A continuum analysis for double layer space frame shells

A Continuum Analysis for Double Layer Space Frame Shells

Méthode de calcul des voiles reticules à deux surfaces considérés comme des systèmes continus

Kontinuumsmethode für doppelschalige Raumfachwerke

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Introduction

Surface structures, shells, plates and folded plates, constructed as continua, usually in reinforced concrete, are aesthetically pleasing, structurally efficient, and interesting in behaviour. In parallel with the development of a construction technology in the past few decades that has provided the techniques for building such structures, there has been a development of a new branch of structural mechanics to cope with the analytical problems of three-dimensional structures. Notwithstanding the success and popularity of reinforced concrete surface structures, some practical limitations are evident: whilst the structures themselves are efficient and economic, construction costs may be prohibitive especially in countries with high labour costs; and larger spans present difficult problems.

In the past five years or so there has been developed, in several forms and countries, a technology for the construction of three-dimensional frameworks which may be built as approximations to the familiar forms of surface structures [1]. It has been found that these three-dimensional frameworks possess most of the virtues of the more familiar plate and shell structures, and in addition offer special advantages in respect of low weight, large spanning capacity, and economy. These circumstances suggest that three-dimensional structural frameworks, while yet uncommon, will necessarily come to be widely used.

A great variety of arrangements of members may be used in three-dimensional structural frameworks, usually termed *space frames*. Nevertheless, some simple classifications and descriptions are possible: see Figs. 1 and 2.

The *single layer space frame* usually termed *reticulated shell* may be flat (as in a grillage) or more commonly may be built to some curved surface, as seen in the examples of Fig. 1. It is important to note that the individual member segments between nodes or joints are straight, and that a variety of patterns of surface division may be employed (the patterns in Fig. 1 all have six members

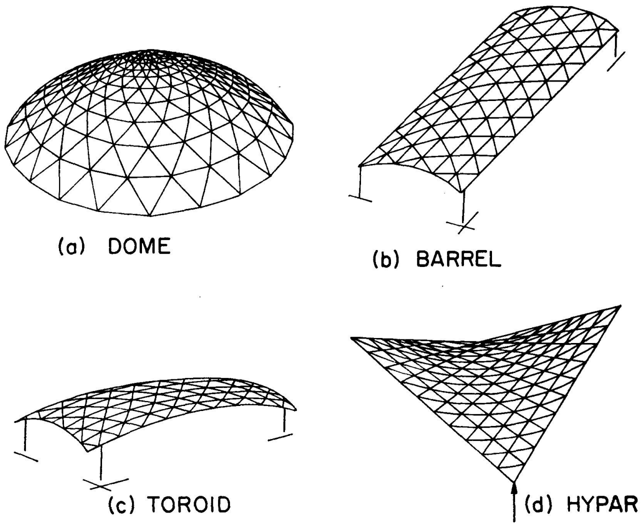

Fig. 1. Examples of reticulates shells.

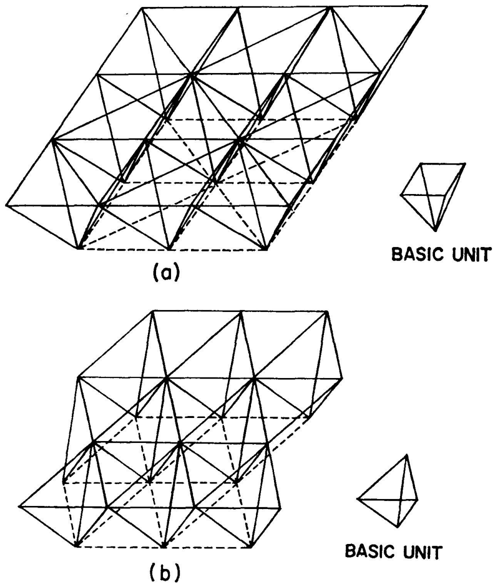

Fig. 2. Examples of double layer space frames (shown without curvature).

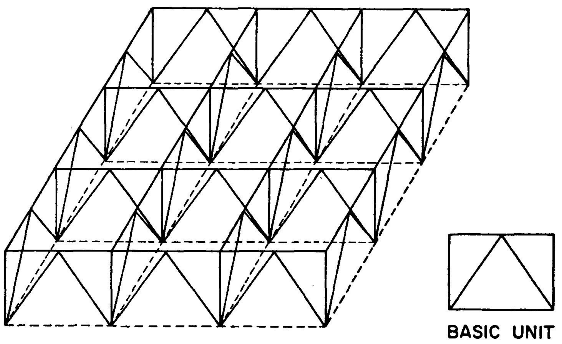

Fig. 3. Orthogonal System of plane trusses.

meeting at a node, with angles between them of approximately 60°). For reasons of stability, connections in reticulated shells must be fully rigid or at least have some substantial resistance against bending out of the shell surface (resistance to bending *in* the surface is not important). A paper dealing with the analysis and stability of reticulated shells has recently been published [2].

The *double layer space frame* (see Fig. 2) may be thought of as a close coupling of two reticulated surfaces. Some subtle but important differences in form may lead to substantial differences in behaviour. The two forms shown in

Fig. 2, when constructed with pinned joints, will resist twisting moments through axial loadings in individual members. Space frames which are developed as systems of plane trusses, as shown in Fig. 3, cannot resist twisting moments without rigid joints and without twisting individual members.

