terms of an equivalent modulus of elasticity, this becomes

$$R' = E(1-R) = E\frac{E\alpha}{E\alpha+1}(1-e^{-rt(E\alpha+1)}). \tag{29}$$

This is the function which corresponds to $\phi(K, T-K)$ in eqs. (19) through (23).

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[2]) The conversion from (27) to (28) may de made by direct solution as in integral equation, by the method of Laplace transforms, or by the use of an appropriate rheological model.

The general nature of the progress of creep and relaxation is illustrated by Fig. 1. It will be noted that relaxation tends toward its terminal value more rapidly than does creep because, of course, the stress which is producing the relaxation is continuously diminishing.

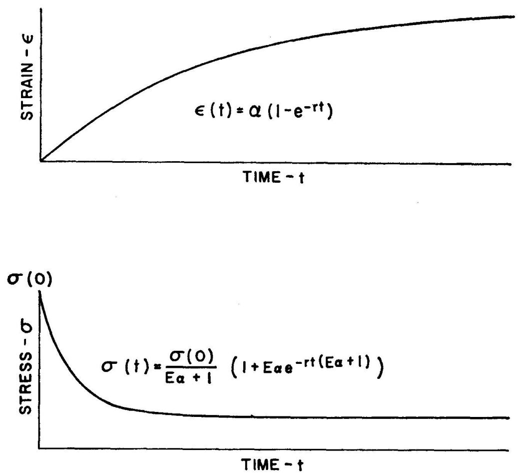

Fig. 1. Progress of creep strain and stress relaxation for the exponential function.

Another concept exists which is certainly of only limited accuracy, but which is sometimes extremely effective in reducing intractable problems to relatively simple terms. This approach derives from the circumstance that the structural designer is frequently not concerned with a complete stress-strain-time history, but only with the final state of the structure. This so-called "final state" will ordinarily be approached within a few years after construction, but in analysis it is reached by letting $t \to \infty$. For a certain class of problems this end result, so far as concerns creep, is independent of the path by which it is reached, *provided* we can accept the assumption that time-dependent changes in the properties of the material may be neglected. The class of problems referred to includes (and is restricted to) those in which the imposed loading, regardless of its source, eventually reaches a stable value. The concept may be illustrated by an example from a previous paper [11]. Adopting as the creep function the exponential expression $Q = \epsilon(t) = \alpha(1 - e^{-rt})$, let us apply the time-dependent uniaxial stress $\sigma(\tau) = \beta(1 - e^{-m\tau})$ which has the limiting value $\sigma(\infty) = \beta$. Substituting the expressions for $\epsilon(t)$ and $\sigma(\tau)$ into (27) and integrating yields a fairly complex equation of the form $\epsilon(t) = F(\alpha, \beta, r, m, t)$. Substitution of $t = \infty$ into that equation reduces it to simply $\epsilon(\infty) = \alpha\beta$ in which the rate factors r and m disappear. If the ultimate stress, β, is applied instantaneously at $t = 0$ and maintained constant thereafter, the ultimate creep is likewise given by $\epsilon(\infty) = \alpha\beta$. Fig. 2 illustrates the progress of creep for the two cases, and shows how the two approach equality.

It appears that the same treatment may be applied to any stress history which eventually reaches a stable value. Within the limitations imposed by the assumptions, a stress which reaches and then maintains a constant value will produce the same ultimate creep as if the ultimate value of the stress were applied at $t=0$ and maintained constant thereafter.

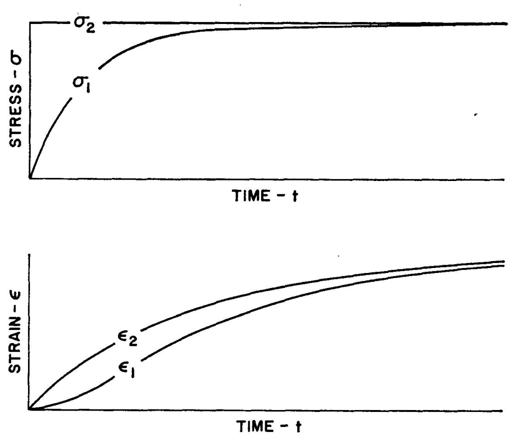

Fig. 2. Creep strain corresponding to two stress histories.

This may also be illustrated by using the relaxation function, eq. (28) derived from $Q=\alpha(1-e^{-rt})$. From (28) if the strain is constant at $\epsilon=\sigma_0/E$, then at $t=\infty$

$$\sigma(\infty)=\frac{\sigma_0}{E\alpha+1}.$$

According to the concept developed above, the ultimate stress $\sigma(\infty)$ if applied at $t=0$ should yield the ultimate strain $\epsilon(\infty)=\sigma_0/E$. This is readily shown, for at $t=\infty$ the creep component of the strain is

$$\epsilon_c(\infty)=\alpha\sigma(\infty)=\frac{\sigma_0\alpha}{E\alpha+1}$$

and the elastic component is

$$\epsilon_1(\infty)=\frac{1}{E}\sigma(\infty)=\frac{1}{E}\frac{\sigma_0}{E\alpha+1}.$$

The total strain is then

$$\epsilon(\infty)=\frac{\sigma_0}{E}\left[\frac{E\alpha}{E\alpha+1}+\frac{1}{E\alpha+1}\right]=\frac{\sigma_0}{E}.$$

This simplifying concept for the ultimate condition has been noted previously for restricted applications by others, particularly F. DISCHINGER and J. N. DISTEFANO.

