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Yield Line Analysis of Punching Failures in Slabs

Analyse des lignes de répartition des fissures découpantes dans des plaques de béton

Bruchlinienberechnung für Durchstanzen von Decken

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Introduction

The subject of concentrated loads acting on reinforced concrete slabs has interested many investigators in the past. This was partly due to the fact that it is of interest in itself, but largely also because it can, in some ways, be equated with the very common problems of the slab supported on columns or with columns supported on spread footings. In this research, the major interest was usually directed toward the so-called punching shear strength or toward the diagonal tensile strength of the slab. Many experiments were conducted and many empirical and semi-empirical formulas were devised in an attempt to relate apparent shearing stresses in the slab, caused by concentrated loads, to the strength of the concrete.

Other, and particularly more recent, investigators, have recognized that the flexural strength of the slab may have a strong influence on the punching resistance. For instance, as early as 1943, JOHANSEN [1], in his original book on yield line theory, analyzed some early slab tests by BACH and GRAF [2] in which several concentrated loads, arranged in a small circle, were applied to slabs supported along their boundaries. JOHANSEN mentioned that at the time of test, failure was attributed to punching shear. However, he analyzed the slabs and loading according to yield line theory and stated that the punching shear was actually a secondary phenomenon, "... since the primary cause – yielding of the reinforcement – had already largely taken effect at lower load stages and produced wide cracks". His calculations correlated well with the experimental results. Another attempt to link punching shear and flexural strength was made by HOGNESTAD [3] in 1953, when he re-analyzed RICHART's [4] footing tests. HOGNESTAD did this by including a term in his empirical shear equations, which he called P_{flex} and defined as "the ultimate flexural capacity the slab would have had if it had not failed in shear". P_{flex} was taken as being "governed by the full static moment at the edge of the column", i.e. by the relation between the yield moment of the footing along a line crossing the entire footing and lying along one face of the column, and the moment applied along this line by the pressure acting on the bottom of the footing.

Similar empirical shear equations were later used by ELSTNER and HOGNESTAD [5] to analyze their slab tests and also some tests conducted by GRAF [6]. However, in these calculations, P_{flex} was calculated by yield line theory using simplified yield line patterns based on the crack patterns observed in the specimens. WHITNEY [7] carried the relationship between punching shear and the flexural strength of slabs a step further. Re-analyzing RICHART's footing tests, as well as the slab tests conducted by ELSTNER and HOGNESTAD, he devised a very simple empirical formula, in which the shear strength was directly related to the "... ultimate resisting moment per inch width of slab within the base of the pyramid of rupture".

MOE [8], in analyzing his own and others' tests, devised a new set of empirical formulas in which the punching shear strength of slabs and footings was partially related to a shear V_{flex} which was obtained from the P_{flex} of ELSTNER and HOGNESTAD. MOE's formulas, or variations thereof, were also used by TASKER and WYATT [9], HOGNESTAD, ELSTNER and HANSON [10], MOWRER and VANDERBILT [11], and HANSON and HANSON [12], in analyzing their own and others' test results. Other investigators [13–16], while not using MOE's formulas, also recognized that some relation exists between the bending strength of the slab or the amount of reinforcement, and the punching shear resistance. Some [14, 15, 17] equations have also been devised to relate the combination of bending and shearing stresses in the concrete to the punching failure.

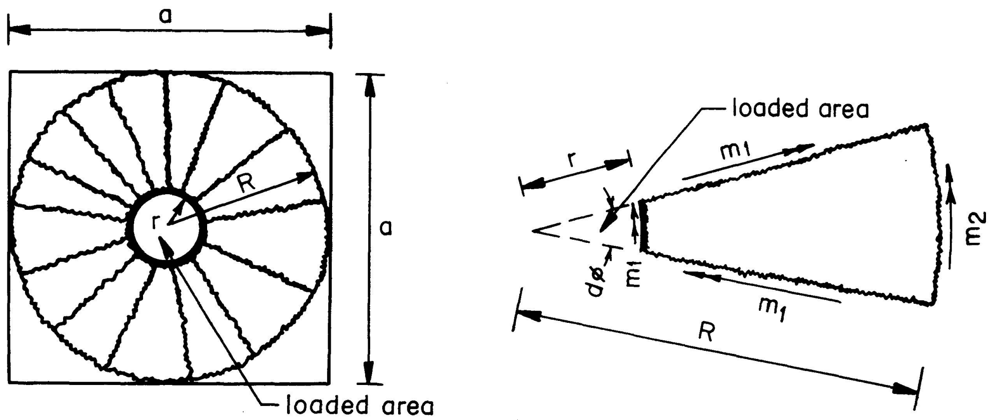
The results of some earlier tests [18] of small slab models, carried out under the supervision of the senior author, indicated that yield line theory might give good correlation with the punching resistance of uniformly loaded slabs supported on columns. Also, examination of the crack patterns of a large number of specimens pictured in the literature indicates that in many cases the reinforcement must have yielded in tension either prior to, or simultaneously with, the failure. Further, it is apparent from MOE's calculations, that in the case of several of his and others' specimens V_{flex}/V_{test} is close to unity, which means that for those cases a considerably simplified yield line approach predicted the failure strength of punching specimens. It is therefore the purpose of this paper to examine some of the punching tests reported in the literature from the standpoint of yield line theory in order to determine what patterns and equations may be applicable, and the range of their validity.

Development of Equations

Yield line theory is an upper bound theory. It is therefore always necessary to look for the lowest possible yield line load, consistent with geometrical and physical constraints, which the structure can resist. If several different yield line patterns are admissible by these constraints, the one providing the lowest load resistance will be the critical one. Yield line theory only predicts a stage of slab behavior characterized by yielding of the reinforcement, formation of extensive cracks, and appreciable deflections. It may or may not predict the collapse load, depending on the membrane or other non-flexural action of the slab. However, this method of analysis is useful even when it underestimates the collapse load, since it then serves to predict the limit of ordinary structural usefulness, and gives warning of impending structural distress.

Several different yield line patterns which model many of the punching test specimens and results available in the literature will now be analyzed:

Case I: A square slab, with length of side a, is supported along its perimeter in such a way that the corners cannot lift. The reinforcement is isotropic (a square grid will essentially satisfy this requirement) and the positive yield moment is m_1 while the negative yield moment is m_2. A vertical load P, distributed over a small circle of radius r, is applied at the center of the slab. See fig. 1a. It has been shown [1] that a fan shaped yield line pattern will form. Fig. 1b represents one of the pie shaped segments and shows the moments acting on it.



(a) (b)

Fig. 1. Yield line pattern for case 1 – (Slab does *not* crack beneath load).