It is the purpose of this paper to present a method of analysis for double layer space frames employed as shells, with particular reference to the type of frame shown in Fig. 2b. While other types of frames may be used this form is readily adapted to any shell shape (the patterns of Fig. 1 may be seen as the bottom surface of a double layer system), and the analysis developed here may be modified to suit other space frame patterns.

Methods of Analysis

Two methods of analysis, entirely different in character, are available for space frame structures. Space frames may be treated as systems of discrete elements, through extensions of the ordinary theories for plane frameworks, or they may be treated as continua. In the latter case one deals with an equivalent continuum, possessing properties like those of the framework, and at the end one transposes results from the continuum analysis to the frame. One of the features of a continuum analysis is that it permits a clearer focussing on the behaviour of the composite structural system (valuable for considerations of buckling, for instance). While the availability of powerful computers may facilitate discrete member analysis, it must be acknowledged that space frames can be very large: an entirely practicable structure presented later in an example has about 375,000 members and over 84,000 joints! In electing a continuum analysis in this paper, it is recognized that some current investigations of continua are being developed through the use of equivalent frameworks.

One of the principal advantages of a continuum analysis lies, of course, in the fact that such analyses are already available for a great variety of shell types [3, 4]. In shell analysis the following are the customary steps:

1. A membrane force system able to resist all loads is determined.
2. Corrections to the membrane system (comprising moments and additional forces) are determined to suit boundary conditions, special perturbations, and superimposed on the membrane force system.
3. Stability is investigated.

In a continuum analysis for a space frame the same steps are followed once the elastic properties of the continuum have been established, the results are transposed back to the frame, and additional checks on individual member behaviour and joint strength must be added.

It follows then that it is necessary only to determine the properties of the equivalent continuum and the functions for transposition of continuum stress resultants to the frame to establish a continuum analysis for the space frame.

Space Frame System

Fig. 4 shows an element of the space frame system of Fig. 2b, orientated with respect to a set of coordinate axes. Six node points are identified, and the nine characteristic members, three each in the top and bottom surfaces and three diagonals coupling the two faces, may be identified by their end points.

The length of the face members (AB, BC, CA, DE, EF, FD) may be represented as L, and the length of the diagonal members (DA, DB, DC) may be represented as kL. The distance face to face of the frame may be represented as KL. It is seen that

$$K = \sqrt{k^2 - \frac{1}{3}}. \tag{1}$$

In the following it will be considered that all members have the same cross sectional area, A, and the same modulus of elasticity, E. Lengths are considered as centre to centre of nodes and it is assumed that joints are inextensible and pinned so that individual members may only be loaded axially.

Where joints have some significant extensibility and/or where members are not all of the same section, suitable modifications may be made in the following analyses. In most practical space frames, joints are very nearly inextensible, and it is not appropriate to vary member sections frequently. The diagonal members may often be of a lighter section than the face members, for reasons of economy, but such a variation would have only a slight influence in the following analyses, and this could be determined readily.

Stress Resultant Transforms

Fig. 5 shows the ten stress resultants that must be considered. Of these, eight are independent: the three components of the membrane force field N_x, N_y and $N_{xy} = N_{yx}$, the two shears Q_x and Q_y, and the three moments M_x, M_y and $M_{xy} = -M_{yx}$. The membrane stress resultants are internally statically determinate, while the others are usually statically indeterminate requiring a knowledge of the elastic properties of the shell and its supports for their determination. As already noted, stress resultants for most shell types may be readily determined using published analyses. Where complete solutions are not available (e.g. moments in hypars) the problem is not peculiar to the space frame shell. Some special problems do arise because of the anisotropy or non-homogeneity that is found in most practical space frame shells. Where variations are modest, solutions for isotropic behaviour can probably be used effectively. Most commonly non-homogeneity takes the form of a smooth variation in stiffness, as from the zenith point in a dome, that may be related to available solutions for shells of varying thickness.

Relationships will now be established between forces in the bars of the space frame (see Fig. 4) and the stress resultants. These transforms will be based on appeals to conditions for statical equilibrium under conditions of uniform levels of the particular stress resultants. For convenience and clarity the separate stress resultants will be dealt with separately; with linearly elastic behaviour the total effect may then be determined by superposition.

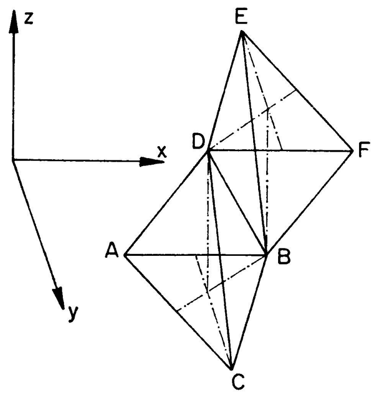

Fig. 4. Space frame element.

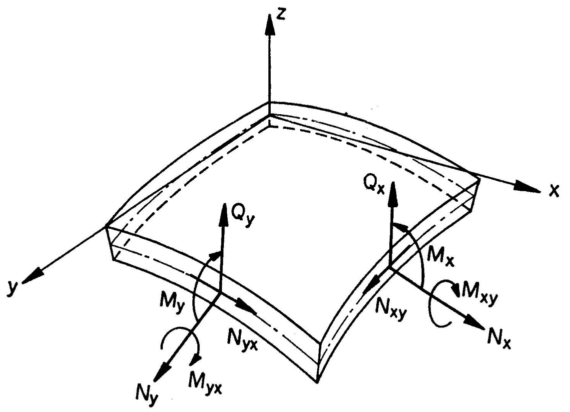

Fig. 5. Stress resultants.

