Lateral buckling of beams without axial loads

Lateral Buckling of Beams without Axial Loads

Déversement latéral des poutres sans forces axiales

Trägerkippen ohne Axialkraft

S. O. ASPLUND
Sweden

Axial Buckling

Lateral buckling can be analyzed by extending the same argumentation as was used for simple beams in S. O. ASPLUND, Structural Mechanics: Classical and Matrix Methods, Prentice-Hall, Englewood Cliffs, 1966. Letters N and F below refer to chapters in that book.

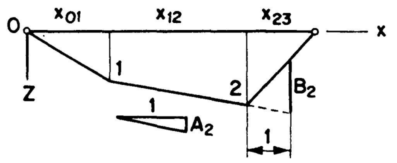


Fig. 1. Polygon angle breaks.

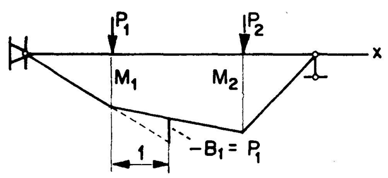


Fig. 2. Moment polygon.

Shortly, a polygon 0123, Fig. 1, is defined by abscissa intervals $\Delta x = [x_{01}\, x_{12}\, x_{23}]^D$ (diagonal matrix) between its end-points, and by its corner ordinates $[z_1 z_2]^*$ (a column vector, * denotes transposition). The slopes A of the polygon sides and their corner angle breaks $-B$, see Fig. 1, are given by

$$d = \begin{bmatrix} 1 & \\ -1 & 1 \\ & -1 \end{bmatrix}, \quad A = (\Delta x)^{-1} dz, \quad -B = d^* (\Delta x^{-1})\, dz = -X z. \qquad (1)$$

X is the second order difference operator corresponding to the differential operator d^2/dx^2. When a simple beam is loaded by transverse loads P, Fig. 2, the angle-breaks $-XM$ of its moment polygon equal the beam loads P. By inversion of $-X$ we find M and the moment influence coefficients M^i

$$-X M = P, \; M = -X^{-1} P = C P, \; C = -X^{-1} = M^i \text{ because } M = M^i P, \qquad (2)$$

The force transformation C transforms structure loads P into member forces M. Its transpose C^* is the displacement transformation that transforms member deformations m into structure displacements p, or

$$p = C^* m. \tag{3}$$

In this case the flexibility matrix f, transforming member forces M into member deformations, is a three diagonal matrix. The influence of both bending and shear deformations may be included in f. In a dense member subdivision f can be approximated by a one-diagonal matrix.

$$m = f M, \quad p = C^* m = C^* f M = C^* f C P. \tag{4}$$

Thus a beam with deflections p will carry in bending the loads

$$P^B = (C^* f C)^{-1} p = X F X p, F = f^{-1}. \tag{5}$$

Whenever a one-diagonal approximation for f is satisfactory, F also will be one-diagonal and $X F X$ a five diagonal band matrix that can be rapidly inverted.

We assume that the *segments* of the beam are subjected to axial forces H (a diagonal matrix). These, acting against the corners of the deflection polygon p, will carry a "bar-chain load" of

$$P^H = X^H p, \quad X^H = -d^* (H/\Delta x) d. \tag{6}$$

X^H being a three-diagonal band matrix. For a continuously variable compression that has the values H (a diagonal) at the bar-chain *joints* we write

$$X^H = H X. \tag{7}$$

In case of a constant compressure force, H in (7) should denote its scalar magnitude.

Finally, side springs of spring constants S connected to the joints between the beam segments, will carry the "side-spring load" of

$$P^S = S p. \tag{8}$$

For separate springs S is a one-diagonal matrix, but by making S a three- or five-diagonal matrix representing a connected spring action, an elastic body side support can be closely approximated.