Applying the principle of virtual work and equating the energy input of a unit downward displacement of the load to the energy simultaneously absorbed in the yield line hinges, one obtains

$$\int_0^{2\pi} (m_1 + m_2) \frac{R\, d\phi}{R - r} = P, \qquad (1)$$

where R is the radius of the fan, and remains to be determined. $d\phi$ is defined in fig. 1. Solving, one finds that

$$P = \frac{2\pi}{1 - \frac{r}{R}} (m_1 + m_2). \qquad (2)$$

From this, it is obvious that P will be smallest for the largest physically possible radius of the fan. Therefore one would expect the fan to spread to the boundaries of the slab, so that $R = \frac{1}{2}a$. This derivation was based on the assumption that the slab would not crack beneath the load. If this assumption is not correct, i.e. if the radial yield lines continue under the load (see fig. 2), it can be shown that

$$P = \frac{2\pi}{1 - \frac{2r}{3R}}(m_1 + m_2). \tag{3}$$

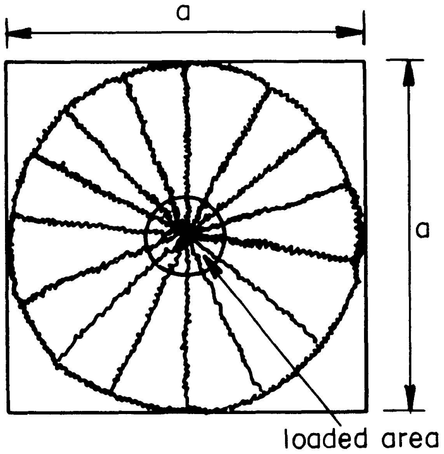


Fig. 2. Yield line pattern for case I – (Slab *does* crack beneath load).

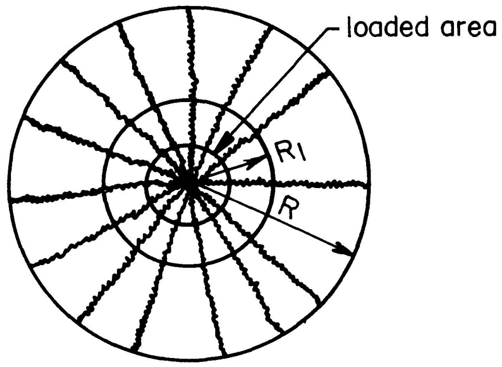


Fig. 3. Yield line pattern for case II.

Case II: A circular slab of radius R is simply supported along its entire perimeter. The reinforcement is isotropic and the positive yield moment is $k\,m_1$ for the central portion of the slab with radius R_1, while for the rest of the slab it is m_1. There is no negative moment reinforcement. A load P, distributed over a small circle of radius r, is applied at the center of the slab. It is assumed that the slab will be able to crack beneath the load. See fig. 3. Applying the principle of virtual work as before,

$$\int_0^{2\pi} \frac{m_1(R - R_1 + k\,R_1)\,d\phi}{R} = \int_0^{2\pi} \frac{P}{\pi r^2}\,\frac{r^2\,d\phi}{2}\,\frac{R - \frac{2}{3}r}{R}. \tag{4}$$

This gives

$$P = \frac{2\pi m_1}{1 - \frac{2}{3}\frac{r}{R}}\left[1 + \frac{R_1}{R}(k-1)\right]. \tag{5}$$

If, on the other hand, the slab does not crack beneath the load, then

$$P = \frac{2\pi m_1}{1 - \frac{r}{R}}\left[1 + \frac{R_1}{R}(k-1)\right]. \tag{6}$$

Case III: A square slab, with length of side a, is simply supported along the perimeter with the corners free to lift. Reinforcement is basically isotropic, but additional reinforcement is provided as shown in fig. 4 so that two bands, of

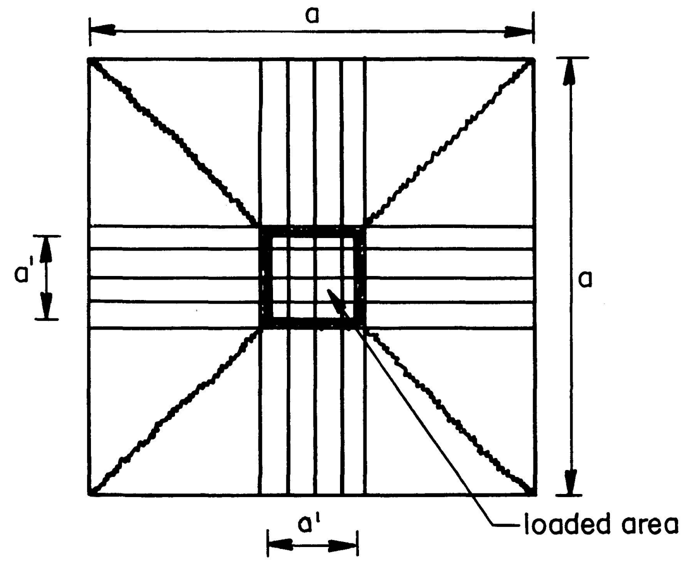


Fig. 4. Yield line pattern for case III.

approximately the width of the loaded area, are created which contain additional reinforcement in the direction indicated. Loading is through a square column stub of side length a' at the center, and it is assumed that the concrete under the column does not crack. The magnitudes of the positive yield moments are m_1 in the major portions of the slab and $k\,m_1$ in the bands, perpendicular to the direction of the steel. A simplified yield line pattern is employed for the analysis, see figure 4, for which it can easily be determined that

$$P = 8\,m_1\left(1 + k\frac{a'}{a-a'}\right). \tag{7}$$

Case IV: A square slab, with length of side a, is supported along the perimeter, with the corners held down. Reinforcement is isotropic with positive

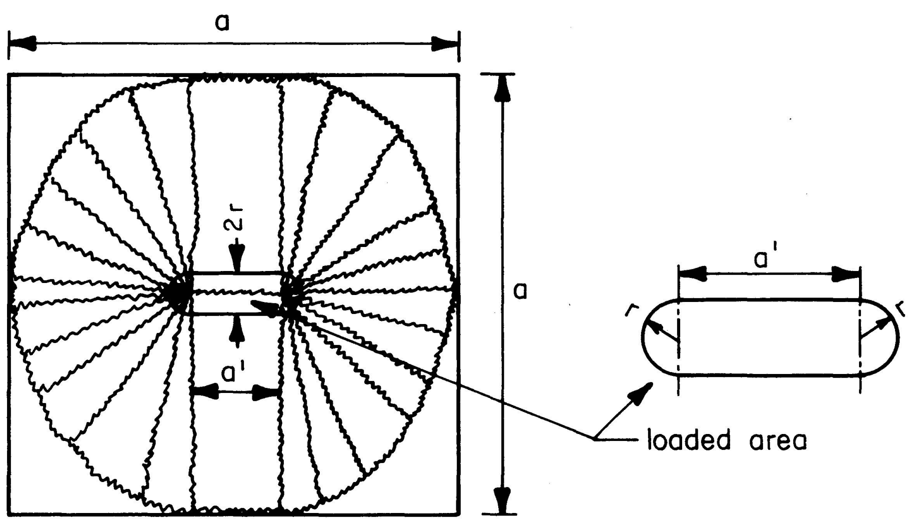


Fig. 5. Yield line pattern for case IV.

yield moment m_1 and negative yield moment m_2. Loading is through an elongated column stub of width $2r$ and length $a'+2r$. See fig. 5 for the shape of the loaded area and the assumed yield line pattern. The total load is P, and it is assumed that the slab will crack beneath the loaded area. For simplicity, and since it provides the lowest value of P, it will also be assumed in the derivation of the load equation that the entire load deflects the same distance. The virtual work equation then becomes (note that R is different for each wedge shaped segment):

$$P = (m_1+m_2)\left[2a'\frac{1}{\frac{1}{2}a}+\int_0^{2\pi} R\,d\phi\frac{1}{R}\right]. \quad (8)$$