This concept, which is consistent with (but also dependent on) the acceptance of superposition and delayed elasticity, will be used in the examples which follow.

As a fairly simple illustration, we may consider a composite section consisting of a prestressed concrete girder with a cast-in-place deck slab as shown in Fig. 3. This example has been selected because it has been investigated in depth both experimentally and analytically by A. H. MATTOCK [12], and thus some confirmation of the solution is available.

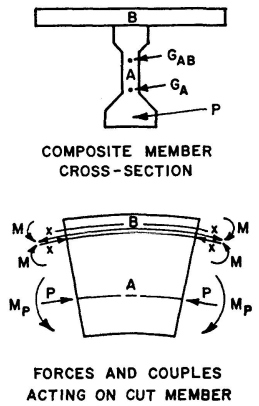

Fig. 3. Prestressed girder with cast-in-place slab.

The prescribed condition for Fig. 3 is that the girder A has been prestressed before casting the slab B, and it is desired to investigate the effect of creep due to the prestressing force P on the restraint moments and shears, M and X. Following MATTOCK's treatment, P will be taken as a constant force. We may start by considering the section as having been cast monolithically, then subjected to the force P, and finding the solution for true elasticity. The moments and shears corresponding to this solution are designated M' and X'. The elastic problem may be solved by introducing an imaginary cut along the junction between the slab and the girder. M' and X' are then determined by considering the conditions which are necessary to restore compatibility of deformations. The solutions will take the form

$$M' = P(J),$$
$$X' = P(K).$$

Where (J) and (K) involve only the dimensional properties of the sections, the location of P, and the modulus of elasticity (which enters the solution only if $E_A \neq E_B$). The elastic solution is thus available, so a solution for delayed elasticity can be developed.

Let Q be the creep function for the concrete (assumed the same for girder and slab), i. e., Q = creep strain for unit stress. Let R be the corresponding relaxation function, i. e., R = stress due to that constant strain which corresponds to an initial stress of unity.

Following the concepts developed previously, we find that the deformations produced in the composite section as a result of creep due to P will be the same as those produced in a fully elastic member by a force $P' = P\,Q\,E$. For this simple case of uniaxial loading, this relationship is apparent from elementary considerations; but it also conforms to eq. (17) for the creep effect only due to a constant load. These deformations will in turn develop the shear X and the moment M. Since the restraints develop as a result of displacements, rather than stresses, their magnitude will depend on the relaxation function R. We may then write for the terminal condition, at $t \to \infty$,

$$M = P\,Q\,E\,R(J),$$
$$X = P\,Q\,E\,R(K).$$

Comparing these with M' and X' from the elastic solution yields

$$M = M'\,Q\,E\,R,$$
$$X = X'\,Q\,E\,R.$$

As stated previously, we shall adopt for illustrative purposes the simple exponential creep function given by eq. (26) and the corresponding relaxation function of eq. (28). For $t \to \infty$ these become, respectively, $Q = \alpha$, $R = 1/E\alpha + 1$

$$Q\,E\,R = \frac{E\,\alpha}{E\,\alpha + 1}.$$

In MATTOCK's notation ϕ is the ratio of creep strain to elastic strain for unit stress; thus $E\,\alpha = \phi$, so

$$M = M'\frac{\phi}{1+\phi}, \qquad X = X'\frac{\phi}{1+\phi}.$$

These relations for the terminal conditions are the same as those arrived at by MATTOCK using an "effective modulus" method of analysis. The effective modulus method, using $E' = E/(1+\phi)$ is consistent with the concepts presented here provided the load is constant, and it is also consistent for the ultimate condition provided the loads reach an ultimate stable value. MATTOCK arrived at a different answer when he used the "rate of creep" method.

The effect of differential shrinkage between the slab and the girder of Fig. 3 may be similarly investigated. Let ϵ_s represent the difference in the free (unrestrained) shrinkage between B and A. Designate the restraint moments and shears due to ϵ_s by M_s and X_s; and let these be M'_s and X'_s if creep were not acting. It is apparent that the elastic solution may be derived in the form

$$M'_s = \epsilon_s(V),$$
$$X'_s = \epsilon_s(W),$$

where (V) and (W) involve only the dimensional properties and moduli of elasticity of the members.