Consider first the membrane force field. It is seen by inspection that identical patterns of strain in the upper and lower faces leave the diagonals unstrained, and further that with the stated conditions the upper and lower surfaces share the membrane forces equally. From a consideration of equilibrium requirements and taking tensile bar forces as positive it is seen [2] that:

$$P_{AB} = P_{FD} = \frac{L}{4\sqrt{3}}(3N_x - N_y), \tag{2a}$$

$$P_{BC} = P_{DE} = \frac{L}{2\sqrt{3}}(N_y - \sqrt{3}N_{xy}), \tag{2b}$$

$$P_{CA} = P_{EF} = \frac{L}{2\sqrt{3}}(N_y + \sqrt{3}N_{xy}). \tag{2c}$$

Consider next the shear forces, only with respect to the diagonals. The shears are of course associated with changes in moments and the resulting influences on face members may be found in terms of levels of the moments themselves. For Q_y,

$$P_{CD} = -\frac{k\,L\,Q_y}{K}, \tag{2d}$$

$$P_{AD} = P_{BD} = \frac{k\,L\,Q_y}{2K}. \tag{2e}$$

For Q_x,

$$P_{CD} = 0, \tag{2f}$$

$$P_{AD} = \frac{\sqrt{3}\,k\,L\,Q_x}{2K}, \tag{2g}$$

$$P_{BD} = -\frac{\sqrt{3}\, k\, L\, Q_x}{2K}. \tag{2h}$$

Consider next the moments M_x and M_y. These must be resisted by face members alone, and in fact the result is as if an additional component of membrane direct force were added in the x and y directions, with intensities in each single surface

$$N_m = \left|\frac{M}{K\,L}\right|$$

with appropriate signs. It follows that

$$P_{AB} = \frac{3\,(M_x - M_y)}{2\sqrt{3}\,K}, \tag{2i}$$

$$P_{BC} = P_{CA} = \frac{M_y}{\sqrt{3}\,K}, \tag{2j}$$

$$P_{DF} = -\frac{(3\,M_x - M_y)}{2\sqrt{3}\,K}, \tag{2k}$$

$$P_{DE} = P_{EF} = -\frac{M_y}{\sqrt{3}\,K}. \tag{2l}$$

Consider now the twisting moment M_{xy}. The resulting influences may be studied by loading a rectangular portion of the frame in such a fashion as to

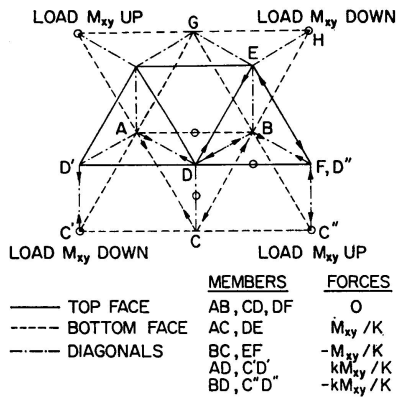

Fig. 6. Effects of twisting moments.

introduce pure twisting moment (see Fig. 6). The forces in the space frame element are statically determinate, and results are shown in Fig. 6. Interpreting these with respect to the element in Fig. 4, the following result:

$$P_{AB} = P_{CD} = P_{DF} = 0 \tag{2m}$$

except at corners where

$$P_{CD} = \pm\frac{k\,M_{xy}}{K}, \tag{2n}$$

$$P_{AC} = P_{DE} = \frac{M_{xy}}{K}, \tag{2o}$$

$$P_{BC} = P_{EF} = -\frac{M_{xy}}{K}, \tag{2p}$$

$$P_{BD} = -\frac{k\,M_{xy}}{K}, \tag{2q}$$

$$P_{AD} = \frac{k\,M_{xy}}{K}. \tag{2r}$$

Combining the results expressed in (2a–q) the nine bar forces in the space frame may be written out completely for the full system of stress resultants:

$$P_{AB} = \frac{K\,L\,(3\,N_x - N_y) + 2\,(3\,M_x - M_y)}{4\sqrt{3}\,K}, \tag{3a}$$

$$P_{BC} = \frac{K\,L\,(N_y - \sqrt{3}\,N_{xy}) + 2\,(M_y - \sqrt{3}\,M_{xy})}{2\sqrt{3}\,K}, \tag{3b}$$

$$P_{CA} = \frac{K\,L\,(N_y + \sqrt{3}\,N_{xy}) + 2\,(M_y + \sqrt{3}\,M_{xy})}{2\sqrt{3}\,K}, \tag{3c}$$

$$P_{DA} = \frac{k\,(Q_y\,L + \sqrt{3}\,Q_x\,L + 2\,M_{xy})}{2\,K}, \tag{3d}$$

$$P_{DB} = \frac{k\,(Q_y\,L - \sqrt{3}\,Q_x\,L - 2\,M_{xy})}{2\,K}, \tag{3e}$$

$$P_{DC} = -\frac{k\,L\,Q_y}{K} \pm \text{(at corners)}\,\frac{k\,M_{xy}}{K}, \tag{3f}$$

$$P_{DE} = \frac{K\,L\,(N_y - \sqrt{3}\,N_{xy}) - 2\,(M_y - \sqrt{3}\,M_{xy})}{2\sqrt{3}\,K}, \tag{3g}$$

$$P_{EF} = \frac{K\,L\,(N_y + \sqrt{3}\,N_{xy}) - 2\,(M_y + \sqrt{3}\,M_{xy})}{2\sqrt{3}\,K}, \tag{3h}$$

$$P_{FD} = \frac{K\,L\,(3\,N_x - N_y) - 2\,(3\,M_x - M_y)}{4\sqrt{3}\,K}. \tag{3i}$$