Solving,

$$P = \left(2\pi+4\frac{a'}{a}\right)(m_1+m_2). \quad (9)$$

Case V: A square slab is supprted on two opposite edges only. Reinforcement is isotropic with positive and negative yield moments m_1 and m_2 respectively. Loading is through a square column stub at the center. See fig. 6. It can easily be determined that

$$P = \frac{4(m_1+m_2)}{1-\frac{a'}{a}}. \quad (10)$$

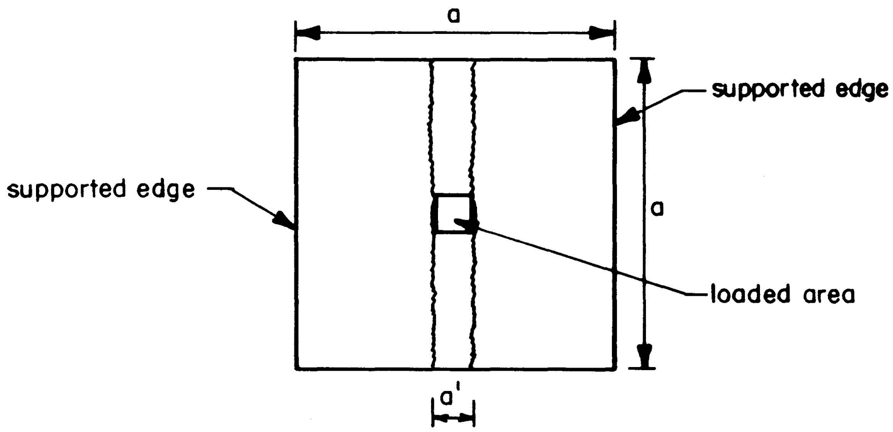


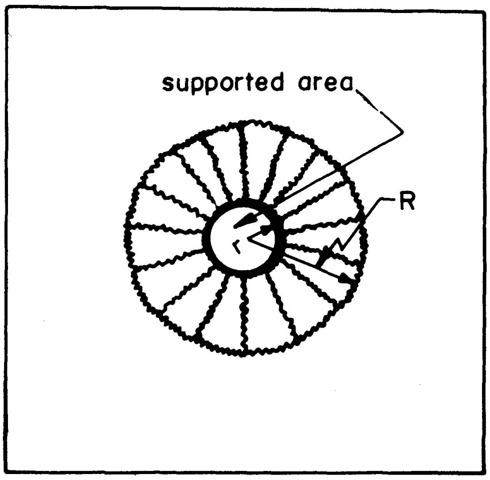
Fig. 6. Yield line pattern for case V.

Case VI: A square or circular slab of area A_{slab} is supported on a circular column, of radius r, located at its center. It is loaded with a uniformly distributed load w per unit area. This loading produces an axial force P in the column. The reinforcement is isotropic and the positive and negative yield moments are m_1 and m_2 respectively. It is assumed that the slab will not crack above the column, but that a yield fan will form as shown in fig. 7, which will permit the entire slab outside the fan to drop the same distance when failure occurs. Let the average shear per unit length on the perimeter of the fan be q. Then from statics,

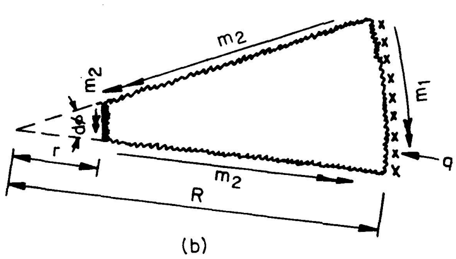
$$q = \frac{P - \pi R^2 w}{2 \pi R}. \tag{11}$$

Dropping the rim of the fan and the entire slab outside it a unit distance, one obtains the virtual work equation by summing the effects on the individual segments of the fan:

$$\int_0^{2\pi} q R\, d\phi + w\left[\tfrac{1}{2} r\, d\phi\, (R-r) + \tfrac{1}{2}(R-r)(R-r)\, d\phi\, \tfrac{2}{3}\right] = \int_0^{2\pi} \frac{R\, d\phi}{R-r}(m_1 + m_2). \tag{12}$$



(a)



(b)

Fig. 7. Yield line pattern for case VI.

Substituting the value of q from equation (11) into equation (12), writing $P = w A_{slab}$ and solving for the sum of the moments,

$$m_1 + m_2 = \frac{w}{2\pi}\left(1 - \frac{r}{R}\right)\left[A_{slab} - \frac{\pi R^2}{3}\left(1 + \frac{r}{R} + \frac{r^2}{R^2}\right)\right] \tag{13a}$$

or

$$m_1 + m_2 = \frac{P}{2\pi}\left(1 - \frac{r}{R}\right)\left[1 - \frac{\pi R^2}{3 A_{slab}}\left(1 + \frac{r}{R} + \frac{r^2}{R^2}\right)\right]. \tag{13b}$$

To determine R, one must maximize the sum of the moments. Differentiating both sides of either of equations (13) with respect to R and setting the resulting partial derivative of the right side equal to zero, one obtains

$$R = r \sqrt[3]{\frac{3}{2}\frac{A_{slab}}{\pi r^2} - \frac{1}{2}}. \tag{14}$$

Substituting this value of R back into equations (13), one can then solve for the sum of the moments, or, alternatively, if one is dealing with a slab of known properties, one can then find P or w.

Correlation Between Theory and Test Results

The equations derived above represent simplified mathematical models of certain idealized physical models which, in turn, are supposed to represent the conditions found in actual buildings. This discussion will only concern itself with the relationship between the mathematical and the experimental models for the time being.

The mathematical models differ from most of the physical models reported in the literature in several respects: First, for convenience in calculation, it was assumed that the punching loads were distributed over circular areas, while in most specimens square plates or column stubs were used. Pictures in references [4, 8, 12, 18 and 19] show that the circular fan cracking patterns developed in specimens with square column stubs were indistinguishable from those developed by the circular column stubs of reference [14], except that the cracks on the tension side, around the periphery of the loaded area, had a slightly different shape. However, even in the case of square column stubs these cracks were frequently roughly circular. Second, it will be desirable to apply the equations derived in cases I and IV to specimens in which the corners were free to lift. Photographs in reference [12] show that the assumed circular yield line patterns will form under those conditions also. It is only necessary then to set the negative yield moment of these slab equal to zero. Third, in pictures of specimens in which case I might be expected to apply, one can observe that cracking usually took place under the loaded area even when the column stub was cast monolithically with the slab, but that a crack seemed to form around the periphery of the load at the same time. This indicates that the actual failure yield pattern was an intermediate between fig. 1 and 2. As one can see by examining equations (2) and (3) they may be expected to give values of P which differ very little. Since, as mentioned earlier, yield line theory gives an upper bound on the load, it seems more conservative to use equation (3) in on all cases except those in which the column stub was so large that cracking beneath it was unlikely. Finally, for the same reason, one should also choose the smallest feasible r for the equivalent circular loaded area when dealing with square column stubs. This would be the radius of the largest circle which could be inscribed in the loaded area. The physical equivalent of this is to assume that the corners of the loading device exert little or no pressure on the slab. By reference to equations (2), (3), (13) and (14), it can be seen that possible inaccuracies in r of the magnitude involved in this assumption can have relatively little effect on P. (For a rough estimate, in most practical cases the change in P will be approximatelx 10% of the change in r.) The slightly low value of $P_{yield\ line}$ resulting from the foregoing approximations compensates at least partly for the fact that some yielding of reinforcement and consequent wide cracking will actually occur at loads less than $P_{yield\ line}$.