To sum up, the beam carries altogether the transverse load $P = P^B + P^H + P^S$, that is,

$$(X F X + X^H + S) p = P. \tag{9}$$

In our present study concerning buckling we can omit the transverse loads P. The homogeneous equations in p

$$(A - \lambda' B) p = 0, \quad A = X F X + S, \quad B = -X^H \tag{10}$$

for $\lambda' = 1$ generally have no solution $p \neq 0$, but when the axial forces are varied by a factor λ', (10) may have non-trivial solutions p for discrete "eigenvalues" of λ'. When A is invertible we premultiply (10) by $A^{-1}\lambda$

$$(C-\lambda I)p = 0, \qquad C = A^{-1}B, \qquad \lambda = 1/\lambda'.$$

We call λ' or $1/\lambda$ the (elastic) buckling safety.

The numerical computation of eigenvalues and eigenvectors by an iterated vector method will be explained in the treatment of lateral buckling below.

On p. 67 in S. P. TIMOSHENKO and J. M. GERE, Theory of Elastic Stability, 2nd Ed., McGraw-Hill, New York, 1961, a continuous column is exemplified with a buckling force of $14.9\,EI/l_2^2$. Treating that case by subdividing the column into 12 segments and applying the present beam theory, results in a buckling load of $H = 15.2\,EI/l_2^2$. This value was obtained by introducing a very stiff spring at joint 5.

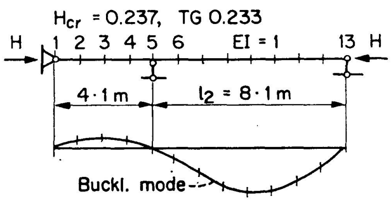


Fig. 3. Continuous column.

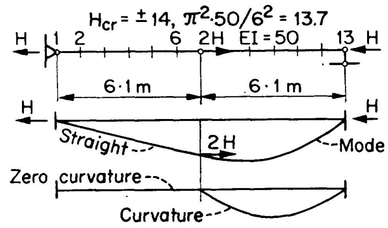


Fig. 4. Column, half in tension.

Testing the problem of Fig. 4 with an axial load at the middle, 7, of the column 1 to 13, creating equal tensions and compressions in both column halves, revealed that the tensioned half of the column remains straight. This result is singular, compare Fig. 5. We further observe that the eigenvalue problem of Fig. 4 has two equal and opposite dominant roots.

The problem of Fig. 5 also has two nearby dominant buckling loads and modes. One is indicated in the figure. The part in tension, 1 to 6, is also bent.

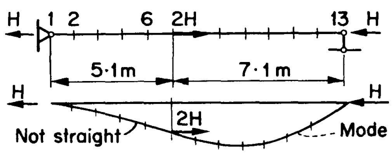


Fig. 5. Column partly in tension.

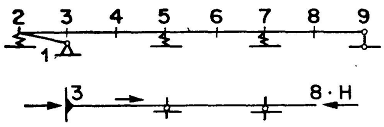


Fig. 6. Boundary conditions.

Various Boundary Conditions

In our treatment of plane beams, both beam ends are fundamentally considered as hinged, but other end conditions can be imitated in several ways. An elastic left end clamping can be obtained by adding as in Fig. 6, two perfectly rigid beam segments 12 and 23 and by installing a side spring of proper stiffness at 2. Making this spring perfectly rigid produces full clamping of the beam at 3.

A continuous beam in serveral spans can be reproduced by installing very rigid springs at the supports, for instance at 5 and 7, Fig. 6. A free end at 8 can be obtained by inserting zero flexibility in an added beam segment 89.

In the following treatment of lateral buckling other boundary conditions than hinged ends can be treated by similar devices to those just explained. Therefore what is just said needs not to be repeated in the subsequent discussion of lateral buckling.

