Computer analysis of grillages curved in plan

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Computer Analysis of Grillages Curved in Plan

Calcul par ordinateur des réseaux de poutres courbes dans le plan

Berechnung gekrümmter Trägerroste mittels Elektronenrechner

F. SAWKO
Professor of Civil Engineering
The University of Liverpool

List of Symbols

E Young's modulus.
μ Poisson's ratio.
G Rigidity modulus $= \frac{E}{2(1+\mu)}$.
I Flexural constant.
J Torsional constant.
$\alpha = \frac{EI}{GJ}$ Ratio of flexural to torsional stiffness of a member.
w Uniformly distributed load.
P Concentrated point load.
r Radius of curved member.
θ Angle subtended by curved member.
ϕ Angle defining position of point on curved member.
M Bending moment.
T Torsion moment.
Q Shearing force.
b, d Width and depth of rectangular member.
t Wall thickness of box member.
A Area of cell of hollow section.
a Depth to breadth ratio $= \frac{b}{d}$.
M_x Moment in x direction.
M_y Moment in y direction.
F_z Vertical force.

f_{11}, f_{12} etc. Flexibility factors for curved member.
s_{11}, s_{12} etc. Stiffness factors at end of 1 of curved member.
t_{11}, t_{12} etc. Stiffness factors at end of 2 of curved member.
r_{11}, r_{12} etc. Restraint factors for curved member.
γ Angle of inclination at end of 1 of curved member with respect to frame axes.
c $\cos\gamma$.
s $\sin\gamma$.
c' $\cos(\gamma+\theta)$ } cosine and sine of angle of inclination of end 2 of
s' $\sin(\gamma+\theta)$ } curved member with respect to frame axes.
F Flexibility matrix of a member.
S Stiffness matrix of member.
f Numerical factor from graphs.
k Multiplying factor for torsional constant.

Introduction

The problem of grillage analysis has always presented design engineers with considerable analytical difficulties. This is due to the high statical indeterminancy that these structures invariably possess. For a torsionless system this would be approximately one for every internal node of the grid, and analytical techniques using the "interacting force" principle have for a long time been the only solution. The success of the analysis has largely depended on the skill of the operator in setting up the simultaneous equations without arithmetic errors, and utilising lines of symmetry and skew symmetry wherever possible to reduce their number.

A torsionally stiff system, presented an even great obstacle for the analysis. The interacting force method had to be abandoned and attempts were made at a moment distribution analysis of grillages, using unit normal deformations at joints with a series of propping forces. With this device the number of equations was reduced to the set equal to that in the torsionless case at the expense of considerable effort in hand computing.

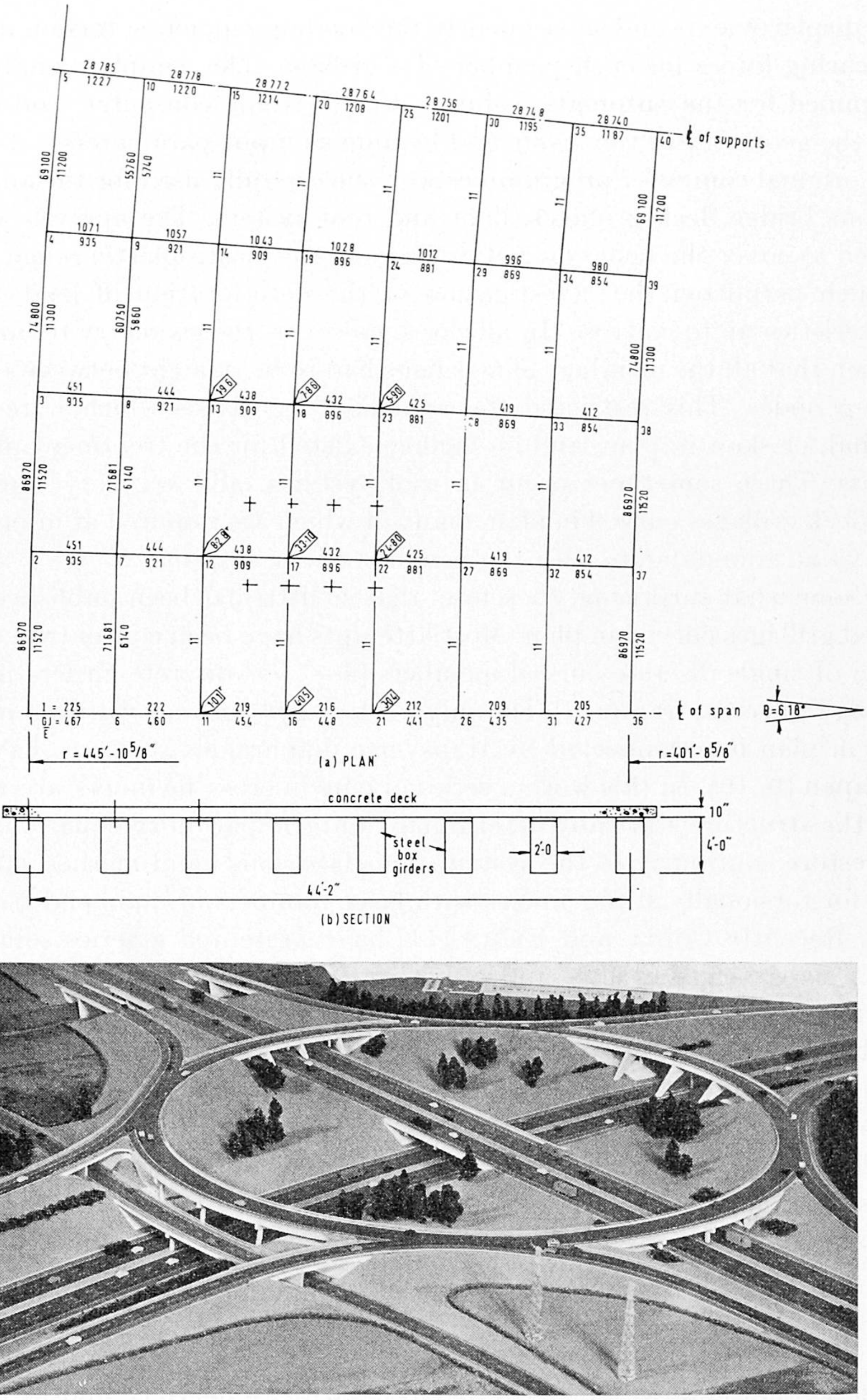
The advent of electronic computers ensured not only an easy and accurate solution of the simultaneous equations, but enabled another great step forward. Thus following the example of plane frame analysis LIGHTFOOT and SAWKO [1, 2] demonstrated that an automatic setting up of slope deflection equations could be achieved for any grillage irrespective of geometric arrangement of members. Equations expressed the interrelation between nodal forces and the corresponding displacements which consisted of two rotations about a mutually perpendicular arbitrary set of in-plane axes and a transverse displacement. Thus three equations were automatically set up at each node of the grillage for the torsionless and torsionally stiff cases and were solved to produce first the un-known displacements and subsequently the bending moments, torsion moments and shearing forces for each member of a grillage. The complete analysis was programmed for the automatic solution by electronic computer, and only required the geometry of the frame and loading as input parameters.

