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# Continuous Girders with Distributed Live Load

*Pièces continues soumises à une charge répartie en mouvement*

*Durchlaufbalken mit verteilter Verkehrslast*

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## 1. Introduction

During the last years, the problems of moving loads on continuous beams have been examined by different authors, see Ref. [1–5]. In these papers, it is a general assumption, that the load is moving slowly which means that dynamic effects are not taken into account.

The main problem in the investigations mentioned has been the determination of the maximum value of a single force, which can traverse the beam repeatedly without causing incremental collapse. It has been found for a single load, that the shakedown load is in most cases only 2–4% less than the collapse load, however the author has found [4] that in some cases it may be 7–8% less than the collapse load.

The difference between the shakedown load and the collapse load is much greater for uniformly distributed live loads. This has been proofed in the author's thesis [4] with the assumption, that the live load can be located over a continuous section of arbitrary length. These results will be repeated here and will be supplied with informations of the shakedown load when the demand for continuity of the live load is given up.

## 2. Basic Assumptions and Notations

We shall assume, that the beams are of double-symmetric, constant cross-section, and that the axis of the beams are horizontal.

The moment-curvature relation is assumed to be an idealized elastic-plastic curve (Fig. 1) and the influence of shear-forces upon this curve is neglected.

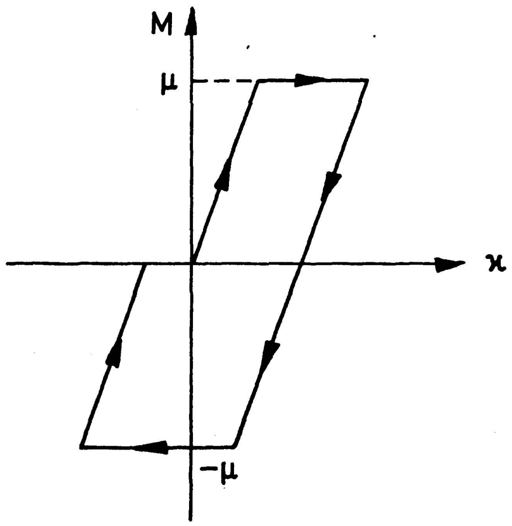


Fig. 1. Idealized moment-curvature diagram.

The total curvature $\kappa$ can be taken as the sum of an elastic part $\kappa^e$ and a plastic part $\kappa^p$

$$\kappa = \kappa^e + \kappa^p. \tag{1}$$

The bending moment is called $M$ and the full plastic moment is called $\mu$. The flexibility is called $B$, and we then have the following relations:

$$\kappa^e = BM, \tag{2}$$

$$|M| \leqq \mu, \tag{3}$$

$$d\kappa^p \geqq 0 \quad \text{for} \quad M = \mu, \tag{4}$$

$$d\kappa^p = 0 \quad \text{for} \quad |M| < \mu, \tag{5}$$

$$d\kappa^p \leqq 0 \quad \text{for} \quad M = -\mu. \tag{6}$$

For mild steel beams, these assumptions seem to form a reasonable basis for the calculation of both the collapse load and the shake-down load.

From the assumption of constant cross-section follows, that $\mu$ and $B$ have constant values along the beam.

In the following we shall consider beams over 2, 3 and 4 spans. The total length is called $L$ and the length of the first span is called $\lambda L$. The beams over 3 and 4 spans are supposed to be symmetrical.

The dead load per unit of length is called $g$ and the live load is called $p$. It is convenient to express the loads by dimensionless quantities $\gamma$ and $\psi$ defined as

$$\gamma = \frac{g L^2}{\mu}, \tag{7}$$

$$\psi = \frac{p L^2}{\mu}. \tag{8}$$

### 3. Collapse Load

In cases, where an end-span is critical, the collapse-mechanism has two hinges located as shown on Fig. 2. The value of $\psi$, which corresponds to collapse is then

$$\psi_c = \frac{6+4\sqrt{2}}{\lambda^2} - \gamma. \tag{9}$$

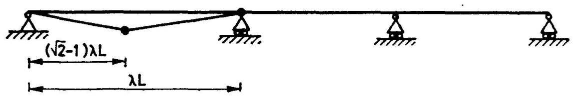


Fig. 2. End-span collapse mechanism.

