Strength of cylindrical shells with imperfections

Strength of Cylindrical Shells with Imperfections

Résistance de coques cylindriques avec imperfections

Festigkeit mit Mängeln behafteter zylindrischer Schalen

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Abstract

Shells in the shape of circular cylinders, acted upon by uniform axial forces or uniform lateral pressure fail by instability, when the loads reach critical intensity. For shells with some simple edge conditions and stresses within the elastic range exact solutions of the instability problems of this kind are available [1]; at the same time close approximate solutions of the same problems may be obtained by the finite element method irrespective of the boundary conditions [2]. Attempts to verify theoretical solutions by experiments have however been unsuccessful, and the main cause for this disagreement seems to be the existance in the shell of some imperfections of shape [3]. In the present study based on finite element procedure a method is proposed for determination of strength of imperfectly shaped shells, fabricated of stiff elasto-plastic material, like structural steel or duralumin. The application of the method is illustrated on examples.

Nature of Shell Failure

The behavior of an imperfectly shaped cylindrical shell under load is quite different from that of a perfect one. A perfect structure resists uniform axial or lateral loading exclusively by membrane stresses, and fails suddenly by snap buckling, involving flexural action, when the load intensity reaches the critical value. On the other hand, imperfect shell develops flexural stresses from the very beginning, and thus presents a typical case of non-linear deformation. For such shell it is reasonable to assume the failure load as the one whose compression stress at the most highly stressed location, produced by both compression and flexure, equals the yield stress σ_y of the material. In most cases deformation of the shell beyond the failure point so defined is likely to proceed under a reduced load, except when the imperfection is very minute.

The allowable load for an imperfect shell, as for a perfect one, should be found by dividing the failure load by the factor of safety.

Shell Instability by the Finite Element Method

The finite element procedure used for analysis of instability of an elastic circular cylinder shell is described in Ref. [2]. It makes use of the model of the cylinder composed of rectangular bar elements or cells. The theory is based on the Rayleigh-Ritz energy principle, by means of which the problem is reduced to an eigenvalue equation, solved by computer. The structure has many critical buckling loads, the lowest of which has usually the greatest practical importance. The results are given in the form of the critical load intensity (the eigenvalue) and the mode of buckling (the eigenvector), i. e. the buckling displacements of all nodes of the model, five for each node: three displacements along the coordinate axes and two angles of rotation about the axes lying in the plane of the shell. The computer also prints out the nodal forces and moments acting on all cells. These data form the basis for computation of strength of an imperfect shell. Computer can also obtain if necessary several other critical loads, above the first one, together with their eigenvectors.

It may be pointed out that, since the buckling mode of a complete shell consists of several waves along the circumference and one or more half-waves along the length, the shell model may be formed of only a segment of the full shell, but this part must be of a proper angle and length.

Imperfect Shell

It is obvious that the strength of an imperfect shell, as defined above, would depend both on the mode of deviation from the theoretical form and on the extent of this deviation. Imperfection of an actual structure is likely to be unintentional and its shape accidental. Designer, committed to safety, should, naturally, be interested in the most unfavorable form of imperfection, and intuition suggests, that this form is one of the buckling modes of a perfect shell, most likely, the first one. The degree of imperfection is just as important as its mode, but is more definite of the two, and may be established, with some exercise of judgment, from the knowledge of equipment and the methods used in fabrication of the shell. Known imperfection in the shape of the first buckling mode of the perfect shell is the basis for determination of the reduced strength of the imperfect shell.

Shell Behavior Under Load

Comprehension of reasoning leading to derivation of strength relations in an imperfect shell may be assisted by reference to a familiar structure, the pin-ended axially loaded column of a constant cross-section, symmetrical about the axis of buckling. The similarity of behavior under load of the shell and the column is very close in spite of physical difference of these structures.

Several cases of loading of the column (or the shell) are illustrated in Figs. 1 to 9, and they are described here one by one. In all cases the state of the structure is fully elastic.

Fig. 1 represents a perfect straight column under a unit load. Its stress, caused by axial compression alone, is σ_1.

In Fig. 2 the load in the same column is brought up to the critical intensity f. The column still remains straight, although it is ready to buckle, if moved sideways. The stress in it, described as the critical stress, is $f\sigma_1$.