Bending and Twist of a Beam in Space

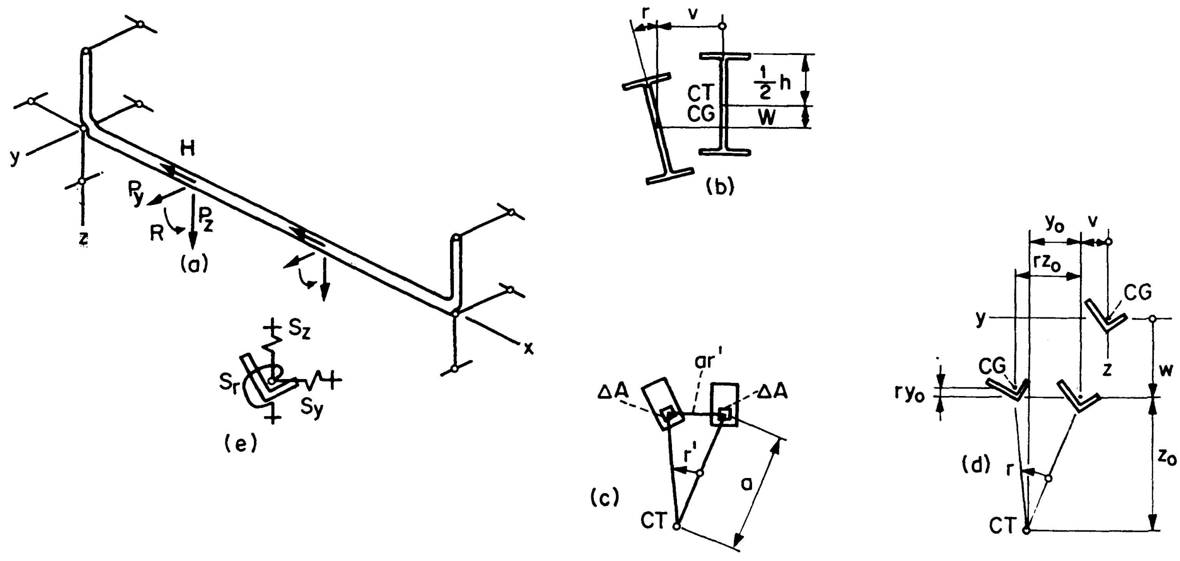


Fig. 7. Beam in space, general case.

Consider an unloaded beam, Fig. 7, whose centroid CG, Fig. b or d, is along the x-axis. A principal axis of inertia of the cross sections is directed along the vertical z-axis. We subdivide the beam into members as in Nc. Transverse horizontal loads P_y at the member joints cause moments $M_z = CP_y$, $C = -X^{-1}$, and deflect the centroid by $v = Cf_zCP_y$. Hence, bending carries the horizontal loads XF_zXv with $F_z = f_z^{-1}$. Vertical bending carries the vertical loads XF_yXw.

In a doubly symmetric beam, Fig. b, joint rotations of r about the centroid will deflect the beam flanges by $v = \pm\frac{1}{2}hr$. These deflections cause the beam flanges to carry loads $P_{yr} = \pm XF_wX\cdot\frac{1}{2}hr$ that combine into a couple $R_w = hP_{yr} = X\,2F_wX(1/4)h^2r = XF_rXr$. The bending stiffness $F_w = f_w^{-1}$ is calculated for one flange, using the bending stiffness $\frac{1}{2}EI_z$. We denote $(1/4)h^2EI_z$ by $EC_w = F_r$ of dimension tm^4, see (Fx 3), l. c.

Saint-Venant's torque is by (Fu 11): $M = GJr'$ thus $R = M' = (GJr')'$. For variable GJ we write $R = -X^{GJ}r$ with $X^{GJ} = -d^*GJ\Delta x^{-1}d$, where GJ is Saint-Venant's torsional stiffness. Constant GJ makes $R = -GJXr$.

A cross section element of area ΔA at the distance a from the center of twist will be stressed by $\sigma = H/A$, Fig. c, H being the compressive force, Fig. a. This stress on the element will form an angle of $a\,r'$ with its normal and have a shear component of $\sigma\, a\, r'\, \Delta A$. All such shears will carry a torque of

$$M = -\int_A \sigma a^2 r' dA = -H i_0^2 r', \quad i_0^2 = \int a^2 dA/A \tag{11}$$

and torque loads of $M' = -i_0^2 (H r')'$. For this we shall add $i_0^2 X^H r$ to the left hand side of (12).

After the beam is displaced by r, v, w, the y- and z-coordinates of the centroid will be $v + r z_0$ and $w - r y_0$ by Fig. d. When axial loads H_k are applied to the beam joints we find by (Ni 3) that further loads of $P_y^H = X^H (v + r z_0)$ and $P_z^H = X^H (w - r y_0)$ can be carried by bar-chain action, see *Ni*.