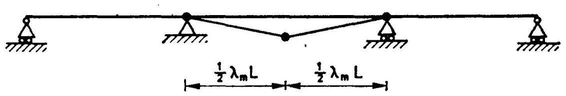


Fig. 3. Intermediate span collapse mechanism.

If an intermediate span is critical, the collapse mechanism has 3 hinges as shown on Fig. 3. Let the length of the span be $\lambda_m L$, then the value of $\psi$ corresponding to collapse will be

$$\psi_c = \frac{16}{\lambda_m^2} - \gamma. \tag{10}$$

For beams over 3 or 4 spans we shall take the smaller of the values obtained from the formulas (9) and (10).

For a 3-span beam, Fig. 4, where both end-spans are $\lambda L$, we find $\lambda_m = 1 - 2\lambda$, which means, that $\psi_c$ is to be found

$$\text{from (9) if } \lambda > \frac{1+\sqrt{2}}{2+4\sqrt{2}} = 0.315,$$

$$\text{from (10) if } \lambda < \frac{1+\sqrt{2}}{2+4\sqrt{2}} = 0.315.$$

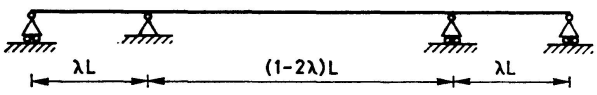


Fig. 4. Symmetrical 3-span beam.

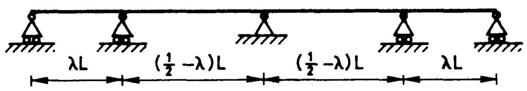


Fig. 5. Symmetrical 4-span beam.

For a 4-span beam, where both of the end spans, Fig. 5, are $\lambda L$ and the 2 intermediate spans are of equal length, $\lambda_m = \frac{1}{2} - \lambda$. This means that $\psi_c$ is to be found

$$\text{from (9) if } \lambda > \frac{5+2\sqrt{2}}{34} = 0.230,$$

$$\text{from (10) if } \lambda < \frac{5+2\sqrt{2}}{34} = 0.230.$$

Values of $\psi_{c0}$, i.e. $\psi_c$ corresponding to $\gamma = 0$, are shown as functions of $\lambda$ on Figs. 8, 13 and 18.

## 4. Shakedown Load

The shakedown load is defined as the maximum value of the live load for which it is possible to find a system of residual moments with the characteristic that the numerical value of the bending moments nowhere exceeds the full plastic moment, regardless of the location of the load on the structure.

We shall denote the bending moment over support no. $i$ corresponding to a unit load intensity in span no. $j$ by $m_i[j]\,L^2$ (span no. $j$ is between the supports no. $j-1$ and $j$).

Further we introduce $M_i^{gr}$ as the sum of bending moments over support no. $i$ from dead load and residual moment.

### *2-Span Beam*

It is most convenient to start with the loading conditions decisive for the negative moment over the intermediate support, Fig. 6.

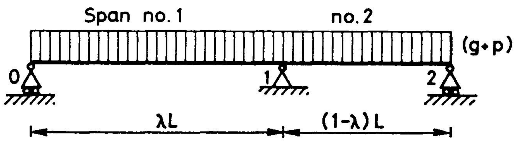


Fig. 6. Loading decisive for bending moment at support no. 1.

We shall take the residual moment to that value, which will give $M_1 = -\mu$ with the loading shown on Fig. 6, i. e.

$$M_1^{gr} + m_1[1]\,p\,L^2 + m_1[2]\,p\,L^2 = -\mu. \tag{11}$$

In order to find the most unfavourable conditions for positive moments, the live load is removed from span no. 2, see Fig. 7.

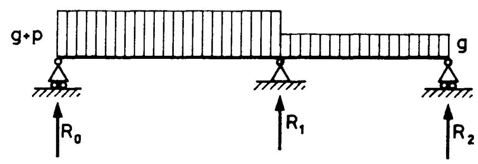


Fig. 7. Loading decisive for maximum bending moment.