The original computer programmes were successfully used for the analysis of numerous bridge deck grillages, floor and roof system. The analysis was also extended to cover the behaviour of grillages in the elasto-plastic range (SAWKO [3]) which permitted the investigation of the deterioration of load-deflected characteristics up to collapse. In all cases, however, the geometry requirements were such that all the members of a grillage had to be straight between their terminating nodes. This restricted the analysis to grillages which were square, polygonal, or skew in plan, and no facility existed for the treatment of curved members. These sometimes occur in roof systems, and are very common in bridge deck grillages curved in plan, many of which are required at interchanges and elevated roundabouts with the new motorway systems.

It is somewhat surprising, therefore, that so little has been published on the analysed grillages curved in plan. Most attempts have been concentrated on the analysis of single discrete curved members [4—7] or discrete girders interconnected by rigid cross frames [8]. The only work on grillages consisting of members curved in plan interconnected by transverse diaphragms seems to have come from Japan [9, 10]. In this work a series of cuts in cross members are made to render the structure statically determinate and compatibility equations are set up to restore continuity to the system. This laborious hand method is impracticable for torsionally stiff grillages with large numbers of main end transverse girders. Recently COULL and ERGIN [11] have presented a series solution for isotropic slabs curved in plan, and demonstrated a sufficient degree of accuracy of the method compared with perspex models. Their work is interesting but somewhat limited, since an isotropic slab is not the most efficient medium for resisting the high torsion moments set up in curved structures.

It is clearly possible to analyse a grillage curved in plan by a number of equivalent straight sections. This, somewhat dangerous procedure, as will be demonstrated in the following paragraphs, is sometimes used indiscriminately by consulting engineers seeking rapid answers to their immediate problems.

The analysis of the curved grillage was attemped by the author for the West Riding County Council (Yorkshire) for the curved decks of Lofthouse Interchange at the intersection of M 1 and M 62 motorways (Fig. 1). This consisted of a number of prefabricated steel box girders with an insitu composite concrete slab, which also acted as the transverse distributing medium in the bridge. Analysis was required for the abnormal vehicle loading in the central and eccentric positions for maximum transverse moments in the concrete slab and bending and twisting moments in the curved longitudinal members respectively. The analysis based on the straight segmental approximation corresponding to the joint numbering of Fig. 1 gave very reliable answers for moments in the longi-



(c)

Fig. 1. Plan and section of Lofthouse Interchange South Roundabout Deck.

tudinal girders. Transverse moments in slab, however, were most difficult to interpret, since the resolved components of the longitudinal moments in the transverse direction at the centre of the bridge were of the same order as transverse moments printed out by the computer. The deck was re-analysed by doubling the number of segments, but this merely halved the included angle, and hence the resolved component, whilse also halving the transverse moments. The dilemma was thus unresolved, and the designers adopted conservative values for design purposes.

The above example demonstrated to the author the importance of investigating the degree of approximation involved in simulating members curved in plan by a series of equivalent straight members, to decide under what conditions this treatment was reasonable.

Approximation in Curved Member Simulation

Fig. 15 shows a straight uniform member of the same section properties as the curved member with subtended angle. It was desirable to obtain a graphical

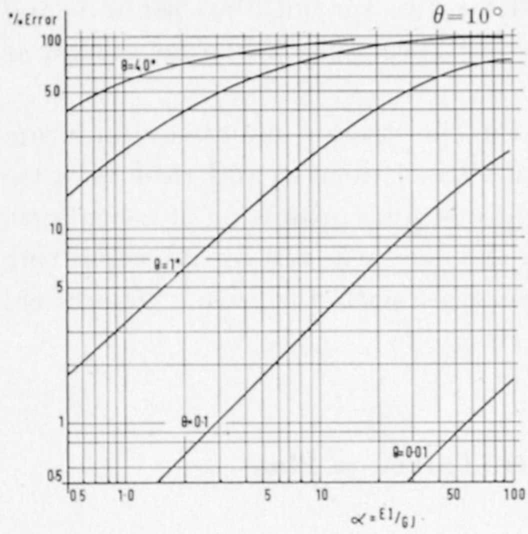


Fig. 2. Simulation of curved member by an equivalent straight uniform member. Percentage error in bending stiffness.

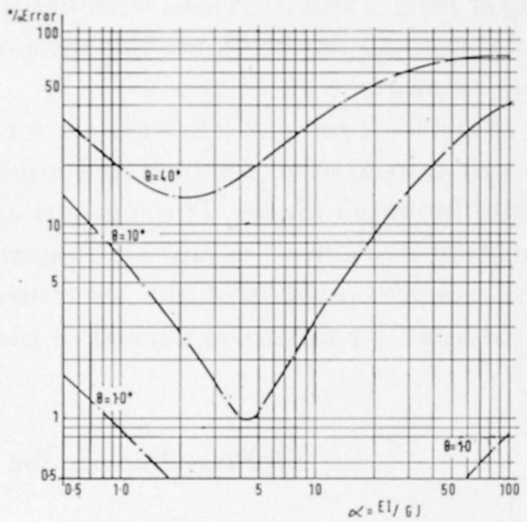


Fig. 3. Simulation of curved member by an equivalent straight uniform member. Percentage error in torsion stiffness.

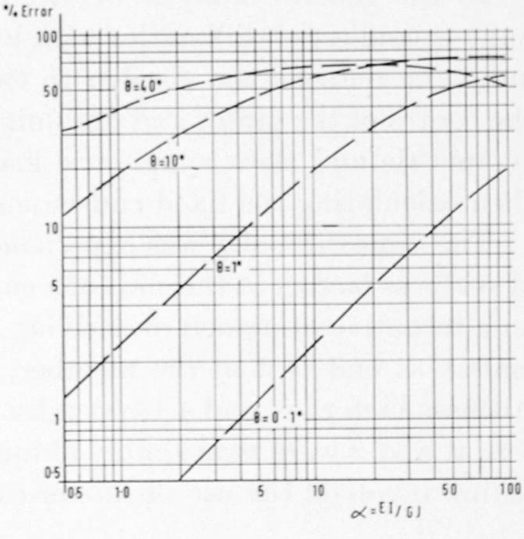


Fig. 4. Simulation of curved member by an equivalent straight uniform member. Percentage error in shear stiffness.