106 specimens of slabs supported along their boundaries and subjected to punching loads, reported in the literature, were analyzed by the yield line theory. The arithmetic mean of $P_{yield\ line}/P_{test}$ was 1.015, where P_{test} is the load at which the specimen was reported to have "failed". The standard deviation was 0.248. In addition, 128 footings tested by RICHART [4] were also analyzed, using

equations (13). The analogy between these footing tests and the analysis of a slab carrying a uniformly distributed load and supported on columns, seems quite evident, particularly since RICHART noted that "... the deflections of the footing were small in comparison to the amount of closure of the car springs until the yield point of the reinforcement was reached, so that an essentially uniform support was provided over the bottom of the footing". The arithmetic mean of $P_{yield\,line}/P_{test}$ for the footing tests was 1.05, with a standard deviation of 0.32. Further details of the comparisons, including yield line patterns used, values of $P_{yield\,line}/P_{test}$ for all specimens examined and arithmetic means and standard deviations for individual test series are given in Table I.

Table 1. Summary of Test and Calculation Results

Source	Specimen Designation	Critical Yield Line Pattern	$\frac{P_{Y.L.}}{P_{test}}$	Q	Remarks
MOWRER and VANDERBILT [11]	M-2-1-0	Fig. 2	0.789	1.36	Series M specimens contained both tension and compression reinforcement and were made from normal weight concrete.
	M-3-1-0	Fig. 2	0.835	1.07	
	M-4-1-0	Fig. 2	0.715	0.92	
	M-5-1-0	Fig. 1	0.714	0.60	Very large column stub
	M-6-1-0	Fig. 1	0.706	0.46	Very large column stub
	M-7-1-0	Fig. 1	0.633	0.40	Very large column stub
	M-8-1-0	Fig. 1	0.643	0.36	Very large column stub
	M-2-2-0	Fig. 2	1.080	5.90	
	M-3-2-0a	Fig. 2	1.057	4.56	
	M-3-2-0b	Fig. 2	0.796	2.68	
	M-4-2-0	Fig. 2	0.964	2.82	
	M-5-2-0	Fig. 1	0.905	2.44	Very large column stub
	M-6-2-0	Fig. 1	0.948	1.89	Very large column stub
	M-7-2-0	Fig. 1	0.837	1.67	Very large column stub
	M-8-2-0	Fig. 1	0.903	1.43	Very large column stub
	M-3-1-2	Fig. 2	1.026	0.93	2 edges fixed
	M-3-1-4a	Fig. 2	1.022	1.06	4 edges fixed
	M-3-1-4b	Fig. 2	0.904	1.09	4 edges fixed
	JL-0-1.7a	Fig. 2	1.016	3.54	Series J specimens contained only tension reinforcement.
	JL-3-1.7a	Fig. 2	0.870	3.77	
	JL-0-1.7b	Fig. 2	0.977	2.57	Series JL specimens were manufactured from lightweight aggregate concrete, series JN specimens from normal weight concrete.
	JL-3-1.7b	Fig. 2	0.828	2.99	
	JN-0-1.7	Fig. 2	0.700	3.97	
	JN-3-1.7	Fig. 2	0.774	3.91	J series slabs with a -3- designation contained a $1\frac{1}{2}''$ square hole at the center of one edge of the column stub.
	JL-0-2.2a	Fig. 2	0.937	6.60	
	JL-3-2.2a	Fig. 2	1.122	6.77	
	JL-0-2.2b	Fig. 2	1.374	5.30	
	JL-3-2.2b	Fig. 2	0.996	5.30	
	JN-0-2.2	Fig. 2	0.757	6.09	
	JN-3-2.2	Fig. 2	0.794	6.77	
	JL-0-1.1	Fig. 2	0.642	0.92	
	Arithmetic Mean = 0.880		Standard Deviation = 0.143		
ELSTNER and HOGNESTAD [5]	B-1	Fig. 2	0.723	0.45	Series B contained only tension reinforcement.
	B-2	Fig. 2	0.711	0.25	
	B-3	Fig. 2	0.806	1.70	Contained. Shear reinforcement
	B-4	Fig. 2	0.781	0.91	
	B-5	Fig. 2	1.086	9.40	Shear reinforcement
	B-6	Fig. 2	1.057	7.84	Shear reinforcement

Source	Specimen Designation	Critical Yield Line Pattern	$\frac{P_{Y.L.}}{P_{test}}$	Q	Remarks
ELSTNER and HOGNESTAD [5]	B-9	Fig. 2	1.101	4.38	
	B-10	Fig. 2	1.038	4.18	Shear reinforcement
	B-17	Fig. 2	0.941	4.30	Shear reinforcement
	B-11	Fig. 2	1.187	21.25	
	B-12	Fig. 2	0.991	9.33	Shear reinforcement
	B-13	Fig. 2	1.000	9.06	Shear reinforcement
	B-14	Fig. 2	1.346	8.73	
	B-15	Fig. 2	1.142	9.06	Shear reinforcement
	B-16	Fig. 2	1.066	9.63	Shear reinforcement
	A-1a	Fig. 2	1.015	2.63	Series A contained both tension and compression reinforcement.
	A-1b	Fig. 2	0.912	1.95	
	A-1c	Fig. 2	0.948	1.83	
	A-1d	Fig. 2	0.973	1.62	
	A-1e	Fig. 2	0.908	2.19	
	A-4	Fig. 2	0.867	1.93	
	A-2a	Fig. 2	1.475	11.23	
	A-2b	Fig. 2	1.430	9.40	
	A-2c	Fig. 2	1.343	6.78	
	A-7b	Fig. 2	1.160	7.87	
	A-5	Fig. 2	1.180	7.87	
	A-7	Fig. 6	1.000	7.77	
	A-8	Fig. 6	0.939	8.61	
	A-3b	Fig. 2	1.75	19.60	
	A-3c	Fig. 2	1.47	18.13	
	A-3d	Fig. 2	1.54	15.88	
	A-6	Fig. 2	1.71	16.65	
	A-13	Fig. 2	0.620	0.41	
	Mean = 1.098		Standard Deviation = 0.272		
MOE [8]	S 5–60	Fig. 2	0.968	2.52	Steel plate shear reinforcement
	S 6-60	Fig. 2	1.027	2.56	Steel plate shear reinforcement
	S 7-60	Fig. 2	0.794	2.48	$\frac{1}{4}''$ diameter bar shear head
	S 8-60	Fig. 2	0.912	2.48	
	S 5-70	Fig. 2	1.034	3.00	
	S 6-70	Fig. 2	1.048	2.91	Steel plate shear reinforcement
	S 1-60	Fig. 2	0.878	1.96	
	S 1-70	Fig. 2	1.026	2.32	
	R 1	Fig. 2	0.934	2.09	
	R 2	Fig. 2	1.128	4.28	
	S 2-60	Fig. 4	1.122	4.23	
	S 3-60	Fig. 4	0.900	9.33	
	S 4-60	Fig. 4	0.706	0.12	
	S 3-70	Fig. 4	1.011	11.78	
	S 4-70	Fig. 4	0.829	0.12	
	S 4-70A	Fig. 4	0.748	0.16	
	Mean = 0.942		Standard Deviation = 0.127		
HOGNESTAD, ELSTNER and HANSON [10]	H 1	Fig. 2	0.833	2.54	All specimens in this series were made of lightweight concrete
	H 1 L 3	Fig. 2	1.013	2.38	
	H 1 L 4	Fig. 2	1.006	2.50	
	R 1	Fig. 5	0.934	2.09	
	R 1 L 13	Fig. 5	1.261	2.19	
	R 1 L 4	Fig. 5	1.113	2.04	
	Mean = 1.027		Standard Deviation = 0.148		
GRAF [6]	1355	Fig. 2	0.876	8.83	All of these specimens contained shear reinforcement, except 1362 and 1375.
	1356	Fig. 2	0.839	8.82	
	1361	Fig. 2	0.861	11.01	
	1363	Fig. 2	0.920	11.78	