We find

$$M_1 = M_1^{gr} + m_1[1]\,p\,L^2 = -\mu - m_1[2]\,p\,L^2. \tag{12}$$

The last expression was found by use of (11). Next the reaction $R_0$ at support no. 0 is calculated

$$R_0 = \tfrac{1}{2}(g+p)\,\lambda\,L + \frac{M_1}{\lambda\,L}. \tag{13}$$

Finally, we have the maximum bending moment equal to the full plastic moment $\mu$,

$$\frac{R_0^2}{2(p+g)} = \mu. \tag{14}$$

From (12), (13) and (14) we get a quadratic equation in $p$, and we notice, that $p$ must be equal to the shakedown load. Here it is convenient to find $\psi_s$ (value of $\psi$ corresponding to the shakedown load). As $m_i[j]$ depends only upon $\lambda$, $\psi_s$ will depend upon $\lambda$ and $\gamma$.

The correct solution for $\psi_s$ is given as formula (2.1.3.2–9) in Ref. [4], but it is shown, that this may be substituted by the much easier formula

$$\psi_s \cong [1-2.89(1-\lambda)^4]\left(\frac{6+4\sqrt{2}}{\lambda^2}-\gamma\right). \tag{15}$$

We deduce from (15) that corresponding to $\gamma=0$ we find $\psi_s=\psi_{s0}$ as

$$\psi_{s0} \cong [1-2.89(1-\lambda)^4]\frac{6+4\sqrt{2}}{\lambda^2} \tag{16}$$

and then

$$\psi_s \cong \psi_{s0}\left(1-\frac{\lambda^2}{6+4\sqrt{2}}\gamma\right). \tag{17}$$

The formulas (15)–(17) should be used only for $\lambda \geqq \frac{1}{2}$. Values of $\psi_{s0}$ and $\psi_{c0}$ (collapse load corresponding to $\gamma=0$) are shown on Fig. 8 as functions of $\lambda$.

In this case, the two-span beam, it should however be more convenient to express the values in terms of the longest span instead of the total length.

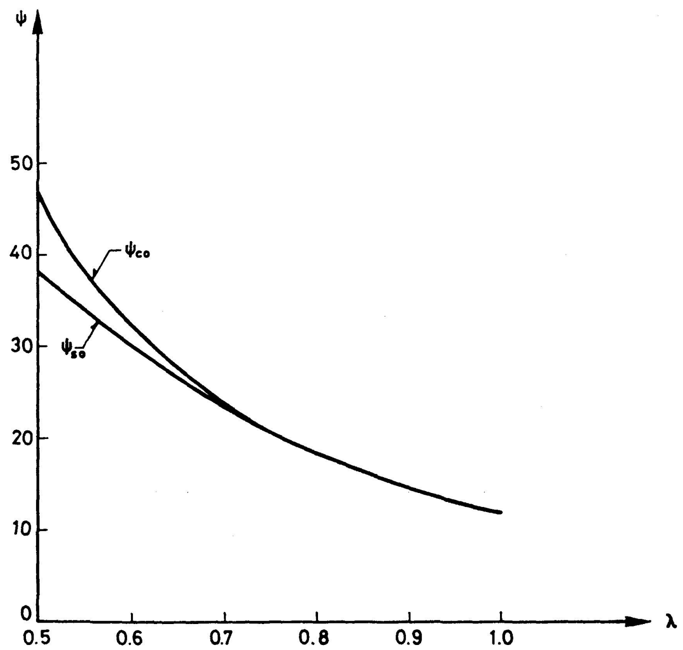


Fig. 8. 2-span beam. $\psi_{c0}$ and $\psi_{s0}$ as functions of $\lambda$.

This is easily done by quantities $\psi'$ and $\gamma'$ defined as

$$\psi' = \lambda^2 \psi, \tag{18}$$

$$\gamma' = \lambda^2 \gamma. \tag{19}$$

(15), (16) and (17) may then be rewritten

$$\psi_s' = [1 - 2.89(1-\lambda)^4][(6+4\sqrt{2}) - \gamma'], \tag{20}$$

$$\psi_{s0}' = [1 - 2.89(1-\lambda)^4](6+4\sqrt{2}), \tag{21}$$

$$\psi_s' = \psi_{s0}'\left(1 - \frac{\gamma'}{6+4\sqrt{2}}\right). \tag{22}$$

$\psi_{c0}'$ will have the constant value $(6+4\sqrt{2})$; $\psi_{s0}'$ is shown on Fig. 9. It follows from Figs. 8 and 9 that the difference between $\psi_s$ and $\psi_c$ is greatest for $\lambda = \frac{1}{2}$, in which case we find $\psi_s/\psi_c = 0.819$.