In Fig. 3 the axial load is removed, and the column is bent by externally applied moments to the shape of the buckling mode (the sine curve in case of the column) with a small central deflection δ_0, described as being of unit normality. The stress condition in the column is flexure, and the maximum flexural compression, occurring at mid-span, is σ_2. It must be pointed out, that the corresponding stress in the shell, although mostly flexural, contains also minor membrane components.

In Fig. 4 the column is bent in the same buckling mode to normality n, with the mid-span deflection $n\,\delta_0$ and the maximum flexural stress $n\,\sigma_2$. The value of the latter is assumed here to remain always in the elastic range, with the normality n, if necessary, being less than one, even though the stress σ_2 may actually be beyond the elastic limit.

In Fig. 5 the critical load f is placed on the column, bent as in Fig. 3. The eccentricity of the load f creates at all sections the moments needed to keep the column in its deformed shape without the assistance of any external agency. The combined maximum compression stress is $(f\sigma_1 + \sigma_2)$.

Fig. 6 illustrates combination of the conditions of Figs. 2 and 4 with the resultant lateral deflection $n\,\delta_0$ and the maximum compression stress $(f\sigma_1 + n\,\sigma_2)$.

Should the mid-span deflection be increased to $(n+n_1)\,\delta_0$, as in Fig. 7, the maximum compression would be raised to $[f\sigma_1 + (n+n_1)\,\sigma_2]$.