representation of the degree of error involved in the bending stiffness, torsion stiffness and shearing stiffness at end [1] of the curved member. The percentage errors, given by:

$$\% \text{ error} = \frac{\begin{matrix}\text{actual stiffness of} \\ \text{curved member}\end{matrix} - \begin{matrix}\text{stiffness of equivalent} \\ \text{straight member}\end{matrix}}{\text{actual stiffness of curved member}} \times 100\%$$

are plotted in Fig. 2—4. Stiffness values of curved members, which will be discussed in the following section, are obtained in Appendix 1, and the corresponding values for the equivalent straight member, in the same directions, in Appendix 2. The evaluation of these errors over the whole range of values of the included angle and factors was achieved by a short program written in ALGOL 60 for the Leeds University English Electric Leo-Marconi KDF 9 computer. Appendix 5 shows that the range of values of α is approximately 0.6 at the lower range, and increases with depth to width ratios fpr both box sections and solid rectangular sections. For torsionally weak (I) sections it can be as high as 100 or more.

Figs. 2—4 indicate that the error involved in the straight member simulation is substantial even for included angles as small as 1 degree, and rises very rapidly for larger angles. The simulation, therefore is not a promising practical proposition if accurate values are required, and the need was felt for incorporating the exact geometry of the structure for analysis with the exact thoeretical treatment for members curved in plan.

Stiffness Factors for Members Curved in Plan

Attention will be confined to the analysis of grillages with members of a constant radius of curvature which are by far the most common in practice, although the proposed treatment is by no means restricted to these cases.

In the general analysis of circular bow girders, PIPPARD and BAKER [4] have encountered difficulties with a direct solution involving three unknowns, and have reduced the number to two, by considering the redundant forces in the centre of the girder and obtaining the solution in two stages by considering symmetric and skew symmetric loading to determine the central forces, and then calculating the fixed end moments and shears.

The same difficulty was experienced in an attempt at a direct determination of stiffness factors at the ends of a curved member. Thus the author has adopted an alternative approach of deriving the influence coefficients for the flexibility matrix at end of 1 of the member, and inverting this to obtain the required stiffness factors at end 1 (Appendix 1). The restraint factors, and stiffness factors at and 2 were then simply obtained from consideration of statics. The procedure involved the use of numerical values of θ and α to form the flexibility coefficients, and the inversion of a 3×3 matrix, and both operations were carried out inside the computer to yield numerical values of stiffness factors.

Fixed End Forces and Moments

The additional difficulty with curved members is the fact that fixed end forces and moments cannot be obtained in a simple way as for a straight member under any loading. Appendix 3 gives PIPPARD and BAKER's solution for a uniformly distributed load, and this is plotted graphically in Figs. 5 and 6 for

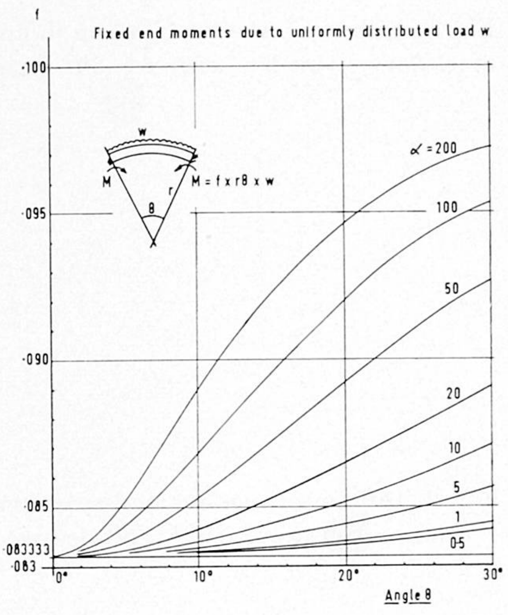


Fig. 5. Values of fixed end moments in a curved girder under uniformly distributed load.

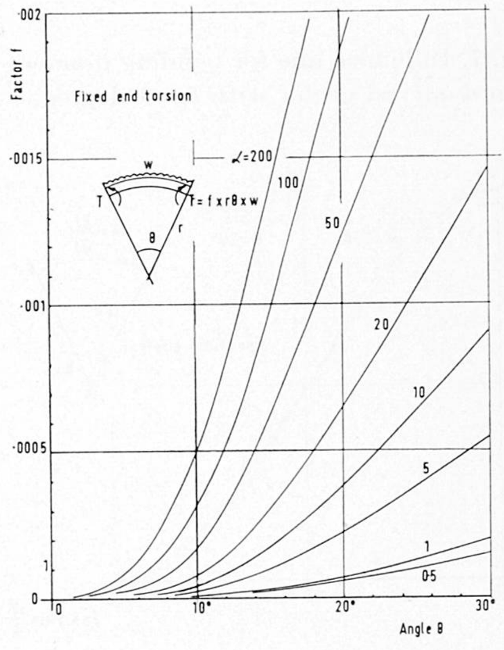


Fig. 6. Values of fixed end torsions in a curved girder under uniformly distributed load.

quick reference. Treatment of point loads presents even more difficulty not so much in the analysis (Appendix 4) as in the presentation of data in the form of graphs or tables. In addition to the two variables θ and α, the position of the point load P varies along the beam. A full range of values has been obtained by the author, but their presentation would require considerable space. Some typical results are presented in the form of influence lines in Figs. 7—9 and these are sufficiently representative of the whole range. As would be expected, large values of θ affect moments considerably over the range of α. Shearing forces (Fig. 9) are relatively insensitive to these charges. For small angles it is sufficiently accurate to induce only the effects of shearing forces and fixed end moments based on a beam of equivalent length, and to neglect torsion effects. This is unlikely to lead to excessive errors if the curve is flat. When θ is large the actual values can be calculated from the formulae in Appendix 4.

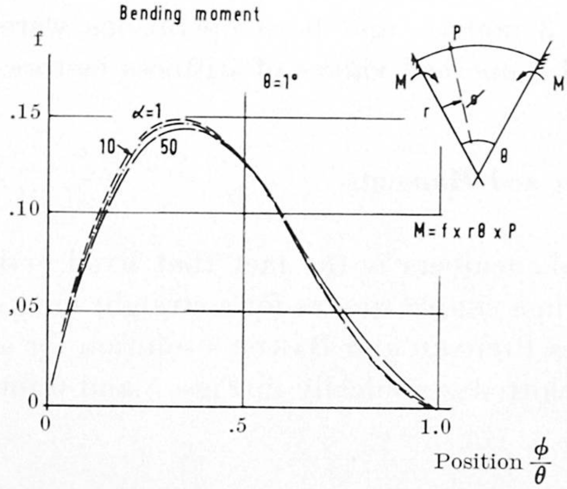


Fig. 7. Influence line for bending moment in a curved girder with $\theta = 1$ degree.