Source	Specimen Designation	Critical Yield Line Pattern	$\frac{P_{Y.L.}}{P_{test}}$	Q	Remarks
GRAF [6]	1376	Fig. 2	0.901	9.48	
	1377	Fig. 2	0.915	9.82	
	1362	Fig. 2	2.02	15.91	
	1375	Fig. 2	2.47	15.87	
	Mean = 1.225		Standard Deviation = 0.641		
KINNUNEN and NYLANDER [14]	5089	Fig. 2	0.831	0.71	All of these specimens were circular in shape, as was the loaded area.
	5098	Fig. 2	0.830	0.67	
	5215	Fig. 2	1.228	3.72	
	5224	Fig. 2	1.152	3.68	
	5107	Fig. 2	0.989	2.99	
	5117	Fig. 2	1.048	3.25	
	5125	Fig. 3	0.981	2.88	
	5134	Fig. 3	0.962	2.89	
	5269	Fig. 3	1.239	12.60	
	5270	Fig. 3	1.288	12.83	
	5281	Fig. 3	1.232	13.75	
	5290	Fig. 3	1.122	14.05	
	Mean = 1.075		Standard Deviation = 0.159		

Arithmetic Mean of $P_{Y.L.}/P_{test}$ for 106 specimens of slabs supported along their boundaries and subjected to punching loads = 1.015, Standard Deviation = 0.248.

Source	Specimen Designation	Critical Yield Line Pattern	$\frac{P_{Y.L.}}{P_{test}}$	Q	Remarks
RICHART [4]	105a	Fig. 7	0.87	1.84	Footing tests
	105b	Fig. 7	1.14	2.20	
	106a	Fig. 7	0.81	1.62	
	106b	Fig. 7	0.89	1.66	
	107a	Fig. 7	0.87	1.61	
	107b	Fig. 7	0.97	1.67	
	108a	Fig. 7	0.72	1.19	
	108b	Fig. 7	0.76	1.34	
	109a	Fig. 7	1.40	4.64	
	109b	Fig. 7	1.63	4.56	
	110a	Fig. 7	1.39	4.10	
	110b	Fig. 7	1.34	4.47	
	111a	Fig. 7	1.42	4.67	
	111b	Fig. 7	1.19	4.40	
	112a	Fig. 7	2.00	7.33	
	112b	Fig. 7	1.85	8.13	
	109Ra	Fig. 7	1.13	3.41	
	109Rb	Fig. 7	1.18	3.36	
	110Ra	Fig. 7	1.44	4.36	
	110Rb	Fig. 7	1.23	4.15	
	201a	Fig. 7	1.85	6.43	
	201b	Fig. 7	1.63	6.43	
	202a	Fig. 7	1.32	4.72	
	202b	Fig. 7	1.38	4.93	
	203a	Fig. 7	1.32	3.30	
	203b	Fig. 7	1.46	3.77	
	204a	Fig. 7	2.05	10.02	
	204b	Fig. 7	2.05	10.10	
	205a	Fig. 7	1.77	7.67	
	205b	Fig. 7	1.77	7.43	
	206a	Fig. 7	1.58	5.29	
	206b	Fig. 7	1.35	5.53	
	207a	Fig. 7	1.23	5.07	
	207b	Fig. 7	1.30	5.23	
	208a	Fig. 7	1.22	3.24	
	208b	Fig. 7	1.15	3.31	
	209a	Fig. 7	0.97	2.57	

Source	Specimen Designation	Critical Yield Line Pattern	$\frac{P_{Y.L.}}{P_{test}}$	Q	Remarks
RICHART [4]	209b	Fig. 7	1.02	2.81	
	210a	Fig. 7	1.24	5.05	
	210b	Fig. 7	1.39	5.10	
	211a	Fig. 7	1.16	3.80	
	211b	Fig. 7	1.07	3.53	
	212a	Fig. 7	1.07	2.72	
	212b	Fig. 7	1.12	2.64	
	213a	Fig. 7	1.30	4.95	
	213b	Fig. 7	1.30	4.92	
	214a	Fig. 7	0.94	2.97	
	214b	Fig. 7	0.91	2.92	
	215a	Fig. 7	0.89	2.84	
	215b	Fig. 7	0.89	3.07	
	216a	Fig. 7	1.23	4.92	
	216b	Fig. 7	1.23	4.96	
	217a	Fig. 7	1.15	3.36	
	217b	Fig. 7	0.95	2.99	
	218a	Fig. 7	0.89	2.14	
	218b	Fig. 7	0.97	2.06	
	304a	Fig. 7	0.88	1.84	
	304b	Fig. 7	1.13	2.20	
	305a	Fig. 7	0.81	1.80	
	305b	Fig. 7	0.81	1.76	
	307a	Fig. 7	0.98	2.06	
	307b	Fig. 7	0.92	1.94	
	308a	Fig. 7	0.77	1.57	
	308b	Fig. 7	0.92	1.59	
	309a	Fig. 7	0.89	1.60	
	309b	Fig. 7	0.95	1.66	
	310a	Fig. 7	0.85	1.86	
	310b	Fig. 7	0.97	1.90	
	312a	Fig. 7	0.86	1.72	
	312b	Fig. 7	1.17	1.82	
	314a	Fig. 7	0.89	2.03	
	314b	Fig. 7	1.05	1.84	
	315a	Fig. 7	0.98	1.61	
	315b	Fig. 7	0.79	1.51	
	316a	Fig. 7	0.79	1.55	
	316b	Fig. 7	0.82	1.48	
	317a	Fig. 7	0.90	1.73	
	317b	Fig. 7	0.84	1.70	
	319a	Fig. 7	1.03	1.91	
	319b	Fig. 7	0.90	1.94	
	321a	Fig. 7	0.93	1.67	
	321b	Fig. 7	0.87	1.63	
	323a	Fig. 7	0.89	1.67	
	323b	Fig. 7	0.85	1.58	
	324a	Fig. 7	0.96	1.71	
	324b	Fig. 7	0.94	1.68	
	326a	Fig. 7	0.88	1.82	
	326b	Fig. 7	0.87	1.64	
	403a	Fig. 7	0.92	1.63	
	403b	Fig. 7	1.12	2.16	
	501a	Fig. 7	1.29	9.50	500 series footings were rectangular in shape.
	501b	Fig. 7	1.34	9.46	
	502a	Fig. 7	0.91	3.76	
	502b	Fig. 7	0.87	3.89	
	503a	Fig. 7	0.86	3.75	
	503b	Fig. 7	0.92	3.80	
	504a	Fig. 7	1.76	18.91	
	504b	Fig. 7	1.63	18.60	