Spring supports, Fig. e, connected to the centroid at the member joints have stiffnesses S_r, S_y, S_z. They carry the loads $S_r r$, $S_y (v + r z_0)$, $S_z (w - r y_0)$.

We collect all beam loads found into the following matrix equation $K p = P$

$$\begin{bmatrix} X F_r X + X^H i_0^2 - X^{GJ} + S_r & (X^H + S_y) z_0 & -(X^H + S_z) y_0 \\ (X^H + S_y) z_0 & X F_z X + X^H + S_y & 0 \\ -(X^H + S_z) y_0 & 0 & X F_y X + X^H + S_z \end{bmatrix} \begin{bmatrix} r \\ v \\ w \end{bmatrix} = \begin{bmatrix} R \\ P_y \\ P_z \end{bmatrix}. \tag{12}$$

The elements in the second and third columns of the first row are added by symmetry or by Maxwell's theorem.

Eq. (12) can be readily used for analyzing a beam for buckling in bending and twist, but that will be here bypassed because the purpose of the present paper is the lateral buckling of beams.

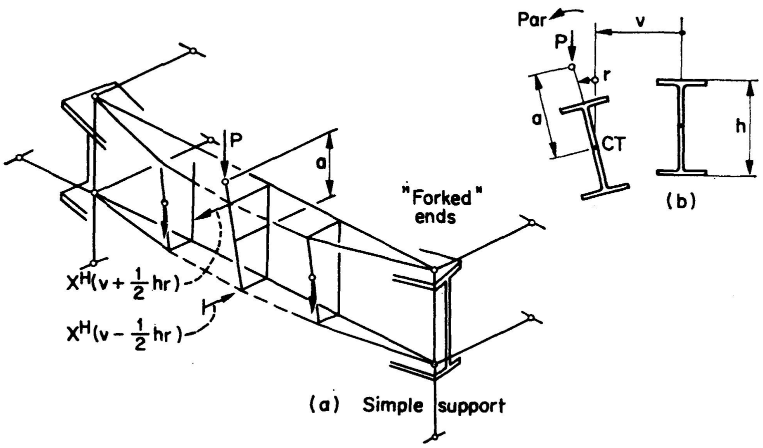


Fig. 8. Lateral buckling.

Lateral Buckling of Beams

A beam can buckle under vertical loads, also in the absence of axial loads on the beam, Fig. 8. A transversely loaded beam failing by twist and lateral deflection will be analyzed. We shall assume that the beam is simply supported and held vertical by forks or equivalent devices at the supports.

We subdivide the beam into segments of non-uniform lengths Δx in such a manner that the vertical loads P act at the segment joints. When the beam deflects by v the loads P will move along laterally by v, so no moment load will be added to the beam when it deflects. In case the load is applied on a seat fixed to the beam at a height of a above its center of gravity, torque loads $P a r$ caused by the rotation r of the cross section are to be added to the right-hand side of the governing Eq. (12). The vertical loads P induce bending moments $M = C P = -X^{-1} P$ in the beam if it is simply supported or other expressions if it is statically indeterminate. Many complicated support conditions can be simulated by $M = C P$.

The moments M induce compressive forces of $H = M/h$ in the top flange of a beam of depth h, and equal tensions in the bottom flange. The flange forces $\pm H$ act on the deflection polygon corners having ordinates $v + \frac{1}{2} h r$ and $v - \frac{1}{2} h r$ and angle breaks of $-X v - \frac{1}{2} h X r$ and $-X v + \frac{1}{2} h X r$. For balancing transverse loads of $-H X v - \frac{1}{2} H h X r$ and $H X v - \frac{1}{2} H X r$ are required which are equivalent to an applied external torque of $R = -H X v \cdot h = -M X v$ and a side force of $V = -\frac{1}{2} H h X r \cdot 2 = -M X r$.

For a doubly symmetric cross-section when all terms for axial loads and spring supports vanish (12) thus reads

$$\begin{bmatrix} X F_r X - X^{GJ} & & \\ & X F_z X & \\ & & X F_y X \end{bmatrix} \begin{bmatrix} r \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ P \end{bmatrix} + \begin{bmatrix} a P & -X^M \\ -X^M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r \\ v \end{bmatrix}, \qquad (13)$$

where a and P are diagonal matrices. For a continuously variable M we replace $-X^M$ by $-M X$. For constant $G J$ we can write $X^{GJ} = G J X$ where $G J$ is a scalar. By the last row of (1) we definitely solve the vertical deflections $w = (X F_y X)^{-1} P$.