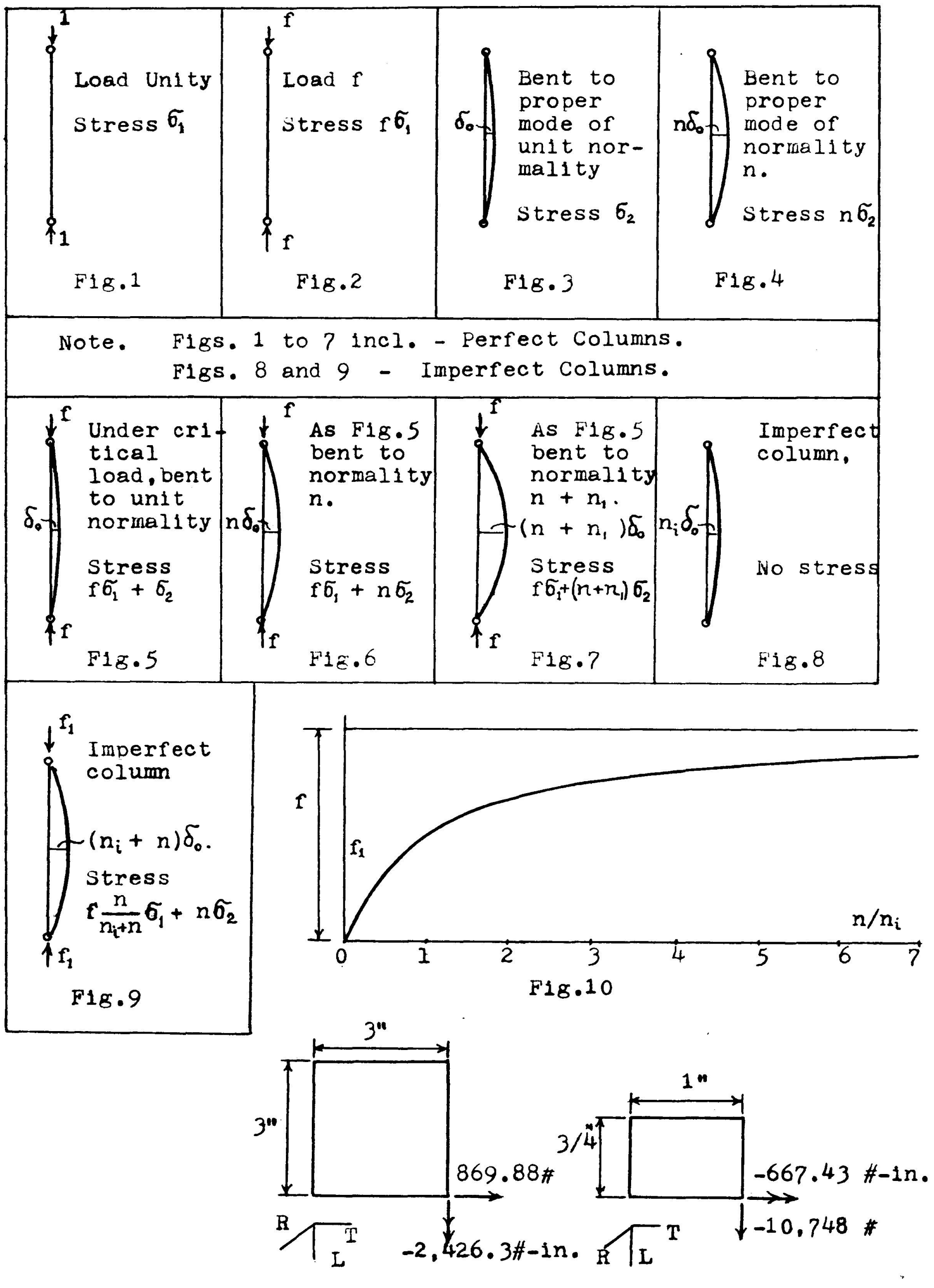


Fig.11 Fig.12

Fig. 8 represents an imperfectly fabricated unstressed column with normality of imperfection n_i, i.e. with the mid-span deviation from straightness $n_i\,\delta_0$. If an axial load, gradually increasing from zero, is applied to this column (Fig. 9), the deviation of the latter from straightness will grow, and its deflected shape will always conform to the same buckling mode. When the mid-span offset becomes $(n+n_i)\,\delta_0$, only a part of it, $n\,\delta_0$, is caused by flexure. At this stage the lever arm of the axial load is $(n+n_i)\,\delta_0$, and so the magnitude of the load f_i corresponding to this condition is less than f; it is

$$f_1 = \frac{n}{n+n_i} f = \frac{n/n_i}{1+n/n_i} f. \tag{1}$$

This equation relates the axial load f_1, causing the deflection of normality n in a column with imperfection n_i, to the buckling load f of a perfect column.

The plot of Eq. (1) is given in Fig. 10.

The maximum compression stress produced in the imperfect structure by the load f_1 is

$$\sigma = \frac{n/n_i}{1+n/n_i} f\sigma_1 + n\sigma_2. \tag{2}$$

The maximum stress σ becomes the yield stress σ_y, when the structure reaches the point of failure; the normality n of its deformation at this point is given by the quadratic (2) and is expressed as follows:

$$\frac{n}{n_i} = \frac{1}{2}\left(\frac{\sigma_y}{\sigma_2'} - \frac{\sigma_{cr}}{\sigma_2'} - 1\right) + \sqrt{\frac{1}{4}\left(\frac{\sigma_y}{\sigma_2'} - \frac{\sigma_{cr}}{\sigma_2'} - 1\right)^2 + \frac{\sigma_y}{\sigma_2'}}. \tag{3}$$

The significance of stresses in this expression with reference to the cylindrical shell is:

σ_y the compression yield stress of the material,
$\sigma_{cr} = f\sigma_i$ the critical compression stress in a perfect shell,
σ_2' a fictitious stress defined by the relation

$$\sigma_2' = n_i\sigma_2, \tag{4}$$

in which σ_2 is the highest compression stress, membrane plus flexural, anywhere in the shell, caused by the buckling deformation of unit normality.

Knowing n/n_i and f, the failure load f_1 of the imperfect shell is found by Eq. (1).

Example 1. Circular cylinder shell, hinged at the ends, under uniform lateral pressure.

Length $L = 96''$, Radius $r = 30.56''$, Thickness $t = 0.4734''$. Material: duralumin, $E = 10{,}000{,}000\,\#/\text{in}^2$, $\sigma_y = 30{,}000\,\#/\text{in}^2$. From the finite element solution of 8×16 model with $3''\times3''$ bar cells, representing a perfect shell.

Critical lateral pressure $f = p_l = 0.10196\,(10)^{-4}\,E = 101.96\,\#/\text{in}^2$. Critical compression stress $\sigma_{cr} = \frac{p_l r}{t} = 6{,}590\,\#/\text{in}^2$.

The maximum compression stress caused by buckling to normality one (maximum transverse deflection $1''$) occurs on the radial plane T (Fig. 11) at the node with the greatest lateral deflection. At the node in question the stress condition is symmetrical about both the horizontal and transverse planes. The normal force here is tension.