Source	Specimen Designation	Critical Yield Line Pattern	$\frac{P_{Y.L.}}{P_{test}}$	Q	Remarks
RICHART [4]	505a	Fig. 7	0.86	5.95	
	505b	Fig. 7	0.89	5.90	
	506a	Fig. 7	0.94	6.22	
	506b	Fig. 7	0.94	5.83	
	701a	Fig. 7	0.94	3.22	
	701b	Fig. 7	0.92	3.34	
	702a	Fig. 7	1.26	2.61	
	702b	Fig. 7	0.79	1.65	
	101a	Fig. 7	0.71	0.55	The following specimens were listed by Richart as failing by tension of the reinforcement.
	101b	Fig. 7	0.73	0.57	
	103a	Fig. 7	0.67	0.50	
	103b	Fig. 7	0.62	0.50	
	104a	Fig. 7	0.84	0.91	
	104b	Fig. 7	0.93	0.92	
	313a	Fig. 7	0.67	1.25	
	313b	Fig. 7	0.65	1.14	
	318a	Fig. 7	0.67	1.13	
	318b	Fig. 7	0.65	1.16	
	320a	Fig. 7	0.72	1.17	
	320b	Fig. 7	0.76	1.32	
	322a	Fig. 7	0.65	1.05	
	322b	Fig. 7	0.63	1.00	
	325a	Fig. 7	0.68	1.23	
	325b	Fig. 7	0.72	1.27	
	327a	Fig. 7	0.69	1.01	
	327b	Fig. 7	0.76	1.02	
	329a	Fig. 7	0.73	1.45	
	329b	Fig. 7	0.71	1.44	
	404a	Fig. 7	0.74	1.31	
	404b	Fig. 7	0.95	1.30	

Arithmetic Mean of $P_{Y.L.}/P_{test}$ for 128 footing specimens subjected to punching loads $= 1.05$, Standard Deviation $= 0.32$.

Differentiation Between Bending and Shear Failure

When the value of $P_{yield\,line}/P_{test}$ is less than one, it is an indication that the failure was preceded and perhaps initiated by yielding of the tension reinforcement along the yield lines, i.e. that it was a primary bending rather than a shear failure. (In fact, as indicated previously, yield line theory only predicts collapse due to formation of a mechanism. It is therefore likely that the reinforcement in the most highly stressed regions of the slab would have yielded at somewhat lower loads than $P_{yield\,line}$.) However, in the case of approximately half of the specimens, $P_{yield\,line}/P_{test}$ was larger than one, and the standard deviations reported above are rather large. It therefore appears that in many tests the failure must have been primarily due to shear. Which of the two types of failure will be critical in any given specimen will naturally be determined by its relative strengths in shear and bending and its relative loading in these two modes.

The relative strengths will be a function of $\frac{p f_y d^2}{\sqrt{f'_c}\, b\, d}$, where b is to be taken as the perimeter of the column stub and the other symbols have their usual mean-

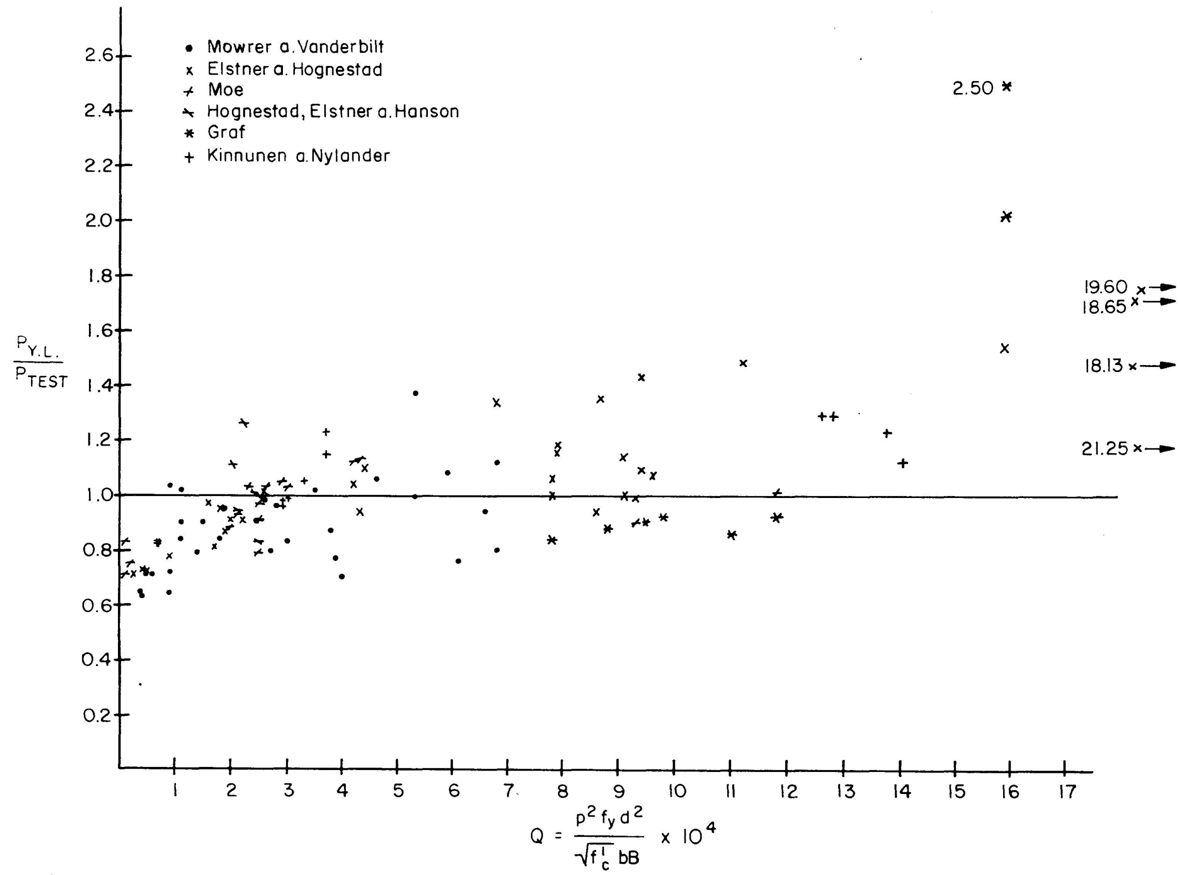


Fig. 8. Yield line load / Test load.