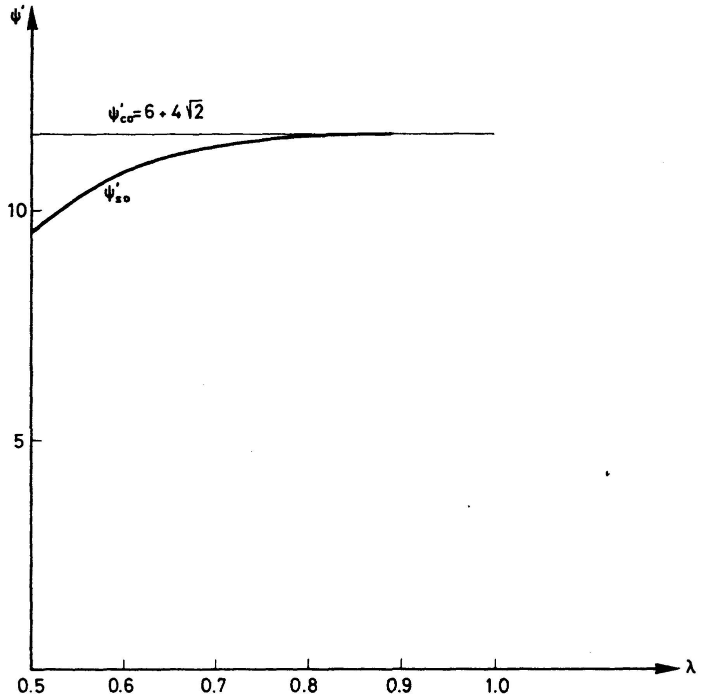


Fig. 9. 2-span beam. $\psi'_{s0}$ and $\psi'_{c0}$ as functions of $\lambda$.

### *3-Span Beam*

For the 3-span beam shown on Fig. 4 the residual moment over support no. 1 is found when the live load is in spans 1 and 2, see Fig. 10. We thus find

$$M_1^{gr} + m_1[1]\,p\,L^2 + m_1[2]\,p\,L^2 = -\mu \tag{23}$$

and due to the symmetry

$$M_2^{gr} = M_1^{gr}. \tag{24}$$

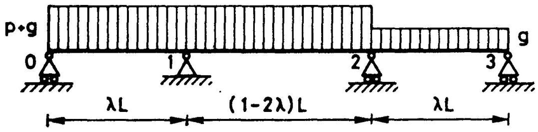


Fig. 10. Loading decisive for bending moment at support no. 1.

The investigations for positive moments become more difficult than was the case for the 2-span beam.

Here it will be easiest to start with the case, that the intermediate span (no. 2) is decisive for the positive moments. The live load must be placed in span 2 only, see Fig. 11.

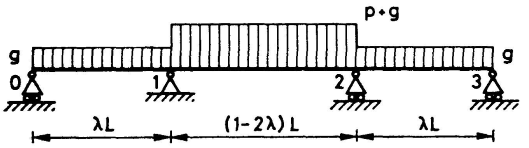


Fig. 11. Loading decisive for maximum bending moment in the mid-span.

The bending moments over the supports 1 and 2 are

$$M_1 = M_2 = -\mu - m_1[2]\,p\,L^2 \tag{25}$$

and the maximum bending moment is found at the center of the span. It must be equal to $\mu$ when $p$ corresponds to the shakedown load, and thus we find

$$M_1 + \tfrac{1}{8}(g+p)(1-2\lambda)^2 L^2 = \mu. \tag{26}$$

It is not very difficult to find $\psi_s$ from (25) and (26), but as the exact formula is somewhat difficult we shall replace it by the more convenient and very accurate

$$\psi_s \simeq (1-86\lambda^5)\left(\frac{16}{(1-2\lambda)^2} - \gamma\right), \tag{27}$$

which holds good for $\lambda > 0.25$, see ref. 4, and smaller values of $\lambda$ don't seem to be of practical importance.

We also have to consider the case that an end span, say span no. 1, becomes decisive for positive moments. In the author's thesis [4] this was treated only under the assumption of a continuous live load, but we shall here also consider the case, where the live load can act in non-adjecent spans. Figures and formulas which assume continuity are indexed $c$ whereas figures and formulas corresponding to the assumption of discontinuity are indexed $d$.

The loading corresponding to maximum bending moment in span no. 1 is shown on Fig. 12.