It must be observed that these have been established for a more or less uniform field of stress resultants. The stress resultants vary, of course, in response to external loadings. In a discrete member analysis loads would be treated as if applied at the nodes, and there would be finite differences between loads in similar members (in the sense of the designations of Fig. 4) in adjacent panels. Although we have used a continuum analysis and equations (3a–i) express bar forces in terms of stress resultants which are continuously variable, it is clear that there must be finite differences in loads between members in adjacent bays, and that loads do not vary along a single member. Results of sufficient precision will of course be obtained if the stress resultants are evaluated at points at the midlength of members. In fact, with the very fine divisions of surface associated with space frame construction, changes in member forces from bay to bay are rarely large.

Continuum Properties

Inspection suggests a high degree of uniformity in the elastic properties of the equivalent continuum, but it is not possible to know what degree of isotropy prevails without studying deformations in some detail. Accordingly it is appropriate to commence with an imaginary continuum which is elastic and homogeneous but not isotropic. For plane stress, Hooke's Law may be written as:

$$\epsilon_x = \frac{\sigma_x}{E'_x} - v'_x \frac{\sigma_y}{E'_y}, \tag{4a}$$

$$\epsilon_y = \frac{\sigma_y}{E'_y} - v'_y \frac{\sigma_x}{E'_x}, \tag{4b}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G'}, \tag{4c}$$

or in a form explicit in stress

$$\sigma_x = \frac{E'_x(\epsilon_x + v'_x \epsilon_y)}{(1 - v'_x v'_y)}, \tag{5a}$$

$$\sigma_y = \frac{E'_y(\epsilon_y + v'_y \epsilon_x)}{(1 - v'_x v'_y)}, \tag{5b}$$

$$\tau_{xy} = G' \gamma_{xy}, \tag{5c}$$

where ϵ_x, ϵ_y, and γ_{xy} are strains, σ_x, σ_y and τ_{xy} are the stresses, E, G and v are the usual elastic constants, subscript x's and y's indicate reference axes, and the superscript primes (′) identify the equivalent continuum.

For the membrane forces,

$$N_x = \sigma_x h', \tag{6a}$$

$$N_y = \sigma_y h', \tag{6b}$$

$$N_{xy} = \tau_{xy} h' \tag{6c}$$

where h' is the thickness of the equivalent continuum.

For bending, the usual assumptions give

$$\epsilon_x = -z \frac{\partial^2 w}{\partial x^2}, \tag{7a}$$

$$\epsilon_y = -z \frac{\partial^2 w}{\partial y^2}, \tag{7b}$$

$$\gamma_{xy} = -2z \frac{\partial^2 w}{\partial x \partial y}, \tag{7c}$$

where w is displacement, and from equations (5a–c) the stresses are seen to be

$$\sigma_x = -\frac{E'_x z}{(1 - v'_x v'_y)} \left(\frac{\partial^2 w}{\partial x^2} + v'_x \frac{\partial^2 w}{\partial y^2} \right), \tag{8a}$$

$$\sigma_y = -\frac{E'_y z}{(1 - v'_x v'_y)} \left(\frac{\partial^2 w}{\partial y^2} + v'_y \frac{\partial^2 w}{\partial x^2} \right), \tag{8b}$$

$$\tau_{xy} = -2 G' z \left(\frac{\partial^2 w}{\partial x \, \partial y} \right). \tag{8c}$$

It is then possible to write expressions for the moments as functions of displacements:

$$M_x = \int_{-h'/2}^{h'/2} \sigma_x z \, dz = -D'_x \left(\frac{\partial^2 w}{\partial x^2} + v'_x \frac{\partial^2 w}{\partial y^2} \right), \tag{9a}$$

$$M_y = \int_{-h'/2}^{h'/2} \sigma_y z \, dz = -D'_y \left(\frac{\partial^2 w}{\partial y^2} + v'_y \frac{\partial^2 w}{\partial x^2} \right), \tag{9b}$$

$$M_{xy} = -\int_{-h'/2}^{h'/2} \tau_{xy} z \, dz = 2 D_{xy} \left(\frac{\partial^2 w}{\partial x \, \partial y} \right), \tag{9c}$$

where the D values are elastic constants:

$$D'_x = \frac{E'_x h'^3}{12 (1 - v'_x v'_y)}, \tag{10a}$$

$$D'_y = \frac{E'_y h'^3}{12 (1 - v'_x v'_y)}, \tag{10b}$$

$$D'_{xy} = \frac{G' h'^3}{12}. \tag{10c}$$

It is now desired to establish the elastic constants and thickness of the equivalent continuum: E'_x, E'_y, v'_x, v'_y, G', D'_x, D'_y, D'_{xy} and h', in terms of the constants that describe the space frame: A, E, L, k, K. The relationships may be established by studying deformations of the space frame under various loadings. While it would be possible to write a set of simultaneous equations containing all the unknowns, it is much easier to proceed selectively.