We assume that all loads P and moments M increase in the same proportions λ while the structure is loaded. The first two Eqs. (13) are homogeneous in $r = [r^* v^*]^*$. Non-zero solutions r exist only for discrete values of the multiplier λ' called the eigenvalues of the problem

$$(A - \lambda' B) r = 0, \qquad (14)$$

$$A = \begin{bmatrix} X F_r X - X^{GJ} & \\ & X F_z X \end{bmatrix}, \quad B = \begin{bmatrix} a P & -X^M \\ -X^M & 0 \end{bmatrix}, \quad r = \begin{bmatrix} r \\ v \end{bmatrix},$$

When the load is smaller than that given by $\lambda' P$ and $\lambda' M$ where λ' is the lowest eigenvalue, the beam remains at $r = v = 0$. At that load the beam buckles proportionally to a set of deflections r, v, called the buckling mode.

In order to find instead a highest eigenvalue we transform (14) by premultiplication by A^{-1}/λ'

$$(C - \lambda I) r = 0, \qquad C = A^{-1} B, \qquad \lambda = 1/\lambda'. \qquad (15)$$

This is of course possible only if A is invertible.

Our problem is now reduced to finding the highest (dominant) eigenvalue of $(C-\lambda I)r$. The buckling load of the structure will then be P/λ, M/λ. We shall proceed to find λ by applying the iterated vector procedure. In our lateral buckling applications we often shall encounter the case that the dominant and subdominant eigenvalues are equal or nearly equal and of opposite signs. In such a case the usual iteration process must be modified to finding each of these two eigenvalues.

As usual we begin iteration with a column vector $r^{(0)}$ of ones. After a sufficient number of iterations the iterated vector r_0 will contain only negligible traces of the eigenvectors x_3 and higher. Therefore we can write $r_0 = a_1 x_1 + a_2 x_2$ where x_1 and x_2 are normalized eigenvectors belonging to the dominant and subdominant eigenvalues λ_1 and λ_2, and a_1 and a_2 are proper multipliers forming r_0. The succeeding iterated vectors are $r_1 = \lambda_1 a_1 x_1 + \lambda_2 a_2 x_2$ etc. To take advantage of Rayleigh's quotient we form the scalar products $r_0^* r_0 = l_0$, $r_0^* r_1 = l_1$, etc. We can further assume that r_0 has been normalized so that $l_0 = 1$. Observing that $x_1^* x_1 = x_2^* x_2 = 1$ while $x_1^* x_2 = 0$ we can summarize

$$\begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \\ \lambda_1^2 & \lambda_2^2 \\ \lambda_1^3 & \lambda_2^3 \end{bmatrix} \begin{bmatrix} a_1 x_1 \\ a_2 x_2 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \\ \lambda_1^2 & \lambda_2^2 \\ \lambda_1^3 & \lambda_2^3 \end{bmatrix} \begin{bmatrix} a_1^2 \\ a_2^2 \end{bmatrix} = \begin{bmatrix} r_0^* r_0 \\ r_0^* r_1 \\ r_0^* r_2 \\ r_0^* r_3 \end{bmatrix} = \begin{bmatrix} 1 \\ l_1 \\ l_2 \\ l_3 \end{bmatrix}$$

and construct the following products and differences

$$\begin{bmatrix} \lambda_1^2 & \lambda_2^2 & 2\lambda_1\lambda_2 \\ \lambda_1^2 & \lambda_2^2 & \lambda_1^2+\lambda_2^2 \\ 0 & 0 & (\lambda_1-\lambda_2)^2 \\ \lambda_1^3 & \lambda_2^3 & \lambda_1\lambda_2(\lambda_1+\lambda_2) \\ \lambda_1^3 & \lambda_2^3 & \lambda_1^3+\lambda_2^3 \\ 0 & 0 & (\lambda_1+\lambda_2)(\lambda_1-\lambda_2)^2 \end{bmatrix} \begin{bmatrix} a_1^4 \\ a_2^4 \\ a_1^2 a_2^2 \end{bmatrix} = \begin{bmatrix} l_1^2 \\ l_0 l_2 \\ l_0 l_2 - l_1^2 \\ l_1 l_2 \\ l_0 l_3 \\ l_0 l_3 - l_1 l_2 \end{bmatrix}, \text{ thus}$$