The moment over support no. 1 is in case of continuity

$$M_1 = M_1^{gr} + m_1[1]\,p\,L^2 = -\mu - m_1[2]\,p\,L^2 \tag{28c}$$

and in case of discontinuity

$$M_1 = M_1^{gr} + m_1[1]\,p\,L^2 + m_1[3]\,p\,L^2 = -\mu - m_1[2]\,p\,L^2 + m_1[3]\,p\,L^2. \tag{28d}$$

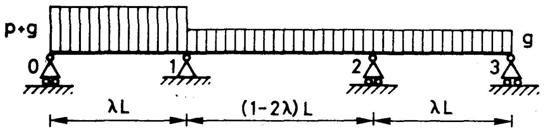


Fig. 12c. Loading decisive for maximum bending moment in span no. 1. Continuity case.

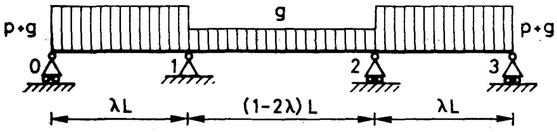


Fig. 12d. Loading decisive for maximum bending moment in span no. 1. Discontinuity case.

From now, we proceed like in the treatment of the 2-span beam, which means that we can again deduct the formulas (13) and (14)

The value of $p$ corresponding to shakedown is then found from formulas (28), (13) and (14). In the case of continuity, the shakedown load is (see Ref. [4])

$$\psi_s \simeq 2.55\lambda\left(\frac{6+4\sqrt{2}}{\lambda^2}-\gamma\right). \tag{29c}$$

In the case of discontinuity of the live load, we obtain as a reasonable good approximation

$$\psi_s \simeq (13.93\lambda^2-20\lambda^3)\left(\frac{6+4\sqrt{2}}{\lambda^2}-\gamma\right). \tag{29d}$$

The approximations (29) should not be used for $\lambda > 0.35$.

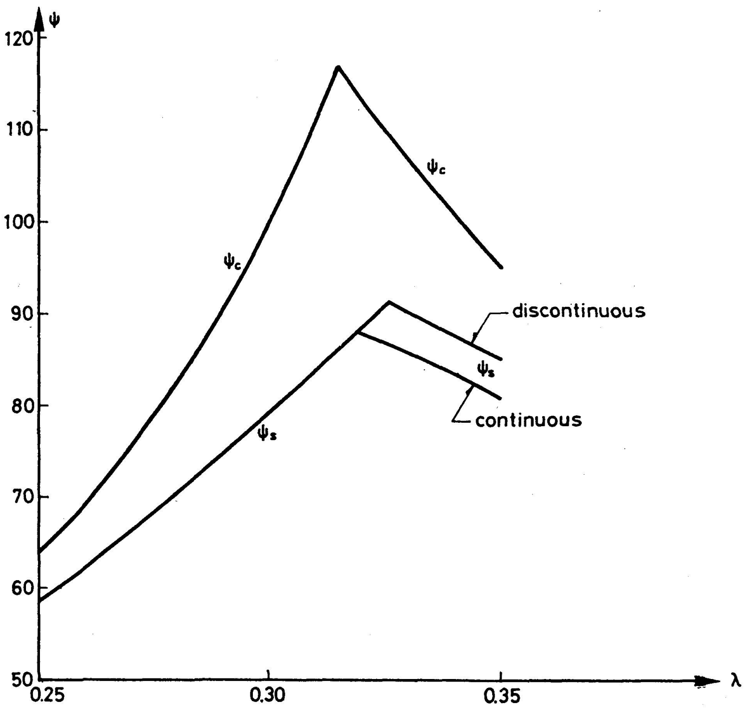


Fig. 13. 3-span beam. $\psi_c$ and $\psi_s$ corresponding to $\gamma=0$ as functions of $\lambda$.

The smaller of the values of $\psi_s$ found from (27) or (29) shall be used. Corresponding to $\gamma = 0$ we find the dependence between $\psi_s$ and $\lambda$ as illustrated in Fig. 13.

### *4-Span Beam*

The residual moment over support no. 1 is found with live load in spans 1 and 2 if continuity is assumed (Fig. 14c). If we don't assume continuity, the live load shall act in spans 1, 2 and 4 (Fig. 14d).