The maximum compression stress is

$$\sigma_2 = -\frac{869.88}{(1.5)(0.4734)} + \frac{2426.32\,(6)}{1.5\,(0.4734)^2} = 42{,}080\,\#/\text{in}^2.$$

Assume imperfect shell with normality of imperfection $n_i = 1$.

$$\sigma_2' = n_i \sigma_2 = 42{,}080 \ \#/\text{in}^2; \qquad \frac{\sigma_y}{\sigma_2'} = 0.7125; \qquad \frac{\sigma_{cr}}{\sigma_2'} = 0.1565;$$

by Eqn. (3), $$\frac{n}{n_i} = 0.651.$$

Lateral pressure producing failure, Eq. (1)

$$f_1 = f\frac{0.651}{1.651} = 101.96\frac{0.651}{1.651} = 40.22 \ \#/\text{in}^2.$$

Example 2. Circular cylinder shell, hinged at the ends, under uniform axial load.

Length $L = 96''$, Radius $r = 30.56''$, Thickness $t = 0.1059''$. Material: duralumin, $E = 10{,}000{,}000 \ \#/\text{in}^2$, $\sigma_y = 30{,}000 \ \#/\text{in}^2$. From the finite element solution of 8×16 model with $1'' \times \frac{3}{4}''$ bar cells, representing a perfect shell:

Critical axial load $f = p_a = 0.2207\,(10)^{-3}\,E = 2{,}207 \ \#/\text{in}$. Critical compression stress $\sigma_{cr} = \frac{p_a}{t} = 20{,}840 \ \#/\text{in}^2$.

The maximum compression stress caused by buckling to normality one (maximum transverse deflection $1''$) occurs on the transverse plane L (Fig. 12) at the node with the greatest lateral deflection. With the stress condition at the node in question being symmetrical about both the transverse and the radial planes, only the nodal force $-10{,}748 \ \#$ and the moment $-667.43 \ \#$-in need be considered. The compression stress produced by them is

$$\sigma_2 = \frac{10{,}748}{0.5\,(0.1059)} + \frac{667.43\,(6)}{0.5\,(0.1059)^2} = 917{,}142 \ \#/\text{in}^2.$$

Assume imperfect shell with normality of imperfection $n_i = 1$.

$$\sigma_2' = 917{,}142 \ \#/\text{in}^2; \qquad \frac{\sigma_y}{\sigma_2'} = 0.03272; \qquad \frac{\sigma_{cr}}{\sigma_2'} = 0.02272.$$

By Eqn. (3): $$\frac{n}{n_i} = 0.0315.$$

Axial load producing failure, Eq. (1),

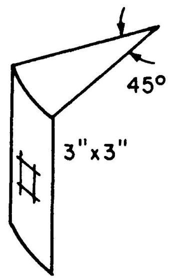
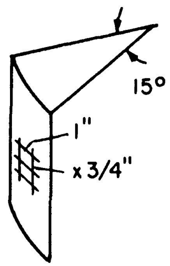
$$f_1 = f\frac{0.0315}{1.0315} = 2{,}207\frac{0.0315}{1.0315} = 67.2 \ \#/\text{in}.$$

The results related to these and other shells with different imperfections are assembled in Table 1.

Comments and Conclusions

1. The procedure described here is equally applicable when the stresses caused by the critical load remain within the elastic range and when they extend

Table 1. Maximum Strength of Hinge-Ended Circular Cylinder Shells

Load	Shell	Model and Cell Size	Material and Properties	Strength p_l #/in² or p_a #/in			
				Shape of Shell			
				Perfect	Imperfect		
					$n_i=0.01$	$n_i=0.1$	$n_i=1$
Lateral Fig. A	$r = 30.56''$ $L = 96''$ $t = 0.4734''$	 Model 9-circum. × 12-length	Duralumin $E = 10^7$ #/in² $\mu = 1/3$ $\sigma_y = 30{,}000$ #/in²	101.9	101.0	87.0	40.2
			Steel $E = 3\,(10)^7$ #/in² $\mu = 1/3$ $\sigma_y = 45{,}000$ #/in²	305.9	291.8	212.8	74.0
Axial Fig. B	$r = 30.56''$ $L = 96''$ $t = 0.1059''$	 Model 9-circum. × 12-length	Duralumin $E = 10^7$ #/in² $\mu = 1/3$ $\sigma_y = 30{,}000$ #/in²	2,207	1,420	479.4	67.2
			Steel $E = 3\,(10)^7$ #/in² $\mu = 1/3$ $\sigma_y = 45{,}000$ #/in²	6,621	2,722	794.0	108.0