$$\lambda_1 + \lambda_2 = (l_3 - l_1 l_2)/(l_2 - l_1^2) = 2S. \tag{16}$$

Using $a_1^2 + a_2^2 = l_0 = 1$ we also transform

$$\begin{aligned} l_1 &= \lambda_1 a_1^2 + \lambda_2 (1 - a_1^2), & l_1 - \lambda_2 &= (\lambda_1 - \lambda_2) a_1^2, \\ l_2 &= \lambda_1^2 a_1^2 + \lambda_2^2 (1 - a_1^2), & l_2 - \lambda_2^2 &= (\lambda_1^2 - \lambda_2^2) a_1^2 = 2S(l_1 - \lambda_2). \\ \lambda_2^2 - 2S\lambda_2 &= l_2 - 2S l_1 & &\text{with the solution (for both roots),} \end{aligned}$$

$$\lambda_{1,2} = S \pm \sqrt{S^2 + l_2 - 2S l_1}. \tag{17}$$

By retrograde substitution it is easily seen that the expression under the square-root sign, $(S-\lambda_2)^2$, cannot become negative.

The dominant and subdominant buckling modes are

$$x_1 = r_1 - \lambda_2 r_0\text{, normalized}, \qquad (18)$$
$$x_2 = r_1 - \lambda_1 r_0\text{, normalized}.$$

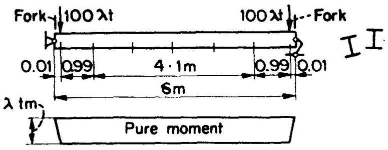
The following flow chart was used in the computations:

In, vectors of Δx, EI_z, GJ, and h. Form matrices X, C, f_z, F_z, F_r, XF_rX, X^{GJ}, $XF_rX - X^{GJ}$, XF_zX, A. In, diag. a, col. P. Form matrices aP, $M = CP$, $-MX$, B, $C = A^{-1}B$. In, delimiter ϵ. Make $l = 0$. Make start vector r_0. Label 10. Normalize r_0. Form $r_1 = Cr_0$, $r_2 = Cr_1$, $l_2 = r_0^* r_2$, $(l/l_2 - 1)^2 > \epsilon^2$? Yes, $r_0 = r_1$, $l = l_2$, go to label 10. No, form $r_3 = Cr_2$, $l_1 = r_0^* r_1$, $l_3 = r_0^* r_3$, $2S = (l_3 - l_1 l_2)/(l_2 - l_1^2)$, $\lambda_{1,2} = S \pm \mathrm{Sqrt}(S^2 + l_2 - 2Sl_1)$, $x_1 = r_1 - \lambda_2 r_0$ normalized, $x_2 = r_1 - \lambda_1 r_0$ normalized.

Out, Δx, EI_z, GJ, h, P, a, ϵ, λ_1, λ_2, x_1, x_2.

This method can be applied to beams with variable bending, warping, and torsional stiffnesses, quite arbitrary loadings and diverse support conditions. Its exactness can be verified on cases already calculated by way of differential equations for constant stiffness and simple cases, see TIMOSHENKO and GERE, l. c., p. 253, p. 262, and p. 264.

$EI_z = 450\,\mathrm{tm}^2$
$GJ = $ (a) $7{,}5\,\mathrm{tm}^2$
(b) $109\,\mathrm{tm}^2$
$h/2 = 0{,}242\,\mathrm{m}$



Value λ	(a)	(b)
6 segm:	± 43	± 122
Cont. TG;	± 43	± 118

Fig. 9. Pure moment.

For that purpose a beam of span 6 m, depth 0,5 m, constant stiffnesses $EI_z = 450\,\mathrm{tm}^2$, and $GJ = 7{,}5\,\mathrm{tm}^2$ (a), or $109\,\mathrm{tm}^2$ (b) was subdivided into members, and loaded as in Fig. 9 by loads causing an almost pure moment of λ tm, or, Fig. 10, by a center load applied at midheight of the beam, or, Fig. 11, on its top flange.