Consider, first, behaviour when all stress resultants are zero saving only N_x. Eq. (6a) gives the stress in the continuum, and bar forces in AB and FD are the same and equal to

$$\frac{\sqrt{3}\, L N_x}{4}$$

according to Eq. (3). The change of length of AB and FD is

$$\frac{\sqrt{3}\, L^2 N_x}{4 A E}$$

and the strain in the x-direction is thus

$$\epsilon_x = \frac{\sqrt{3}\, L N_x}{4 A E}.$$

For the resulting strain in the y-direction see the construction in Fig. 7 which shows a displacement of C with respect to AB of

$$\frac{L^2 N_x}{8 A E}$$

and since the distance from C to AB is $\sqrt{3}\,L/2$, it follows that

$$\epsilon_y = -\frac{L N_x}{4\sqrt{3}\,A E}.$$

Substituting these values of stress and strain in Eqs. (4a) and (4b) it is found that

$$E'_x = \frac{4 A E}{\sqrt{3}\,L h'}, \tag{11a}$$

$$v'_y = \frac{1}{3}. \tag{11b}$$

It is to be noted that Eq. (11a) does not completely give E'_x since h' is not yet known.

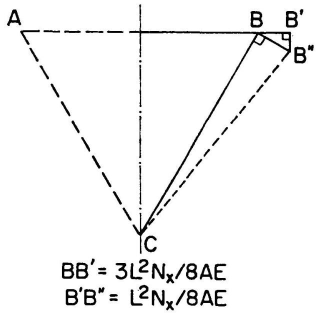

Fig. 7. Displacements in space frame (1).

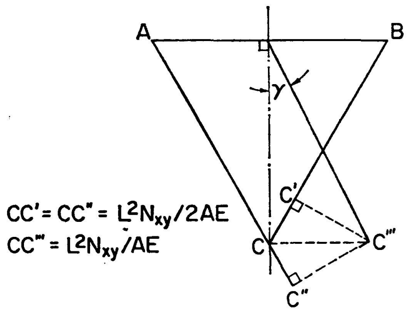

Fig. 8. Displacements in space frame (2).

In a similar fashion consider behaviour when all stress resultants are zero except N_y. Now, from Eq. (3) the bar forces in AB and FD are $(-L N_y/4\sqrt{3})$, and the bar forces in BC, DE, CA, and EF are $L N_y/2\sqrt{3}$. The changes in length are again the products of bar forces and the quantity L/AE. It is seen that

$$\epsilon_x = -\frac{L N_y}{4\sqrt{3}\,A E}$$

and by a Williot construction similar to that in Fig. 7, we find that

$$\epsilon_y = \frac{\sqrt{3}\,L N_y}{4 A E}.$$

Referring again to Eqs. (4a) and (4b), the following results are obtained

$$E'_y = \frac{4 A E}{\sqrt{3}\,L h'}, \tag{11c}$$

$$v'_x = \frac{1}{3}. \tag{11d}$$

It is evident that there is some degree of isotropy.

Now consider behaviour when all stress resultants are zero except $N_{xy} = N_{yx}$. The bar forces, from Eq. (3), are zero in AB and FD, $(-N_{xy} L/2)$ in BC and DE, and $N_{xy} L/2$ in CA and EF. Fig. 8 shows the deformation of the unit element ABC: the point C is displaced along a line parallel AB in the direction of B by an amount $N_{xy} L^2/AE$. The associated shear strain is then readily determined as $\gamma_{xy} = 2 N_{xy} L/\sqrt{3} AE$. From Eqs. (4c) and (6c) we find

$$G' = \frac{\sqrt{3} A E}{2 h' L}. \tag{11e}$$

It may be noted that

$$G' = \frac{E'}{2(1+v')}, \tag{12}$$

so that with respect to forces in the surface we have complete isotropy. It will be seen that if the cross-sections of members in the upper and lower faces were equal, or with other variations in section, behaviour would be anisotropic (see reference [2]).

Turning now to flexural deformations, consider behaviour when all stress resultants are zero except M_x. The bar force in AB is $\sqrt{3} M_x/2K$, and the force in DF is of the same intensity but opposite in sign. The strain ϵ_x in the top face of the space frame is $\sqrt{3} M_x/2KAE$, and in the bottom face is $(-\sqrt{3} M_x/2KAE)$. With the usual approximations

$$\frac{\partial^2 w}{\partial x^2} = \frac{\epsilon_{top} - \epsilon_{bottom}}{\text{depth}} = -\frac{\sqrt{3} M_x}{K^2 A E L}.$$

Now, from Eq. (9b) we see that

$$\frac{\partial^2 w}{\partial y^2} = -v_y' \frac{\partial^2 w}{\partial x^2} = \frac{M_x}{\sqrt{3} K^2 A E L}.$$

Combining these results in Eq. (9a) we find

$$D_x' = \frac{3\sqrt{3} K^2 A E L}{8}. \tag{11f}$$

The effective thickness can now be found using Eq. (10a)

$$h' = \sqrt{3} K L. \tag{11g}$$

This is most interesting since the effective thickness is substantially greater than the actual space frame depth KL; on the other hand E' is very much less than E.