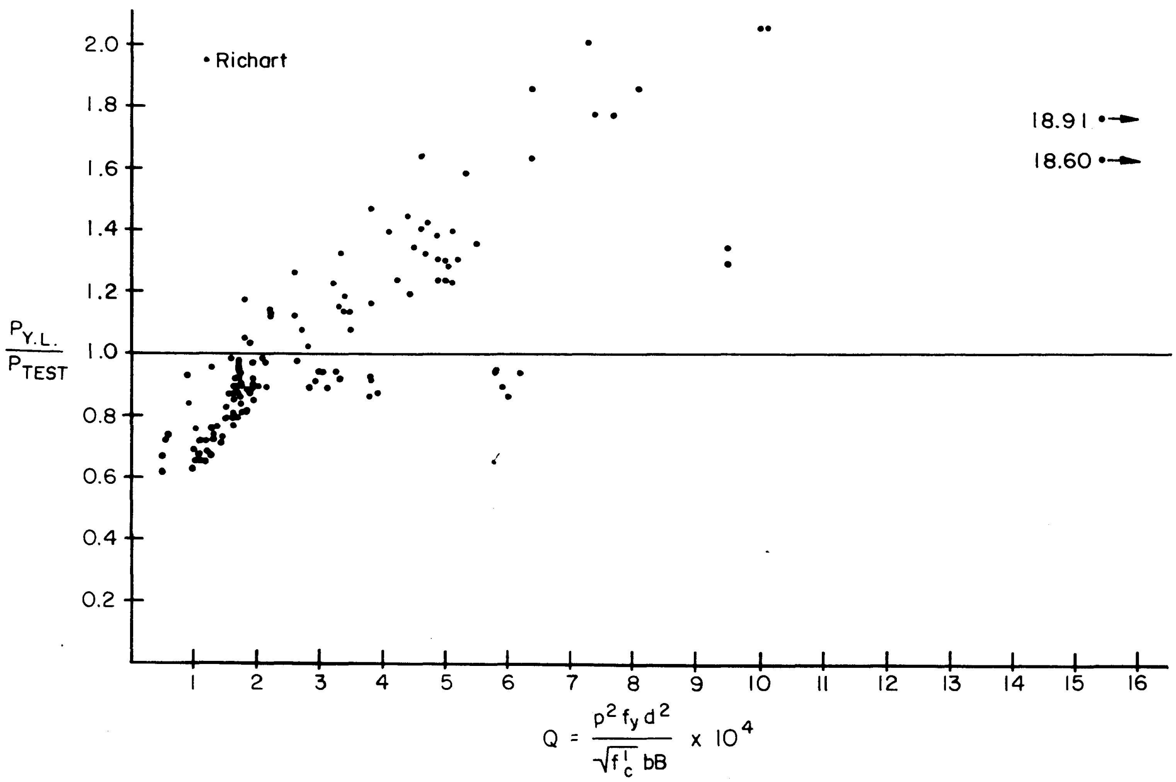


Fig. 9. Yield line load / Test load.

ings. If the slab contains two layers of reinforcement, one near the top and one near the bottom surface, it would seem logical to make p equal to the sum of the reinforcement ratios, since it is the sum of the yield moments in the positive and negative directions which determines the yield line punching strength. However, if the reinforcement is so arranged that the layer acting as compression reinforcement extends into the column, it will provide an appreciable amount of additional shear resistance through dowel action [15] and through strengthening of the shear-compression zone. In this case, therefore, it will probably be more accurate to make p equal only to the ratio of reinforcement acting in tension along the column perimeter. The latter comment will also apply, of course, when the support conditions are such that one of the yield moments will not act.

It is more difficult to establish a criterion for the relationship between the loadings, but, empirically, they seem to be related through $\frac{p\,d}{B}$, where B is the perimeter of the slab. Combining the two relationships, one can empirically define a parameter Q such that

$$Q = \frac{p\,f_y\,d^2}{\sqrt{f_c'}\,b\,d}\left(\frac{p\,d}{B}\right)10^4 = \frac{p^2\,f_y\,d^2}{\sqrt{f_c'}\,b\,B}(10^4). \tag{15}$$

Values of Q for all specimens are included in table I.

The values of $P_{yield\,line}/P_{test}$ for the 106 slab specimens supported along their boundaries are plotted against Q in fig. 8, while fig. 9 displays similar data for the 128 footing specimens. If one regards the $\frac{P_{Y.L.}}{P_{test}} = 1.0$ line as the dividing line between primarily shear and primarily yield line bending failure, then it is apparent that when Q is less than 2, the specimens failed first in yield line bending. When Q is between 2 and 4, approximately half the specimens failed first in bending and half probably failed primarily in shear, while for Q greater than 4 most specimens without shear reinforcement failed in shear.

Application to design

The present design of flat slab or flat plate structures is largely based on the need to prevent shear failures. When one is dealing with interior panels (and these are the only ones to which the above analyses can be applied at this time), it appears that normally the critical loading will be one in which a uniformly distributed load is applied over the entire surface. Therefore equation (13a) is the applicable one, and one should design against punching failure by calculating the required moment sum and providing the appropriate reinforcement and concrete depth. Then one can check for the value of Q and if that is less than 2, the shear check becomes essentially irrelevant since, from the above evidence, the structure will fail first in flexure.

If one applies this criterion to current designs the results are rather startling. For instance, if one looks at the standard design of an interior panel of a slab, supported on circular columns spaced 25 feet center to center and supporting a service live load of 200 pounds per square foot, one finds that the CRSI Handbook [20] requires both a drop panel and a large capital. Using their slab design for this case, but omitting the capital and assuming the slab to be supported directly on 12 inch diameter columns, one finds that $Q = 1.47$. This is based on p calculated from the CRSI design for the column strip by adding twice the number of trussed bars, plus the number of top bars, plus, since the bottom layer of reinforcement is not detailed to extend into the column, the straight bars. If one then calculates v_u according to equations 15–1 and 17–7 of ACI 318–63, one finds that it would be 345 psi, far above the permissible 208 psi. However, according to the experimental evidence, the failure would clearly be of a bending yield line nature, which requires further examination. The Handbook or Code design is based on shear failure, and therefore requires the capital.

Analyzing the Handbook design, but without the capital, according to equations (13a) and (14), one finds that first, the radius of the yield fan, R, is slightly larger than half the side length of the drop panel, and, second, that the sum of the yield moments required for the ultimate load is slightly more than 11% greater than that available. To ensure that the full yield moments can be developed throughout the fan, it would therefore be desirable to enlarge the drop panel by a foot or so in each direction, at the same time extending the bottom reinforcement so that it ends about a foot nearer the column than presently detailed. The sum of the yield moments must be increased by 11% or more, which can be accomplished either by a 15% increase in depth within the drop panel area without changing Q, or, perhaps more easily, by a 12% increase in p (concentrated in the negative reinforcement), which will increase Q to 1.84. This is still within the range in which shear failure apparently cannot occur, which means that the capital is unnecessary. The savings to be obtained by elimination of column capitals are obvious.