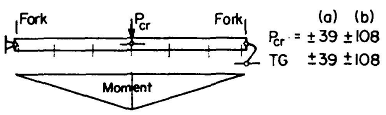


Fig. 10. Load at mid-depth.

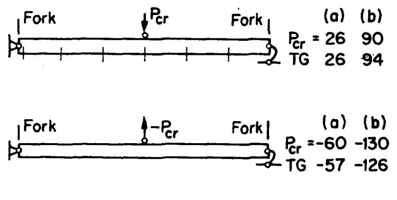


Fig. 11. Beam loaded on top flange.

Comparative results are indicated in the figures. TG denotes values in TIMOSHENKO and GERE, l. c. In spite of a rather rough subdivision into segments, Fig. 9 to 11 show that all results by the presented theory agree well with TIMOSHENKO and GERE's.

We also consider a continuous beam of two spans, Fig. 12. The beam may have the same properties as (b) in Fig. 9. Loads of 1 t applied on the top flange at intervals of 1 m produce a continuity moment above the center support of $-4{,}375$ tm. The case can be treated using one of the spans only with one end moment produced by an additional load of $-437{,}5$ t at a distance of 0,01 m from the support, Fig. 13. The computations disclose two eigenvalues, one for all loads multiplied by 50 and the other for the multiplier -73. The buckling deflections of top and bottom flanges are drawn in the figure. The negative eigenvalue obviously can be used for a positive buckling load that is applied to the bottom flange.

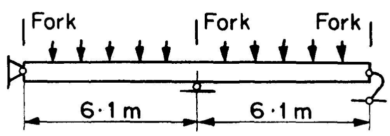


Fig. 12. Cont. beam, load on top flange.

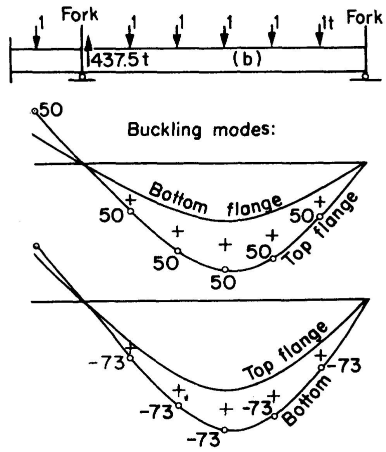


Fig. 13. Buckling modes.

Summary

Finite beam theory in the plane is concisely outlined and applied to the axial buckling of beams under various boundary conditions. Finite beam theory in space is briefly stated, also in order to prepare for lateral buckling theory which is the main subject of the paper. An iterated vector method for finding the dominant and subdominant eigenvalues is explained. The deduced theory is verified on eight lateral buckling cases with known solutions. An application to a uniformly loaded continuous beam concludes the paper.

Résumé

La théorie de la poutre finie dans le plan est esquissée brièvement et appliquée au flambement des conditions aux limites variées. La théorie de la poutre finie dans l'espace est rapidement exposée, aussi pour l'introduction à la théorie du déversement latéral qui est le sujet de ce mémoire. On y explique une méthode vectorielle itérative pour trouver les valeurs caractéristiques dominants et sous-dominants. La théorie démontrée est verifiée par 8 cas connus de déversement latéral. Pour conclure, le mémoire contient une application de la poutre continue uniformément chargée.

Zusammenfassung

Knapp umrissen und aufs axiale Knicken bei Trägern mit verschiedenen Lagerungsbedingungen angewandt wird die endliche Trägertheorie in der Ebene. Diese wird auch für den Raum als Vorbereitung fürs Kippen kurz angetönt, was das Hauptanliegen dieser Abhandlung ist. Zur Auffindung der Haupt- und Nebeneigenwerte wird ein iteratives Vektorverfahren angegeben. Die abgeleitete Theorie wird an acht schon bekannten Kippfällen geprüft. Die Abhandlung schließt mit einer Anwendung auf einen gleichmäßig belasteten Durchlaufträger.