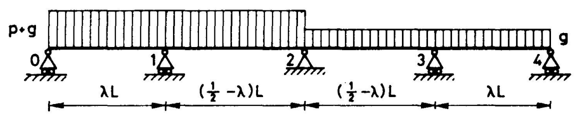


Fig. 14c. Loading decisive for bending moment at support no. 1. Continuity case.

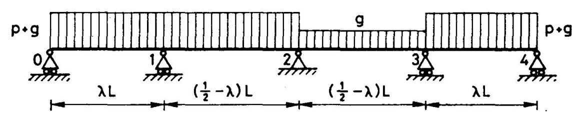


Fig. 14d. Loading decisive for bending moment at support no. 1. Discontinuity case.

The residual moment in point 1 is thus found from (30c) or (30d)

$$M_1^{gr} + m_1[1]\,p\,L^2 + m_1[2]\,p\,L^2 = -\mu, \tag{30c}$$

$$M_1^{gr} + m_1[1]\,p\,L^2 + m_1[2]\,p\,L^2 + m_1[4]\,p\,L^2 = -\mu. \tag{30d}$$

Due to symmetry

$$M_3^{gr} = M_1^{gr}. \tag{31}$$

The residual moment in point 2 is found with live load in spans 2 and 3, see Fig. 15.

$$M_2^{gr} + m_2[2]\,p\,L^2 + m_2[3]\,p\,L^2 = -\mu. \tag{32}$$

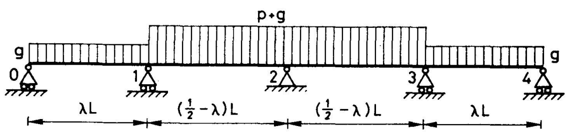


Fig. 15. Loading decisive for bending moment at support no. 2.

If the end-span is most dangerous with respect to positive moments, the live load must be placed in span 1 (Fig. 16c) or in spans 1 and 3 (Fig. 16d).

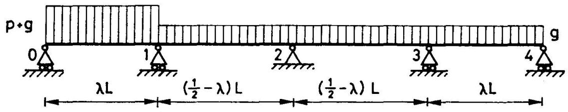


Fig. 16c. Loading decisive for maximum bending moment in span no. 1. Continuity case.

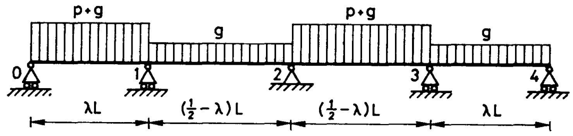


Fig. 16d. Loading decisive for maximum bending moment in span no. 1. Discontinuity case.

The moment over support no. 1 is

$$M_1 = M_1^{gr} + m_1[1]\,p\,L^2 = -\mu - m_1[2]\,p\,L^2, \tag{33c}$$

$$M_1 = M_1^{gr} + m_1[1]\,p\,L^2 - m_1[3]\,p\,L^2 = -\mu - m_1[2]\,p\,L^2 + m_1[3]\,p\,L^2 - \\ - m_1[4]\,p\,L^2. \tag{33d}$$

As before, the reaction in point 0 is given by (13), and the shakedown value of the live load is found from (14). When the live load is continuous, it is found from (33c), (13) and (14), and its value is found to be expressed by

$$\psi_s \cong [1 + 0.184\,(\tfrac{1}{2}-\lambda)^2 - 41\,(\tfrac{1}{2}-\lambda)^4]\left(\frac{6+4\sqrt{2}}{\lambda^2} - \gamma\right), \tag{34c}$$

$$\psi_s \cong [1 - 0.1\,(\tfrac{1}{2}-\lambda) + 0.093\,(\tfrac{1}{2}-\lambda)^2 - 44\,(\tfrac{1}{2}-\lambda)^4]\left(\frac{6+4\sqrt{2}}{\lambda^2} - \gamma\right). \tag{34d}$$

(34c) is taken from Ref. [4].

If an intermediate span, say span 2, is most dangerous for positive moments, the live load must act in span 2 (Fig. 17c) or in spans 2 and 4 (Fig. 17d).

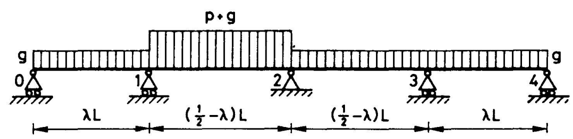


Fig. 17c. Loading decisive for maximum bending moment in span no. 2. Continuity case.