It may be shown that the results so far derived satisfy Eqs. (9b) and (10b) as well. Some interesting problems arise with Eqs. (9c) and (10c). Consider Fig. 6 again. It is possible to find the vertical displacement of point C'' by the application of the principle of virtual work. The calculations are quite conventional and need not be reproduced here; the vertical displacement of C'' with respect to the plane defined by points C, G and H is

$$\frac{6 L M_{xy} (1+k^3)}{K^2 E A}.$$

Knowing this we may find the twist. Note that $\Delta w/\Delta x = 0$ at line GH, and that

$$\frac{\Delta w}{\Delta x} = \frac{6 M_{xy} (1+k^3)}{K^2 E A}$$

at line CC''. The distance from H to C'' in the y-direction is $\sqrt{3}\, L$ so that

$$\frac{\Delta}{\Delta y}\left(\frac{\Delta w}{\Delta x}\right) = \frac{6 M_{xy} (1+k^3)}{\sqrt{3}\, L K^2 E A}.$$

From Eq. (9c) we find

$$D'_{xy} = \frac{\sqrt{3}\, K^2 L A E}{12 (1+k^3)} \tag{11h}$$

and from Eqs. (10c) and (11g) we find

$$G'' = \frac{A E}{3 L^2 (1+k^3)}, \tag{11i}$$

which is clearly *not* the same as G' in Eq. (11e). Thus we observe some special anisotropy in respect of twisting deformations.

Since the elastic constants and properties of the analogous shell have now all been determined explicitly, it is appropriate to present a summary, which will also indicate the simplifications of the isotropy observed:

$$h' = \sqrt{3}\, K L, \tag{13a}$$

$$E'_x = E'_y = E' = \frac{4 A E}{3 K L^2}, \tag{13b}$$

$$v'_x = v'_y = v' = \frac{1}{3}, \tag{13c}$$

$$G' = \frac{A E}{2 K L^2}, \tag{13d}$$

$$G'' = \frac{A E}{3 L^2 (1+k^3)}, \tag{13e}$$

$$D'_x = D'_y = D' = \frac{3\sqrt{3}\, K^2 A E L}{8}, \tag{13f}$$

$$D'_{xy} = \frac{3 L K^2 A E}{12 (1+k^3)} \tag{13g}$$

and

$$G' = \frac{E'}{2(1+v')} \neq G''. \tag{13h}$$

With the elastic constants known we could write out Hooke's Law for the analogous material, and write direct equations relating moments and displacements in terms of the properties of the space frame in a manner similar to the more conventional form of Eqs. (9). For the sake of brevity these expressions will not be presented here. More particularly, with the elastic constants known we are in a position either to attempt solutions of the general shell equations for structures constructed as space frames, or, where possible, to adapt existing solutions for shell problems to space frame shells. Before considering some examples of this latter process, it is necessary to give some attention to the problem of buckling.

Buckling

In single layer space frame shells, shell buckling is the most important design consideration for dome spans much over 100 ft. Reinforced concrete shells of comparable span are not nearly so sensitive to buckling, although time-dependent effects (which do not arise with metal skeletons) may be significant. In 1963 there occurred a catastrophic failure of a single layer space frame dome of 93.5 m. span in Bucharest. In a recent paper [2] that presents methods of analysis, including buckling, for single layer space frame shells, the writer has shown that a continuum analysis provides a remarkably precise estimate of the failure load for the Bucharest dome.

In the double layer space frame, under consideration here, stiffness is much greater than with the single layer space frame. The problem of buckling will clearly recede. However, since our concern with the double layer space frame shell is, implicitly, with very large spans it may be expected that buckling will still be important.

Consider first buckling of an individual member in the face of the space frame. The critical load for such a member is

$$P_{cr} = \frac{\pi^2 E_T I}{L^2}, \tag{18}$$

where E_T is the tangent modulus (introduced to make Eq. (18) applicable to all ranges of slenderness), I is the least second moment of area, and L must be taken as the distance from node to node, independent of connection characteristics, because of the prospect of antisymmetric buckling shapes. For a simple measure of load carrying capacity (critical conditions for individual members must of course be related to local membrane and bending forces) consider a uniform radial pressure q on a spherical shell of radius a. The membrane normal forces are $N = q a/2$ everywhere and the shears vanish. Eqs. (3) and (18) give the critical load as

$$q_{cr} = \frac{4\sqrt{3}\,\pi^2 E_T I}{a\,L^3}. \tag{19}$$

It may be noted that for a given load and span, the parameter to be considered in design is the ratio (I/L^3), since in a space frame members will ordinarily be so slender that E_T will be very close to E.

Consider overall shell buckling. It must first be acknowledged that the problem is hardly satisfactorily solved even for simple homogeneous continuous spherical shells. Theoretical and experimental studies [5, 6] indicate that critical radial pressure for a spherical shell may be expressed in the form

$$q_{cr} = C\left(\frac{E h^2}{a^2}\right), \tag{20}$$

where C is a constant, h is the shell thickness and a the radius of curvature. Estimates of C vary widely, and the value is clearly affected by imperfections, residual stresses, etc. There is some fairly good reason to believe that C is somewhat larger than 0.4, as applied to space frame shells (see reference 2 *and* associated discussion): It is suggested that a value of 0.4 be used in subsequent considerations. Using E' and h' from Eq. (17) we may write

$$q_{cr} = 1.6\left(\frac{A E K}{a^2}\right) \tag{21}$$

for double layer space frame spherical shells. It is of interest to note that the significant design parameter here is the product (AK): buckling strength is independent of L, the member length.