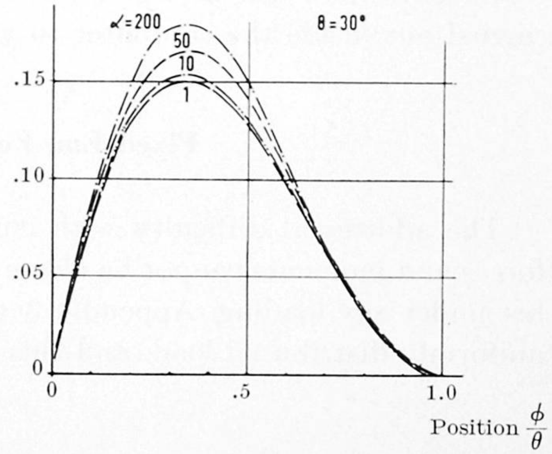


Fig. 8. Influence line for bending moment in a curved girder with $\theta = 30$ degrees.

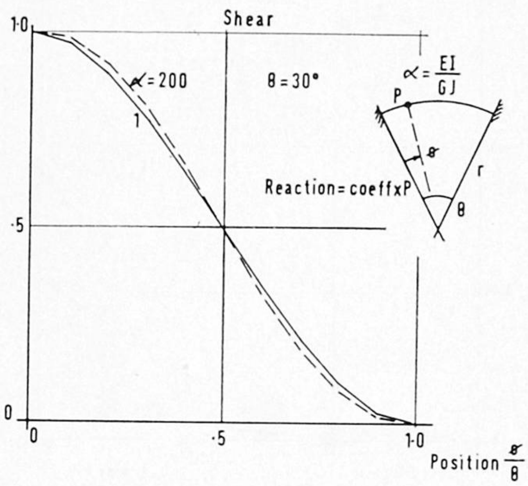


Fig. 9. Influence lines for end reaction in a curved girder with $\theta = 30$ degrees.

The Computer Programme

The programme for the analysis of grillages curved in plan has been written in ALGOL 60 for the KDF 9 computer at Leeds University Electronic Computer Laboratory.

Formulation of the elastic analysis for an automatic solution by the electronic computer is not new, and attention will be confined only to features of interest.

The data for input consists of the properties of straight and curved members, restraint conditions and loading. Curved members are specified by defining joint numbers that they connect, radius r, included angle θ, bending and torsion constants I and J, whether the curvature is positive or negative (Fig. 13) and the angle of inclination at end 1 of the member. Members with the same section properties are specified in groups, and it is not necessary to repeat the data for each member separately.

The computer sets up the stiffness factors for the curved members with respect to the local member axes, and transforms these, using different transformation matrices at ends 1 and 2, to the arbitrary set of axes specified for the grillage (Appendix 1). The stiffness matrix is then assembled in the usual way, only one half of the non-zero band width being stored to reduce storage space and subsequent solution time. The solution of equations is based on T. BANACHIEWICZ [11] square root method, which proved to be very fast and accurate even for large sets of equations. The deflections are then used to calculate, for each curved member, the bending and torsion moments at the two ends and the shearing forces. For straight members only one value of the torsion is given.

Numerical Example

The bridge deck of the Lofthouse Interchange was re-analysed once the programme had been developed, and the results are plotted in Figs. 10—12. Transverse members were formed by the reinforced concrete slab itself, which is wedge-shaped in plan. A uniform member was assumed in its place, since the discrepancies were very small. Account was taken of the reduction of stiffness due to cracking of concrete over the central portion of the bridge.

The wheel loading of the abnormal vehicle was represented by equivalent

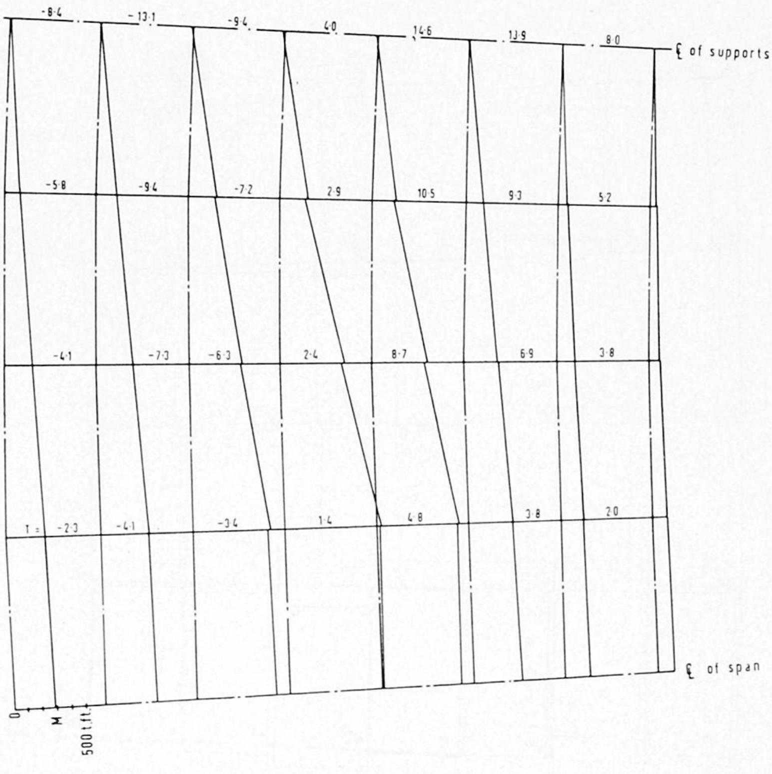


Fig. 10. Lofthouse Interchange — Distribution of moments in longitudinals and torsion in transversals.