p_l #/in²

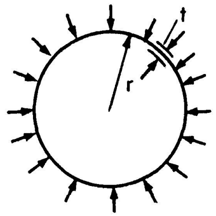


Fig. A

p_a #/in²

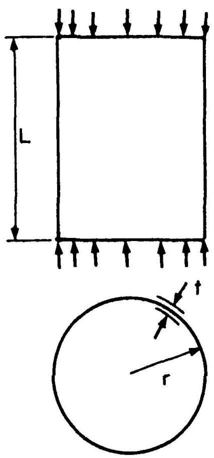


Fig. B

beyond it. Naturally, in the latter case the true failure load in a perfect shell is not f, but a lower load corresponding to the first appearance of yield stress.

2. Under lateral load, failure of shells with high ratio of r/t is affected by small imperfections only slightly. As the imperfections grow bigger this effect increases, particularly in steel, a material with a higher ratio of E/σ_y than duralumin.
3. The effect of imperfections on failure is more pronounced under axial loading. Here the strength of an imperfect shell becomes reduced to a small fraction of that of a perfect shell, especially in case of steel.
4. The proposed theory may be easily extended to cases in which the imperfectly shaped structures contain some residual stresses.
5. Some modes of failure of perfect shells, such as the ones involving several half-waves lengthwise, may appear improbable as modes of imperfections covering the full length of shell. However, it is felt, that even if the imperfections of this kind distort only a part of the shell, the reduction of strength is not likely to be much different.

Notation

E modulus of elasticity
L length of shell
f critical load of a perfect shell or column
f_1 critical load of an imperfect shell or column
n, n_1 normalities of deformation of structures
n_i normality of imperfection
p_a axial uniform pressure at failure
p_l lateral uniform pressure at failure
r radius of shell
t thickness of shell
δ_0 deformation of unit normality
σ compression stress
σ_1 compression stress due to unit load, prior to buckling
σ_2 compression stress caused by flexure of unit normality
σ_2' fictitious stress
σ_{cr} critical stress
σ_y compression yield stress
μ Poisson's ratio

Acknowledgement

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Summary

Shells in the shape of circular cylinders, acted upon by uniform axial forces or uniform lateral pressure fail by instability when the loads reach critical intensity. For shells with some simple edge conditions and stresses within the elastic range exact solutions of the instability problems of this kind are available. In the present study based on finite element procedure a method is proposed for determination of strength of imperfectly shaped shells, fabricated of stiff elasto-plastic material, like structural steel or duralumin.

Résumé

Des coques en formes de cylindres circulaires se trouvant sous l'influence de forces axiales uniformes ou de pressions latérales uniformes présentent des points faibles dus à l'instabilité lorsque la charge atteint des valeurs critiques. Pour des coques à conditions de bords simples et pour des sollicitations dans des régimes élastiques on dispose de solutions exactes des problèmes d'instabilité. Dans la présente étude basée sur le procédé des éléments finis on propose une méthode pour la détermination de la résistance de coques imparfaites en matériel raide et élasto-flexible, comme l'acier de construction ou le duraluminium.

Zusammenfassung

Schalen in Form kreisförmiger Zylinder, die unter Einwirkung gleichförmiger axialer Kräfte oder gleichförmiger seitlicher Drucke stehen, versagen infolge Instabilität, wenn die Belastung kritische Werte erreicht. Für Schalen mit einfachen Randbedingungen und Beanspruchung innerhalb elastischer Bereiche sind genaue Lösungen der Unstabilitätsprobleme verfügbar. In der vorliegenden auf dem Verfahren der endlichen Elemente beruhenden Studie wird eine Methode zur Bestimmung der Festigkeit unvollkommen gestalteter Schalen aus steifem, elastisch nachgiebigem Material, wie Baustahl oder Duraluminium, vorgeschlagen.

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