Some thought of shells in forms other than simple spheres should be introduced. The fact that space frame shells are largely prefabricated and require little or no falsework makes the construction of most shapes as practicable as the simple sphere. The problem of buckling of non-spherical shells has been little studied, and analytical solutions must be remote. SCHMIDT [7] has suggested the use of the product $a_x a_y$, where a_x and a_y are the appropriate radii of curvature, instead of a^2 in Eq. (20) for translation shells. In this event a more general form of (21) may be

$$q_{cr} = 1.6\left(\frac{A E K}{a_x a_y}\right). \tag{22}$$

Examples

To give some indication of the significance and use of the analyses presented herein, and of the potential of double layer space frames, the design of three large span domes will be discussed. Table 1 provides a summary of data and calculations.

The domes, designated A, B and C in the table, are of spans 400 ft., 800 ft. and 1600 ft. The second is somewhat larger than any dome yet constructed. The proportions indicated are not necessarily most suitable or most economic: the three examples are intended to show the nature of space frame analysis

Table 1. Design Examples — Large Span Space Frame Domes

Line	Quantity	Dome A	Dome B	Dome C	Units
Basic Information and Assumptions					
1	Span	400	800	1600	ft.
2	Rise	80	160	320	ft.
3	Radius of curvature, a	290	580	1160	ft.
4	Depth, KL	40	70	100	in.
5	Length L	$40 \leqq L \leqq 80$	$50 \leqq L \leqq 100$	$60 \leqq L \leqq 120$	in.
6	K	$0.50 \leqq K \leqq 1.00$	$0.70 \leqq K \leqq 1.40$	$0.83 \leqq K \leqq 1.67$	—
7	Dead Load	10	15	20	psf
8	Live Load (Snow)	30	30	30	psf
Membrane Forces and Maximum Member Loads					
9	N_ϕ @ Base	-6.0	-13.8	-26.5	k/ft.
10	N_ϕ @ Zenith	-5.8	-13.0	-24.6	k/ft.
11	N_θ @ Base	-0.5	-1.4	-3.5	k/ft.
12	N_θ @ Zenith	-5.8	-13.0	-24.6	k/ft.
13	$P_{CA} = P_{BC}$ @ Base	-11.5	-33.0	-76.5	k
14	$P_{CA} = P_{BC}$ @ Zenith	-11.1	-31.3	-71.2	k
15	P_{AB} @ Base	$+4.4$	$+11.5$	$+23.1$	k
16	P_{AB} @ Zenith	-11.1	-31.3	-71.3	k
Design in Aluminum (structural alloy, yield stress = 35 ksi)					
17	Tubes (dia. × wall thickness)	$3-1/2 \times 0.120$	$4-1/2 \times 0.250$	9×0.50	in.
18	E'	7050	8500	18,300	psi
19	h'	69	121	173	in.
20	Edge Moment, M	0.78	4.66	8.02	k
21	Edge Force, H	0.086	0.132	0.223	k/ft.
22	Load Increment for Edge Effects	1.13	4.29	6.48	k
23	λ for Member Buckling	2.33	2.12	3.64	—
24	λ for Shell Buckling	3.02	2.47	2.46	—
25	Space Frame Weight	2.1	4.7	16.2	psf
Design in Steel (structural grade, yield stress = 50 ksi)					
26	Tubes (dia. × wall thickness)	$2-1/2 \times 0.120$	$4-1/2 \times 0.120$	$5-1/2 \times 0.250$	in.
27	E'	15,000	13,000	18,000	psi
28	h'	69	121	173	in.
29	Edge Moment, M	0.77	4.45	8.45	k
30	Edge Force, H	0.083	0.132	0.235	k/ft.
31	Load Increment for Edge Effects	1.12	4.12	6.86	k
32	λ for Member Buckling	2.14	1.85	2.07	—
33	λ for Shell Buckling	6.4	3.65	2.46	—
34	Space Frame Weight	4.9	7.7	16.8	psf

rather than to serve as models for design. In particular, skeleton design must interact with cladding design. The space frame pattern is that resulting when the equiangular reticular pattern of Fig. 1a is made into a double layer; lengths L diminish from the periphery towards the zenith, and to avoid excessively fine divisions, alternate members may be omitted when lengths reach a magnitude half that at the periphery. In a large shell, this process may be repeated two or three times. The result is that L varies between a maximum value and a value half as great, but with a constant depth of the space frame KL is of course a constant. In a practical design tube sizes would be varied over the surface to suit requirements for local intensities of load and for buckling strength, etc. For the sake of brevity, Table 1 lists only some critical values of membrane forces and individual member loads. Wind loads have not been treated: although they usually produce suction over the entire surface of low-rise domes, and so have negligible influence on direct loads, they may influence boundary conditions significantly.

In Table 1 membrane forces have been calculated according to the usual solutions [3] and resulting bar forces have been determined from Eq. (3). Designs are shown in both aluminum and steel. Tubular members only are considered because of their attractive structural properties. Tubes present substantial problems in the design and fabrication of connections, but satisfactory solutions have been demonstrated [8, 9, 10]. In each case E' and h' are given; it may be noted how small E' is. Edge effects are dependent upon the elastic constants of the shell, but where these are known or can be calculated as demonstrated herein, the edge moments and shears are readily calculated from existing solutions [11]. For the examples given boundary effects are calculated for encastered edges; Eq. (3) has been used again to calculate resulting increments in bar forces. Finally, load factors have been calculated as (a) the ratio of the ultimate axial compressive strength of the most critically loaded individual member to its maximum load, and (b) the ratio of the buckling pressure as calculated from Eq. (21) to the sum of the dead and live loads. The solutions are roughly comparable with least load factors of about 2 for (a), above, and 2.5 for (b). It is seen that for the 1600 ft. span domes overall buckling governs, whilst for the 400 ft. span domes buckling is not important. In the intermediate case of the 800 ft. span, buckling is not significant with steel members, and with aluminum members some balance is achieved between the two modes of failure for the intended load factors of 2 and 2.5.