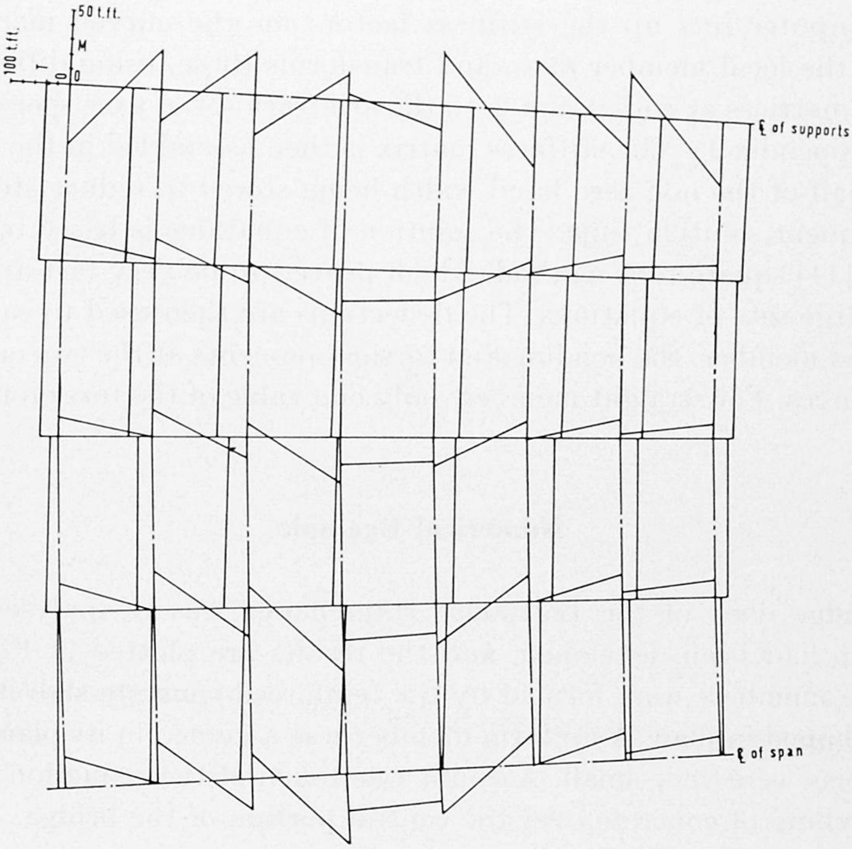


Fig. 11. Lofthouse Interchange — Distribution of torsions in longitudinals and moments in transversals.

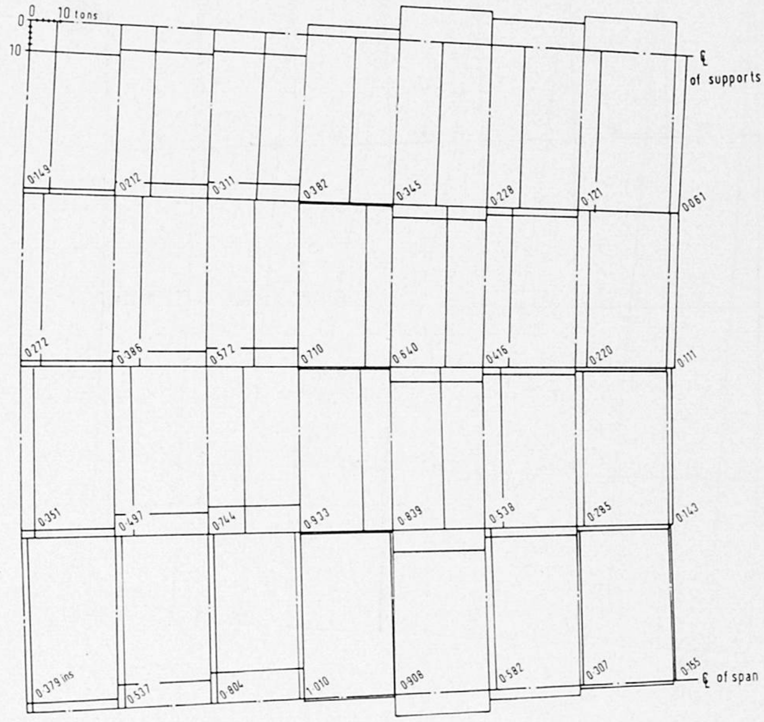


Fig. 12. Lofthouse Interchange — Distribution of shears and deflections.

point loads at the nodes of the curved grid. This did not lead to large errors in longitudinal bending and torsion moments, and local wheel effects were superimposed on the computer solution in design.

The author believes, that the results so obtained are an accurate representation of the actual stress conditions occurring in the bridge under the central vehicle loading. He also feels that the analysis would be difficult, if not impossible, by any other known method.

With the development of the computer programme for grillages curved in plan, another class of structure has been made amenable to a fast and reliable solution.

Acknowledgements

Bridge decks of the Lofthouse Interchange were designed by the Highways and Bridges Section, County Council of West Riding (Yorkshire), S. Maynard Lovell, County Engineer and Surveyor, who also supplied the photograph of the model.

The analysis of the decks was performed on the English Electric Leo Marconi KDF 9 computer operating at the University of Leeds Electronic Computing Laboratory.

Appendix 1

Stiffness and Restrain Factors for Member Curved in Plan

With reference to Fig. 13, end 1 of curved member 1—2 is released and unit values of forces M_{x1}, M_{y1}, F_{z1} are applied in turn. The flexibility factors for

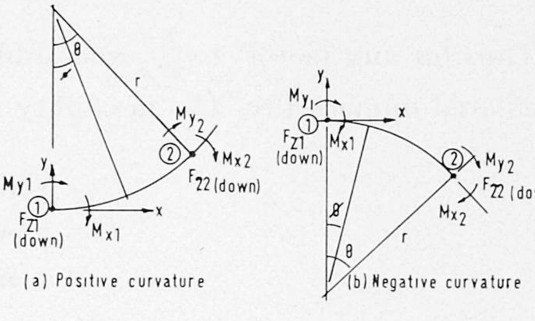


Fig. 13. Member curved in plan with definition of positive (a) and negative (b) curvatures.

member are given below, where the upper sign refers to positive, lower to negative curvature of the curved member as defined in Fig. 13.

$$f_{11} = \frac{r}{E\,I}\int_0^\theta \sin^2\phi\,d\phi + \frac{r}{G\,J}\int_0^\theta \cos^2\phi\,d\phi$$
$$= \frac{r}{E\,I}\left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\,\theta\right) + \frac{r}{G\,J}\left(\frac{1}{2}\theta + \frac{1}{4}\sin 2\,\theta\right), \qquad (1)$$

$$f_{22} = \frac{r}{EI}\int_0^\theta \cos^2\phi\,d\phi + \frac{r}{GJ}\int_0^\theta \sin^2\phi\,d\phi$$

$$= \frac{r}{EI}\left(\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta\right)+\frac{r}{GJ}\left(\frac{1}{2}\theta-\frac{1}{4}\sin 2\theta\right),$$

$$f_{33} = \frac{r^3}{EI}\int_0^\theta \sin^2\phi\,d\phi + \frac{r^3}{GJ}\int_0^\theta (1-\cos\phi^2)\,d\phi$$

$$= \frac{r^3}{EI}\left(\frac{1}{2}\theta-\frac{1}{4}\sin 2\theta\right)+\frac{r^3}{GJ}\left(\frac{3}{2}\theta-2\sin\theta+\frac{1}{4}\sin 2\theta\right),$$

$$f_{12} = \mp\frac{r}{EI}\int_0^\theta \sin\phi\cos\phi\,d\phi \pm \frac{r}{GJ}\int_0^\theta \sin\phi\cos\phi\,d\phi$$