The corresponding answer in terms of delayed elasticity may be written readily by direct use of the relaxation function, but for illustration we shall apply the fictitious conditions defined by eq. (23). That equation is in terms of displacements, but for the present uniaxial case we may substitute strains (e. g. by considering an element of unit length). The stress reduction function

ϕ of eq. (23) is equal to $E(1-R)$, as given by eq. (29). The fictitious strain for $t \to \infty$ is then

$$\epsilon_s' = \epsilon_s - \epsilon_s \frac{E\alpha}{E\alpha+1} = \epsilon_s\left(1 - \frac{E\alpha}{E\alpha+1}\right) = \epsilon_s \frac{1}{E\alpha+1}.$$

The terminal solution for delayed elasticity is therefore

$$M_s = M_s' \frac{1}{E\alpha+1}, \qquad X_s = X_s' \frac{1}{E\alpha+1}.$$

Introducing, as before, $\phi = E\alpha$, the moments and shears including creep effects are obtained by multiplying the corresponding values from the elastic solution by $1/(1+\phi)$. This coincides with MATTOCK's solution using the effective modulus method, but as before he reached a different answer by the rate of creep method.

MATTOKC's laboratory studies of a two-span continuous girder and slab verified his effective modulus calculation for load durations exceeding 250 days, but showed rather large discrepancies for earlier periods.

Acknowledgement

The writer wishes to acknowledge the collaboration of Prof. W. H. JURNEY of the Colorado School of Mines in parts of the development presented above.

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Summary

It appears that for design purposes the creep of concrete may be considered, with reasonable accuracy, as a delayed elastic response to stress. The mathematical theory of delayed elasticity may be developed by methods paralleling those of conventional elasticity. In the general treatment the stress-strain relationship involves two time variables: age of the concrete and elapsed time after loading. Time-dependent loading is introduced through integration based on the principle of superposition. Environmental factors which influence creep may also be introduced. In the resulting equations the time-dependent terms enter in the same way as do the boundary and body forces. In delayed elasticity, therefore, the determination of displacements from prescribed forces, or of stress from given strains or displacements, may be reached by the methods of conventional elasticity. The procedure involves simply replacing the prescribed quantities by fictitious ones which are determined from the given quantities, the nature of their time variations, and the properties of the material. Simplifying assumptions will ordinarily be introduced, and the use of such assumptions is illustrated by examples.

Résumé

Il apparaît que pour le calcul on peut considérer le fluage du béton comme une réponse élastique différée aux efforts, tout en conservant une précision raisonnable. La théorie mathématique de l'élasticité différée peut être établie par des méthodes parallèles à celles de l'élasticité conventionnelle. Dans le procédé général la relation contraintes-déformation implique deux variables temporelles: l'âge du béton et le temps qui s'est écoulé depuis le chargement. C'est par une intégration basée sur le principe de superposition qu'on introduit la fonction de charge liée au temps. On peut aussi prendre en compte les facteurs de l'environnement qui influent sur le fluage. Les termes dépendant du temps entrent dans les équations résultantes de la même façon que les efforts aux limites et les forces massiques. Dans l'élasticité différée, c'est par conséquent en mettant en œuvre les méthodes de l'élasticité conventionnelle qu'on peut déterminer les déplacements à partir des efforts prévus ou les contraintes à partir des déplacements ou des déformations données. Le procédé consiste simplement à remplacer les quantités prévues par des grandeurs fictives que l'on détermine en fonction des grandeurs données, de la nature de leur variation dans le temps et des propriétés des matériaux. On admettra généralement des hypothèses simplificatrices, dont l'emploi est ici illustré sur des exemples.

Zusammenfassung

Für die Berechnung kann das Betonkriechen mit genügender Genauigkeit als eine verzögerte elastische Antwort auf die aufgebrachte Spannung betrachtet werden. Die mathematische Formulierung der verzögerten Elastizitätstheorie kann durch ähnliche Methoden wie bei der klassischen Elastizitätstheorie erfolgen. In der allgemeinen Behandlung müssen bei Formulierung der Spannungs-Dehnungsfunktion zwei zeitabhängige Größen eingeführt werden: das Alter des Betons und die Zeit seit dem Aufbringen der Belastung. Die zeitabhängige Belastung wird durch eine nach dem Prinzip des Superpositionsgesetzes durchgeführte Integration eingeführt. Umweltsfaktoren, welche das Kriechen beeinflussen, werden ebenfalls berücksichtigt. In den resultierenden Gleichungen können die zeitabhängigen Ausdrücke auf die gleiche Weise eingeführt werden wie die Randkräfte und die Massenkräfte. Somit kann bei der verzögerten Elastizitätstheorie die Bestimmung der Verschiebungen infolge gegebener Kräfte, oder der Spannungen infolge gegebener Dehnungen oder Verschiebungen, mit den Methoden der klassischen Elastizitätstheorie erfolgen. Das Vorgehen verlangt einzig den Ersatz der vorgeschriebenen durch fiktive Größen, welche aus den gegebenen Größen bestimmt werden, unter Berücksichtigung ihrer Zeitabhängigkeit und der Eigenschaften des Materials. Im allgemeinen werden vereinfachende Annahmen eingeführt, wobei die Benützung solcher Annahmen durch Beispiele erläutert wird.

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