The Table also indicates skeleton unit weights. These are all quite small, and notwithstanding the prospect of a relatively high cost per unit weight for space frames, the domes used as examples would all be practicably economic. There is some reason to think that roof spans (other than cable suspended structures) much over 500 ft. will not likely be built except as space frames.

It is of interest to speculate on maximum possible spans. The examples suggest that aluminum may not be competitive even at a span of 1600 ft. The proportions selected clearly favour the steel frame; even though it may be possible to produce a more efficient design for an aluminum dome of this span, the presence of the product (AE) in Eq. (21) clearly prejudices prospects for aluminum domes of larger span. Steel domes of larger span are obviously possible, although the influence of the square of the radius of curvature in Eq. (21) indicates a practical limit that may not be much more than 2500 ft., even with higher ratios of rise to span. Domes with spans of the order of a mile, which have been the subject of some serious discussion [12], seem clearly impossible.

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References

1. Z. S. Makowski: «Räumliche Tragwerke aus Stahl.» Verlag Stahleisen mbH, Düsseldorf (1963).
2. D. T. Wright: "Membrane Forces and Buckling in Reticulated Shells." Proc. American Society of Civil Engineers, Structural Division, v. 91, n. ST. 1, p. 173—201 (February 1965).
3. W. Flügge: "Stresses in Shells." Springer-Verlag, Berlin (1960).
4. V. V. Novozhilov: "The Theory of Thin Shells." P. Noordhoff Ltd., Groningen, Netherlands (1959).
5. Y. C. Fung and E. E. Sechler: "Instability of Thin Elastic Shells." Proc. 1st Symposium on Naval Structural Mechanics, Stanford 1958, Pergamon Press, Oxford (1960).
6. W. A. Little: "Reliability of Shell Buckling Predictions Based Upon Experimental Analysis of Plastic Models." Dept. of Civil Engineering, MIT, Cambridge, Mass. (August 1963).
7. H. Schmidt: «Ergebnisse von Beulversuchen mit doppelt gekrümmten Schalenmodellen aus Aluminium.» Proc. of Symposium on Shell Research, North Holland Publishing Company, Amsterdam (1961).
8. "Hub Speeds Space Frame Erection." Engineering News-Record (New York), v. 172, n. 21 (May 21, 1964).
9. F. Lederer: «Gitterschalen aus Stahlrohren.» Proc. of Symposium on Shell Research, North Holland Publishing Company, Amsterdam (1961).
10. S. du Chateau: "The SDC Structural System." Colloquium on Hanging Roofs, Metallic Shell Roofs, and Lattice Roofs, Paris 1962, Published as "Hanging Roofs", Academic Press/Wiley, New York (1963).
11. T. Brondum-Nielsen: "Axisymmetric Bending of Shells." Danish Technical Press, Copenhagen (1962).
12. L. W. Marks: "The Dymaxion World of Buckminster Fuller." Reinhold, New York (1960).

Summary

Three dimensional space frameworks are highly indeterminate structures which may be analysed either as systems of discrete elements or as continua. In this paper a continuum analysis for double layer space frames is established with particular reference to shell structures. The elastic constants of an anisotropic homogeneous continuum having the same deformational characteristics as the space frame are established, and transforms are given through which forces in members in the space frame may be determined from stress resultants for the continuum. Buckling of space frames is discussed, and examples are given of designs of domes of 400 ft., 800 ft. and 1600 ft. spans, in both steel and aluminum.

Résumé

Les ossatures tri-dimensionnelles sont des structures dont le degré d'hyperstaticité est élevé et qui peuvent être calculées par assimilation à des systèmes discrets ou à des systèmes continus. Dans cette contribution, on établit, par référence à un système continu, une méthode de calcul des ossatures tridimensionnelles à deux surfaces, eu égard plus particulièrement aux structures en voile. On détermine les constantes élastiques d'un continuum homogène anisotrope qui a les mêmes caractéristiques de déformation que l'ossature, et l'on donne les transformées à l'aide desquelles on calculera les efforts dans les éléments de l'ossature à partir des résultantes des efforts dans le système continu. On discute le problème du flambement des ossatures tri-dimensionnelles et, à titre d'exemple, on présente le calcul de dômes de 120 m, 240 m et 480 m de portée réalisés en acier et en aluminium.

Zusammenfassung

Dreidimensionale Raumfachwerke sind hochgradig statisch unbestimmte Tragwerke, die entweder als System diskreter Elemente oder als ein Kontinuum behandelt werden können. In diesem Beitrag wird eine Kontinuumsmethode für doppelschalige Raumfachwerke unter besonderer Berücksichtigung von Schalentragwerken aufgestellt. Die elastischen Konstanten eines anisotropen homogenen Kontinuums, das die gleichen Verformungseigenschaften wie das Raumfachwerk aufweist, werden bestimmt sowie die Transformationsgleichungen angegeben, über welche die Kräfte in den Tragwerkselementen aus den resultierenden Spannungen für das Kontinuum berechnet werden können. Anschließend werden Stabilitätsprobleme räumlicher Tragwerke besprochen. Als numerische Beispiele werden die Entwürfe für Schalen aus Stahl und Aluminium mit Spannweiten von 120 m, 240 m und 480 m angegeben.