$$= \mp\frac{r}{EI}\left(\frac{1}{2}\sin^2\theta\right)\pm\frac{r}{GJ}\left(\frac{1}{2}\sin^2\theta\right), \tag{1}$$

$$f_{13} = \pm\frac{r^2}{EI}\int_0^\theta \sin^2\phi\,d\phi \mp \frac{r^2}{GJ}\int_0^\theta \cos\phi\,(1-\cos\phi)\,d\phi$$

$$= \pm\frac{r^2}{EI}\left(\frac{1}{2}\theta-\frac{1}{4}\sin 2\theta\right)\mp\frac{r^2}{GJ}\left(-\frac{1}{2}\theta+\sin\theta-\frac{1}{4}\sin 2\theta\right),$$

$$f_{23} = -\frac{r}{EI}\int_0^\theta \sin\phi\cos\phi\,d\phi - \frac{r}{GJ}\int_0^\theta \sin\phi\,(1-\cos\phi)\,d\phi$$

$$= -\frac{r}{EI}\left(\frac{1}{2}\sin^2\theta\right)-\frac{r}{GJ}\left(1-\cos\theta-\frac{1}{2}\sin^2\theta\right).$$

Thus for any factor $\alpha=\frac{EI}{GJ}$ and subtended angle θ flexibility factors can be calculated numerically. The flexibility matrix for the member

$$F = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}$$

is then inverted, giving numerically the stiffness matrix at end 1 of the member:

$$S = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix}.$$

The basic force displacement relationship for the curved member can then be written as:

$$\begin{bmatrix} M_{x1} \\ M_{y1} \\ F_{z1} \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix},$$
$$\begin{bmatrix} M_{x2} \\ M_{y2} \\ F_{z2} \end{bmatrix} = \begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}, \tag{2}$$

or more briefly

$$F_{12} = S_{12} D_1 + R_{12} D_2,$$
$$F_{21} = R_{21} D_1 + S_{21} D_2, \tag{3}$$

where S_{ij}, R_{ij} are the stiffness and restraint matrices respectively and $F_{1,2}$ and $D_{1,2}$ the forces and corresponding displacements at ends of 1, 2 of the curved member.

The restraint factors r_{ij} are obtained from equilibrium consideration, by applying unit displacements in turn to end 1 of the member.

Thus for $\theta x_1 = 1$ only, all other displacements equal to zero,

$$M_{x1} = s_{11}, \qquad M_{x2} = r_{11},$$
$$M_{y1} = s_{12}, \qquad M_{y2} = r_{12},$$
$$F_{z1} = s_{13}, \qquad F_{22} = r_{13},$$

and from equilibrium:

$$M_x^{+}\circlearrowleft \quad M_{x2} + M_{x1}\cos\theta \pm M_{y1}\sin\theta \mp F_z r(1-\cos\theta) = 0,$$
$$\therefore\ r_{11} = -s_{11}\cos\theta \mp s_{12}\sin\theta \pm s_{13} r(1-\cos\theta),$$
$$M_y^{+}\circlearrowleft \quad M_{y2} - M_{x1}\sin\theta + M_{y1}\cos\theta - F_z r\sin\theta = 0,$$
$$\therefore\ r_{12} = \pm s_{11}\sin\theta - s_{12}\cos\theta + s_{13} r\sin\theta,$$
$$F_z^{+}\downarrow \quad F_{z2} + F_{z1} = 0,$$
$$\therefore\ r_{13} = -s_{13}.$$

Hence the restraint matrix becomes:

$$R_{12} = \begin{bmatrix} s_{11} c \mp s_{12} s + s_{13} r(1-c) & \pm s_{11} s - s_{12} c + s_{13} r s & s_{13} \\ s_{21} c \mp s_{22} s \pm s_{23} r(1-c) & \pm s_{21} s - s_{22} c + s_{23} r s & s_{23} \\ s_{31} c \mp s_{32} s \pm s_{33} r(1-c) & \pm s_{31} s - s_{32} c + s_{33} r s & s_{33} \end{bmatrix}. \tag{4}$$

Stiffness factors at end 2 of the curved member are similar to the corresponding factors at end 1, the only difference being those of sign in the off diagonal terms. It is easy to show that:

$$S_{21} = \begin{bmatrix} s_{11} & -s_{12} & s_{13} \\ -s_{21} & s_{22} & -s_{23} \\ s_{31} & -s_{32} & s_{33} \end{bmatrix}. \tag{5}$$

In assembling the stiffness matrix of the complete structure, equation (2) has to be transformed to a general system of axes x', y', z' (Fig. 14). Thus every curved member will in general be inclined at an angle γ at end 1 to the positive direction of the x' axis. The inclination at end 2 will be $(\gamma+\theta)$, due account being taken of the curvature of member.

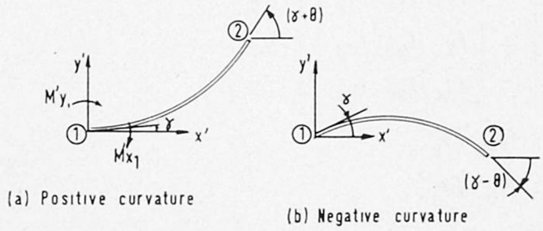


Fig. 14. Angles of inclination of the ends of curved member related to general system of axes. Positive (a) and (b) negative curvatures.

From static and geometric consideration, the force displacement relationships for the curved member referred to the frame system of axes x', y', z' are:

$$\begin{aligned} F_1' &= (T_1^{-1} S_{12} T_1) D_1' + (T_1^{-1} R_{12} T_2) D_2', \\ F_2' &= (T_2^{-1} R_{21} T_1) D_1' + (T_2^{-1} S_{21} T_2) D_2', \end{aligned} \tag{6}$$

where $T_1 = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $T_2 = \begin{bmatrix} \cos(\gamma+\theta) & \sin(\gamma+\theta) & 0 \\ -\sin(\gamma+\theta) & \cos(\gamma+\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

The final equation can be written in the form:

$$\begin{aligned} F_1' &= S_{12}' D_1' + R_{12}' D_2', \\ F_2' &= R_{21}' D_1' + S_{21}' D_2', \end{aligned} \tag{7}$$

where $S_{12}' = \begin{bmatrix} c^2 s_{11} - 2 s c s_{12} + s^2 s_{22} & s c s_{11} + (c^2 - s^2) s_{12} - s c s_{22} & c s_{13} - s s_{23} \\ & s^2 s_{11} + 2 s c s_{12} + c^2 s_{22} & s s_{13} + c s_{23} \\ & & s_{33} \end{bmatrix}$,