A similar analysis will show that column capitals are also unnecessary in a slab supporting a service live load of 200 pounds per square foot with 16 inch diameter columns spaced 40 feet center to center, provided the drop panels are extended slightly, p is increased by 15% and the bottom reinforcement is continued through the column. In lieu of the latter two requirements a 10% increase in column diameter and a 15% increase in depth through the drop panel would also provide an adequate design against yield line punching, or shear failure. Even the most extreme case considered in the CRSI Handbook, 40 feet spacing of columns and 500 psf, service live load will, if one assumes the columns to have diameters of 22 inches and removes the capitals, give a Q of 1.85, and a safe design against punching, provided the bottom reinforcement is extended into the columns, the drop panel area is extended slightly, and the total depth is increased by 16% in the drop panel area. It should be noted that the column sizes chosen for these illustrations are very small for the loads involved. Since Q is inversely proportional to column diameter, any increase in the column sizes would automatically improve the situation. Effects of these changes on design requirements for bending moments in the remainder of the structure were not examined.

Since shear appears not be the primary cause of failure as long as Q is less than 2, this index can also be used to determine the possible requirement for shear reinforcement for a slab when capitals and/or drop panels are undesirable. From test results [5, 6, 8] on specimens containing shear reinforcement, it is apparent that such devices can be very effective in keeping $P_{yield\,line}/P_{test}$ near or below one even for very high values of Q.

Conclusions

1. Yield line patterns and equations can be devised which will model some cases of a load concentrated over a small area of a reinforced concrete slab supported on its perimeter, and also of uniformly loaded reinforced concrete slabs supported on columns.

2. Test results from 234 punching specimens available in the literature were compared with values of punching load predicted by yield line theory. The overall arithmetic mean of $P_{yield\,line}/P_{test}$ was 1.03, with a standard deviation of 0.29.

3. Closer examination of the test results revealed that one could define a parameter $Q = \frac{p^2 f_y d^2}{\sqrt{f'_c}\, b\, B}(10^4)$, which could be calculated for all specimens. It appears that when Q is less than 2, $P_{yield\,line}/P_{test}$ is consistently less than 1, i.e. the punching strength predicted by yield theory is less than the actual punching strength. Therefore it appears that bending strength rather than shear strength is the controlling factor for design whenever Q is less than 2.

4. When Q is between 2 and 4, either strength may control, while for Q larger than 4 the shear strength definitely becomes the critical one.

5. Analysis of some standard designs for flat slab structures shows that under present shear provisions Q is very low and that in most cases column capitals may be omitted without increasing Q above 2, even though this raises the nominal ultimate shear stresses far above the permissible. Some revisions in reinforcement detailing and small increases in either reinforcement percentage or depth in the drop panel area may then become necessary to prevent the possibility of punching yield line failure. Effects on design requirements for bending moments in other parts of the structure were not investigated.

6. Since the Code shear provisions seem to be overconservative, it is likely that shear reinforcement could safely be omitted from some flat plate structures in which it would otherwise seem necessary.

7. Since bending failures are much preferable to shear failures because they are more gradual and give warning of distress, it would seem prudent to keep Q less than two in all designs.

8. The preceding investigation has been solely concerned with symmetrically loaded, symmetrical specimens. Further experimental data will be required to analyze unsymmetrical cases in which appreciable bending moment will be transferred from a column to the slab, or slabs in which the reinforcement is not isotropic.

Notation

a	side length of square slab
a'	side length of square column, or other column dimension
A_{slab}	area of slab supported by one column
b	perimeter of column
B	perimeter of slab supported on one column or supporting one column
d	distance from compression face of reinforced concrete section to the centroid of the tension reinforcement
f_y	yield stress of reinforcement
f'_c	compression strength of concrete
k	an arbitrary constant
m_1	positive yield moment per unit length of slab – numerically equal to M_u as defined by ACI 318–63, with the capacity reduction factor set equal to one.
m_2	negative yield moment per unit length of slab – calculated similarly to m_1
p	ratio of reinforcement
P	axial load in column, column stub or other loading device
P_{flex}	defined in reference [3]
P_{test}	load at which an experimental specimen was reported to have "failed"
$P_{yield\,line}$	load carrying capacity of a slab predicted by yield line theory
$P_{Y.L.}$	abbreviation for $P_{yield\,line}$
q	shear in slab per unit length
Q	parameter defined by equation (15)
r	radius of circle over which load P is, or may be assumed to be, distributed
R	radius of yield line fan
v_u	ultimate shear stress as defined by equation 17–7 of ACI 318-63
V_{flex}	defined in reference [8]
V_{test}	defined in reference [8]
w	distributed load per unit area of slab
ϕ	angle measured in the plane of the slab, around central axis of column, column stub or other loading device

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Summary

Yield line theory equations are derived for various cases of concentrated loads or columns acting on slabs. These equations are then used in the analysis of 234 punching failure specimens presented in the literature. The arithmetic mean of $P_{yield\,line}/P_{test}$ was 1.03, with a standard deviation of 0.29. A semi-empirical formula, based on the test results, is presented which provides a differentiation between yield line bending failures and shear failures and permits one to predict with good accuracy whether the failure in any given case will be due primarily to bending or to shear. Application to standard designs reveals that capitals and shear reinforcement can probably be eliminated in many flat slab and flat plate structures.

Résumé

Les équations de la théorie des lignes de répartition sont obtenues pour différents cas de charges concentrées ou de colonnes agissant sur des plaques. Ces équations sont alors utilisées dans l'analyse de 234 types de fissures découpantes présentés dans différents ouvrages. La moyenne arithmétique du rapport charge théorique/charge expérimentale ($P_{yield\,line}/P_{test}$, voir appendice 11) était égale à 1,03 avec un écart moyen de 0,29. On présente une formule semi empirique, fondée sur des résultats expérimentaux. Cette formule permet de différencier les lignes de répartition des fissures de flexion de celles de cisaillement. Elle permet également de prévoir avec une bonne précision dans un cas donné si la fissure sera plutôt dûe à la flexion qu'au cisaillement. Son application à des constructions courantes révèle que des armatures principales et des armatures de cisaillement peuvent être généralement supprimées dans les dalles et les plaques.

Zusammenfassung

Hergeleitet werden Bruchliniengleichungen für verschiedene Fälle von Einzellasten oder Stützen auf Decken. Diese Gleichungen sind sodann für 234 in der Literatur angegebene Durchstanzproben angewendet worden. Der Mittelwert ergab $P_{berechnet}/P_{Versuch} = 1{,}03$ mit einer Standardabweichung von 0,29. Eine auf Grund der Versuchsergebnisse gewonnene, halbempirische Formel erlaubt die Unterscheidung zwischen Biege- und Schubbruchlinien sowie mit guter Genauigkeit für irgend einen Fall die Voraussage, ob der Bruch ursprünglich durch Biegung oder Schub verursacht wurde. Die Anwendung bei Regelbauten offenbart, daß Stützenköpfe und Schubbewehrung wahrscheinlich in vielen Flachdecken weggelassen werden können.