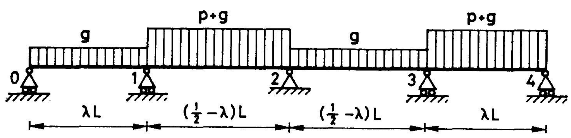


Fig. 17d. Loading decisive for maximum bending moment in span no. 2. Discontinuity case.

The moments in points 1 and 2 are

$$M_1 = M_1^{gr} + m_1[2]\,p\,L^2 = -\mu - m_1[1]\,p\,L^2, \tag{35c}$$

$$M_1 = M_1^{gr} + m_1[2]\,p\,L^2 + m_1[3]\,p\,L^2 = -\mu - m_1[1]\,p\,L^2 + m_1[3]\,p\,L^2 - m_1[4]\,p\,L^2, \tag{35d}$$

$$M_2 = M_2^{gr} + m_2[2]\,p\,L^2 = -\mu - m_2[3]\,p\,L^2, \tag{36c}$$

$$M_2 = M_2^{gr} + m_2[2]\,p\,L^2 + m_2[4]\,p\,L^2 = -\mu - m_2[3]\,p\,L^2 + m_2[4]\,p\,L^2. \tag{36d}$$

The shearing force in the left end of span 2 is given by

$$Q_{2.1} = \frac{2\,(M_2 - M_1)}{(1-2\lambda)\,L} + \tfrac{1}{2}\,(p+g)\,(\tfrac{1}{2} - \lambda)\,L \tag{37}$$