$S_{21}' =$

$$\begin{bmatrix} (c')^2 s_{11} + 2 s' c' s_{12} + (s')^2 s_{22} & s' c' s_{11} + [(s')^2 - (c')^2] s_{12} - s' c' s_{22} & c' s_{13} + s' s_{23} \\ & (s')^2 s_{11} - 2 s' c' s_{12} + (c')^2 s_{22} & s' s_{13} - c' s_{23} \\ & & s_{33} \end{bmatrix},$$

$R_{12}' =$

$$\begin{bmatrix} c c' r_{11} - s c' r_{21} - c s' r_{12} + s s' r_{22} & c s' r_{11} - s s' r_{21} + c c' r_{12} - s c' r_{22} & c r_{13} - s r_{23} \\ s c' r_{11} + c c' r_{21} - s s' r_{12} - c c' r_{22} & s s' r_{11} + c s' r_{21} + s c' r_{12} + c c' r_{22} & s r_{13} + c r_{23} \\ c' r_{31} - s' r_{32} & s' r_{31} + c' r_{32} & r_{33} \end{bmatrix}.$$

The coefficients of the four 3×3 matrices are symmetric about the leading diagonal, and c, s stand for $\cos\gamma$, $\sin\gamma$, and c', s' for $\cos(\gamma+\theta)$, $\sin(\gamma+\theta)$ respectively.

In this form the equations are suitable for an automatic setting up process and solution inside the computer, yielding at each joint three displacement D' in terms of the reference axes of the frame. Member forces are then determined from the following equations:

$$F_1 = S_{12}\,T_1\,D_1' + R_{12}\,T_2\,D_2',$$
$$F_2 = R_{21}\,T_1\,D_1' + S_{21}\,T_2\,D_2', \tag{8}$$

where
$$S_{12}\,T_1 = \begin{bmatrix} c\,s_{11} - s\,s_{12} & s\,s_{11} + c\,s_{12} & s_{13} \\ c\,s_{21} - s\,s_{22} & s\,s_{21} + c\,s_{22} & s_{23} \\ c\,s_{31} - s\,s_{32} & s\,s_{31} + c\,s_{32} & s_{33} \end{bmatrix},$$

and
$$R_{12}\,T_2 = \begin{bmatrix} c\,r_{11} - s\,r_{12} & s\,r_{11} + c\,r_{12} & r_{13} \\ c\,r_{21} - s\,r_{22} & s\,r_{21} + c\,r_{22} & r_{23} \\ c\,r_{31} - s\,r_{32} & s\,r_{31} + c\,r_{32} & r_{33} \end{bmatrix}.$$

It is usually unnecessary to calculate the corresponding matrices for forces at end 2 of the member, since these are obtained far more simply from statics. Thus with reference to Fig. 14 it follows immediately that

$$M_{x2} = -M_{x1}\cos\theta \mp M_{y1}\sin\theta \pm F_{21}\,r\,(1-\cos\theta),$$
$$M_{y2} = \pm M_{x1}\sin\theta - M_{y1}\cos\theta + F_{21}\,r\sin\theta. \tag{9}$$

The force system for every curved member is thus defined by five values — bending and twisting moments at ends 1 and 2 of the member and shearing force.

Appendix 2

Stiffnes Factors for an Equivalent Straight Member

With reference to Fig. 15, stiffness factors for a straight member connecting ends 1 and 2 of member curved in plan can be determined by any of a number of standard procedures. A flexibility approach parallel to the one adopted in Appendix 1 is outlined here, since it makes the procedure easy to follow.

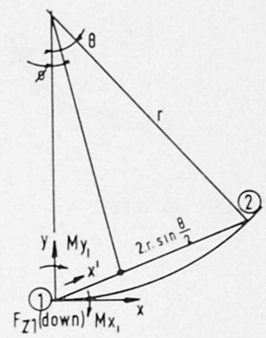


Fig. 15. Simulation of curved member by an equivalent straight member.

End 1 of the member is released, and unit values of forces M_x, M_y, F_{z1} are applied in turn. It is most convenient to use x', the distance along the member, as a variable, rather than positional angle ϕ.

Thus:

$$f_{11} = \frac{1}{EI}\int_0^{2r\sin(\theta/2)} \sin\frac{\theta}{2}\,dx' + \frac{1}{GJ}\int_0^{2r\sin(\theta/2)} \cos\frac{\theta}{2}\,dx' = \frac{r}{EI}\,2\sin^2\frac{\theta}{2} + \frac{r}{GJ}\sin\theta\,,$$

$$f_{22} = \frac{1}{EI}\int_0^{2r\sin(\theta/2)} \cos\frac{\theta}{2}\,dx' + \frac{1}{GJ}\int_0^{2r\sin(\theta/2)} \sin\frac{\theta}{2}\,dx' = \frac{r}{EI}\sin\theta + \frac{r}{GJ}\,2\sin^2\frac{\theta}{2},$$

$$f_{33} = \frac{1}{EI}\int_0^{2r\sin(\theta/2)} (x')^2\,dx' = \frac{r^3}{EI}\,\frac{8}{3}\sin^3\frac{\theta}{2}, \tag{10}$$

$$f_{12} = \mp\frac{r}{EI}\sin\theta\sin\frac{\theta}{2} \pm \frac{r}{GJ}\sin\theta\sin\frac{\theta}{2},$$

$$f_{13} = \pm\frac{r^2}{EI}\,2\sin^3\frac{\theta}{2},$$

$$f_{23} = -\frac{r^2}{EI}\sin\theta\sin\frac{\theta}{2},$$

The inverse of the flexibility matrix for any values of r, α and θ gives numerically the corresponding stiffness factors at end 1 of the member.

Appendix 3

Uniformly Distributed Load over Girder Curved in Plan

The circular arc bow girder has been solved by PIPPARD and BAKER [4] (p. 413), and their notation is modified to comply with that used previously in Fig. 16. Values of moment, torsion and shearing force at any point defined by

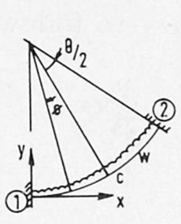


Fig. 16. Uniformly distributed load on curved girder.

angle ϕ are:

$$M = 2wr^2\left[\frac{4\cos\phi\left\{(\alpha+1)\sin\frac{\theta}{2} - \alpha\frac{\theta}{2}\cos\frac{\theta}{2}\right\}}{(\alpha+1)\,\theta - (\alpha-1)\sin\theta} - 1\right],$$

$$T = 2wr^2\left[\frac{4\sin\phi\left\{(\alpha+1)\sin\frac{\theta}{2} - \alpha\frac{\theta}{2}\cos\frac{\theta}{2}\right\}}{(\alpha+1)\,\theta - (\alpha-1)\sin\theta} - \phi\right], \tag{11}$$

$$Q = 2wr\phi\,.$$

Thus end moments at $\phi=\theta/2$ (i. e. end 1) become:

$$\begin{aligned} M_y &= 2\,w\,r^2\left[\frac{4\cos\frac{\theta}{2}\left\{(\alpha+1)\sin\frac{\theta}{2}-r\frac{\theta}{2}\cos\frac{\theta}{2}\right\}}{(\alpha+1)-(\alpha-1)\sin\theta}-1\right], \\ M_x &= 2\,w\,r^2\left[\frac{4\sin\frac{\theta}{2}\left\{(\alpha+1)\sin\frac{\theta}{2}-r\frac{\theta}{2}\cos\frac{\theta}{2}\right\}}{(\alpha+1)-(\alpha-1)\sin\theta}-\frac{\theta}{2}\right], \\ F_z &= 2\,w\,r\frac{\theta}{2}. \end{aligned} \tag{12}$$

Values of M_x and M_y are plotted in Figs. 5 and 6 respectively.

Appendix 4

Point Load on Girder Curved in Plan

Pippard and Baker [4] give the solution to a curved girder under a single point load in two steps. Values at the centre of the girder are first calculated,

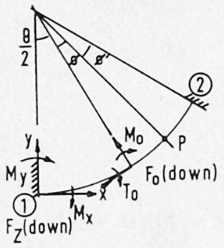


Fig. 17. Point load on a curved girder.

and resolved for end moments torsions and shears. Thus, with the notation of Fig. 17 the central values are:

$$\begin{aligned} M_0 &= P\,r\frac{(\alpha+1)\,\phi'\sin\phi+(\alpha-1)\sin\frac{\theta}{2}\sin\phi'-2\,\alpha\left(\cos\phi-\cos\frac{\theta}{2}\right)}{(\alpha+1)\,\theta-(\alpha-1)\sin\theta}, \\ F_0 &= \\ & P\,r\left[\frac{1}{2}+\frac{2\,(\alpha+1)\left(\phi'\cos\phi\sin\frac{\theta}{2}-\frac{\theta}{2}\sin\phi'-\phi\frac{\theta}{2}\right)+4\,\alpha\sin\phi\sin\frac{\theta}{2}-(\alpha-1)\,\phi\sin\theta}{(\alpha+1)\,\theta^2+(\alpha-1)\,\theta\sin\theta-4\,\alpha\,(1-\cos\theta)}\right], \\ T_0 &= P\,r\frac{1}{2}\left[\phi'-\sin\phi'-\frac{F_0}{P}\left(\theta-2\sin\frac{\theta}{2}\right)\right]\operatorname{cosec}\frac{\theta}{2}. \end{aligned} \tag{13}$$

The end moments and force at end (1) become:

$$\begin{aligned} M_y &= M_0\cos\frac{\theta}{2}+T_0\sin\frac{\theta}{2}+F_0\,r\sin\frac{\theta}{2}, \\ M_x &= -M_0\sin\frac{\theta}{2}+T_0\cos\frac{\theta}{2}-F_0\,r\left(1-\cos\frac{\theta}{2}\right), \\ F_z &= F_0 \end{aligned} \tag{14}$$

and at end (2):

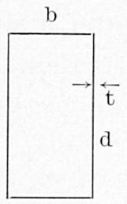
$$
\begin{aligned}
M_y &= M_0 \cos\frac{\theta}{2} - T_0 \sin\frac{\theta}{2} - F_0\, r \sin\frac{\theta}{2} - P\, r \sin\phi', \\
M_x &= M_0 \sin\frac{\theta}{2} + T_0 \cos\frac{\theta}{2} - F_0\, r \left(1 - \cos\frac{\theta}{2}\right) - P\, r\,(1 - \cos\phi'), \\
F_z &= F_0 - P.
\end{aligned}
\tag{15}
$$

Appendix 5

Flexural and Torsional Properties for Hollow and Solid Sections

Hollow Box Sections

For fabricated steel boxes, wall thickness t is very small compared to dimensions of section



Thus:
$$I = \left(2\frac{d^3}{12} + 2\frac{b\,d^2}{4}\right) t + d^2\, t \left(\frac{d}{6} + \frac{b}{2}\right),$$

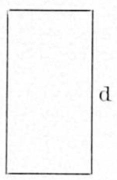
$$J = \frac{4\,A^2\, t}{\text{perimeter}} = \frac{2\,b^2 d^2\, t}{(b+d)}.$$

If $d/b = a$, and μ for steel $= 0.3$,

$$= \frac{E\,I}{G\,J} = 2.6\,\frac{\left(\frac{a}{12} + \frac{1}{4}\right) a^2}{\frac{a^2}{1+a}} = 0.217\,(a+1)\,(a+3).$$

a	α
0	0.65
0.5	1.14
1.0	1.74
3.0	5.21
5.0	10.42

Solid Sections



$$I = \frac{b\,d^3}{12}, \qquad J = k\,d b^3 \text{ for } a > 1, \qquad k\,b\,d^3 \text{ for } a < 1.$$

Thus for μ concrete $= 0.15$,

$$= \frac{E\,I}{G\,J} = 0.192\left(\frac{d}{b}\right)^2 \frac{1}{k} \quad \text{for } a > 1,$$

$$= 0.192\,\frac{1}{k} \qquad \text{for } a < 1.$$

Values of k are given by TIMOSHENKO [12]

a	α
0	0.58
0.5	0.84
1.0	1.36
3.0	6.58
5.0	16.50

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Summary

The paper describes the development of a computer program for the analysis of grillages curved in plan. The basic structural element is a curved member, treated in terms of its "exact" stiffness and restraint factors, which are derived in the Appendix. Results of analysis of a bridge deck curved in plan for the Lofthouse Interchange (an intersection of MI and M62 Motorways, Nr. Leeds) are presented.

Résumé

La présente communication se rapporte à la description d'un programme de calcul électronique des réseaux de poutres courbes dans le plan. L'élément de base du système est une pièce courbe que l'on considère du point de vue des coefficients «exacts» de rigidité et d'encastrement qui lui sont associés; ces coefficients sont déterminés dans l'Appendice. On présente les résultats du calcul du tablier d'un pont courbe dans le plan destiné à l'échangeur de Lofthouse (intersection des autoroutes M 1 et M 62, au nord de Leeds).

Zusammenfassung

In dieser Studie wird die Entwicklung eines Rechnerprogramms für die Analyse von im Grundriß gekrümmten Trägerrosten beschrieben. Das Grundelement ist ein gekrümmter Stab, der mit den genauen Stab- und Kreuzsteifigkeiten, welche im Anhang abgeleitet sind, behandelt wird. Die Ergebnisse der Analyse einer im Grundriß gekrümmten Brückenfahrbahn für die Lofthouse Interchange (eine Kreuzung der Autostraßen M 1 und M 62, Leeds) liegen ebenfalls vor.