and as the maximum of positive moment shall be $\mu$, we finally find

$$M_2 + \frac{Q_{2.1}^2}{2\,(p+g)} = \mu. \tag{38}$$

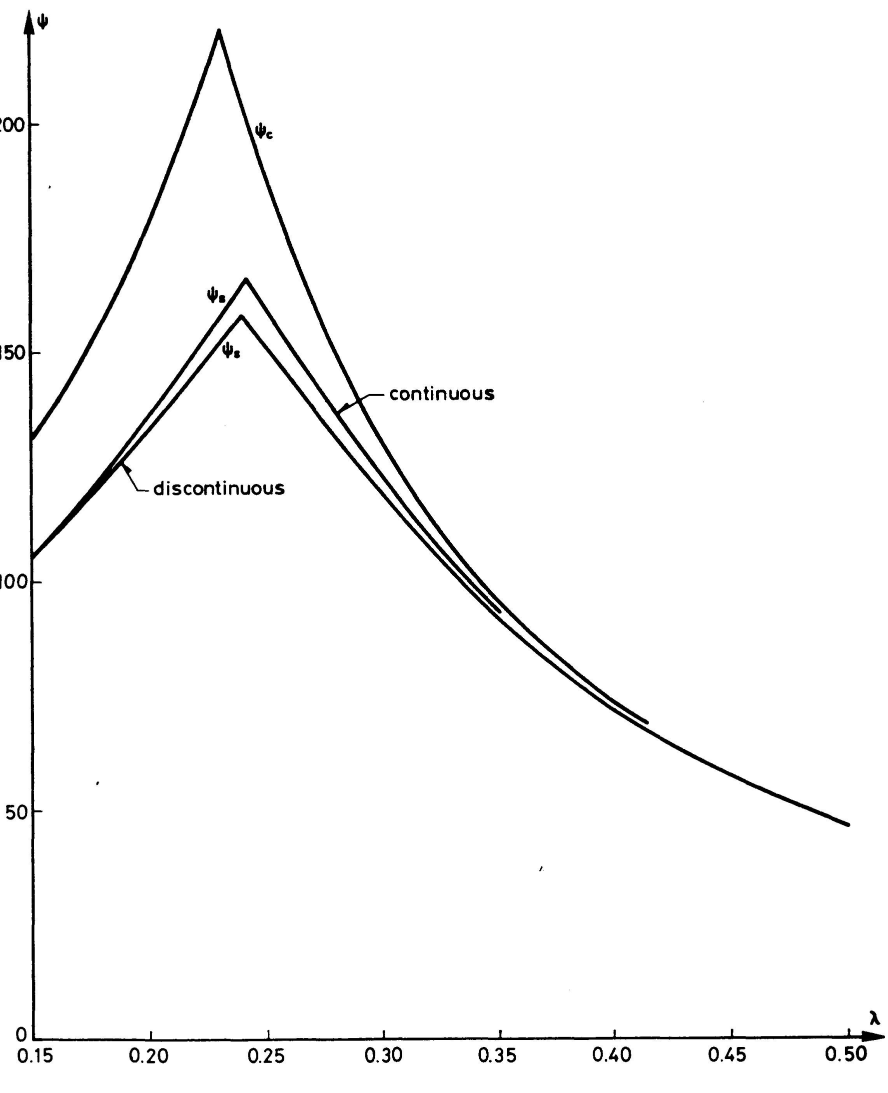


Fig. 18. 4-span beam. $\psi_c$ and $\psi_s$ corresponding to $\gamma = 0$ as functions of $\lambda$.

If the live load is continuous, the shakedown load is found from (35c), (36c), (37) and (38). If the live load cand act in separate spans, the shakedown load is found from (35d), (36d), (37) and (38).

We find the approximate formulas

$$\psi_s = (0.642 + 2.56\lambda - 9.69\lambda^2)\left(\frac{64}{(1-2\lambda)^2} - \gamma\right), \tag{39c}$$

$$\psi_s = (0.674 + 2.34\lambda - 9.80\lambda^2)\left(\frac{64}{(1-2\lambda)^2} - \gamma\right). \tag{39d}$$

(39c) is taken from Ref. [4].

The smaller of the values from (34) or (39) is to be used. A graphical representation of $\psi_s$ is given in Fig. 18.

If $\gamma = 0$, i.e. when the dead load is negligeable compared to the shakedown value of the live load, we find for $\lambda = 0.230$ that $\psi_s = 158.0$ when the live load is continuous and $\psi = 152.3$ when the live load is discontinuous. As $\psi_c = 220.0$ we find $\psi_s/\psi_c = 0.718$ for continuous load and $\psi_s/\psi_c = 0.693$ for discontinuous load.

## Conclusion

The results obtained show, that for some girders the shakedown value of a distributed loading may be smaller when the demand for continuity is cancelled. However, it does not give a great difference, and it seems questionable, if it is reasonable to take this into account in static calculations. This is a question dealing with the probability of the different load conditions and it must be judged for individual structures, what conditions they should be calculated for.

It has often been mentioned, that the difference between collapse-load and shakedown load is so small, that it would be unnecessary to investigate for incremental collapse. For a distributed live load the shakedown load may be appreciably less than the collapse load and it seems to be hazardous not to take this into account.

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**Summary**

For continuous beams with constant cross-section and an idealized moment-curvature relation, the shakedown value of uniformly distributed loading is calculated.

The results are presented as simple formulas for beams with 2, 3 and 4 spans with the limitations, that only beams with symmetrical spans are considered in the case of beams with 3 and 4 spans.

It is found, that the shakedown load may be 30% below the collapse load with certain relations between spanlengths, that are within the practical ranges.

**Résumé**

Considerant des poutres continues à inertie constante et de comportement parfaitement élastoplastique, l'auteur détermine la charge uniforme de stabilisation.

Pour des poutres à 2, 3 et 4 travées, les resultats sont presentés sous forme d'expressions simples; toutefois dans les cas des poutres à 3 et 4 travées, seule des groupes de travées symétriques sont considérés.

L'auteur démontre qu'a condition de respecter certains rapports limites entre les longueurs des travées – restant à l'intérieur du domaine pratique – la charge uniforme de stabilisation sera jusqu'a 30 % inférieure à la charge uniforme d'adaptation plastique.

**Zusammenfassung**

Für kontinuierliche Balken mit konstantem Querschnitt und idealisierter Moment-Krümmungs-Beziehung sind die Stabilisierungswerte einer gleichmäßig verteilten Belastung gefunden worden.

Für Balken mit 2, 3 oder 4 Feldern sind die gefundenen Werte als einfache Formeln gegeben; für Balken mit 3 oder 4 Feldern sind nur die Fälle mit symmetrischen Feldweiten behandelt worden.

Man hat herausgefunden, daß die Stabilisierungslast in den ungünstigsten Fällen bis zu 30% weniger als die gewöhnliche plastische Bruchlast beträgt, wenn man spezielle Verhältnisse zwischen den Spannweiten hat, die innerhalb des Gebietes praktischer Konstruktionen liegen.

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